THE SINGULAR SET OF A NONLINEAR ELLIPTIC OPERATOR

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0. Introduction. The equation

$$\Delta u + \lambda u - u^3 = g \text{ on } \Omega, \quad u \mid \partial\Omega = 0,$$

where $\Omega \subset \mathbf{R}^n$ ($n \le 4$) is a bounded domain, was studied in [9], and we continue that study here. Let H be the Sobolev space $W_0^{1,2}(\Omega)$, define

$$\langle A_\lambda(u), \varphi \rangle_H = \int_\Omega [\nabla u \nabla \varphi - \lambda u \varphi + u^3 \varphi]$$

for all $\varphi \in C_0^\infty(\Omega)$, and define $A: H \times \mathbf{R} \to H \times \mathbf{R}$ by $A(u, \lambda) = (A_\lambda(u), \lambda)$. The present paper investigates the singular set SA of this (real analytic) mapping.

Most results here are actually given for a more general nonlinear operator $A_\lambda: H \to H$ called abstract A_λ (0.1) defined on some Hilbert space, and the map defined in the previous paragraph is called standard A_λ (0.2).

Let $\lambda_j(u)$ be the jth eigenvalue of $DA_\lambda(u)$ (for standard A, of $\Delta v + \lambda v - 3u^2 v = 0$); then the singular set SA is the union of graphs of these eigenvalue functions $\lambda_j: H \to \mathbf{R}$ ($j = 1, 2, \ldots$), each is locally Lipschitzian (1.5), and wherever dim ker $DA_{\lambda_j}(\bar{u}) = 1$, the function λ_j at $\bar{u}$ is locally real analytic (1.8). In particular, for $\lambda < \lambda_2$ (the second eigenvalue of $-\Delta$ with null boundary conditions), SA is the graph of a real analytic function $\lambda_1: U \to \mathbf{R}$, where U is an open star-shaped neighborhood of 0 in H (1.9 and 2.4); moreover the function λ_1 has its only singular point at $u = 0$ (1.10), but it fails to satisfy the Morse lemma (2.10). For standard A the function λ_1 is real analytic for all $u \in H$ (1.9). Moreover, if $\partial\Omega$ is a compact C^∞ manifold, then there are (1.11) an open dense subset V_j of H such that $\lambda_j \mid V_j: V_j \to \mathbf{R}$ is real analytic ($j = 1, 2, \ldots$), and (1.12) an open dense subset W of the singular set SA such that, for every $(u, \lambda) \in W$, dim ker $DA_\lambda(u) = 1$.

If $\lambda < \lambda_{j+1}$ and $0 \ne u \in H$, then the ray $\{cu : c \ge 0\}$ meets SA_λ in at most j points (2.5). On the other hand, given any $\lambda \in \mathbf{R}$, there is a $0 \ne u \in H$ such that the line $\{cu : c \in \mathbf{R}\}$ is disjoint from SA_λ if $\lambda \ne \lambda_j$, and for $\lambda = \lambda_j$ they meet only in $(0, \lambda_j)$ ($j = 1, 2, \ldots$) (2.6). Thus, for any $\lambda > \lambda_1$ there is a $0 \ne u \in H$ such that $\lambda_1 \le \lambda_1(cu) < \lambda$ for all $c \in \mathbf{R}$ (2.7). If $A_\lambda(u) = 0$, $u \in SA_\lambda$, and $\lambda \le \lambda_k$, then (3.1)

$$(u, \lambda) \in \left(\bigcup_{i=1}^{k-2} \text{graph } \lambda_i \right) \cup \{(0, \lambda_{k-1}), (0, \lambda_k)\}.$$

Our ultimate goal is to determine for each g and λ the number of (weak) solutions u of the given boundary value problem, and how this number changes as

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g and λ are perturbed. In the perspective of the series of papers [7]–[10] and the present paper, the numbers of solutions correspond to numbers of point inverses of the nonlinear map A. We wish to determine for each $(\tilde{g}, \lambda) \in H \times \mathbf{R}$ the number of points in $A^{-1}(\tilde{g}, \lambda)$ and how this number changes as $(\tilde{g}, \lambda)$ is varied. In some related cases ([7], [8], [25]) the map A is actually identified up to global coordinate change, while in others (e.g. [24], [12], [13], [23], [27], [28]) the singularities of A are studied.

Almost all of the results in the present paper are used in a manuscript [14] in preparation. In that work with E. N. Dancer our goal is to identify up to global coordinate change the map A for λ near λ_1, the first eigenvalue of $-\Delta u = 0$ on Ω with $u \mid \partial\Omega = 0$.

The present paper is done in an abstract context (0.1). Although this increases the difficulty, because of the abstraction the results apply to several other examples [9: (1.4), (1.7), (1.8)] and relate to the similar von Kármán equations [9: (1.9), §4].

0.1. DEFINITION [9: (1.2)]. *The abstract map A.* Consider any Hilbert space H over the real numbers and a map $A_\lambda : H \to H$ defined by

$$A_\lambda(u) = u - \lambda L u + N(u),$$

where L and N have the following properties:

(1) L is a compact, self-adjoint, positive linear operator ($\langle Lu, u\rangle_H \geq 0$ and $= 0$ only if $u = 0$). It follows [18, pp. 349–350] that H is separable and the eigenvalues λ_m ($m = 1, 2, \dots$) of $u = \lambda L u$ are positive, $\lambda_m \leq \lambda_{m+1}$, and (if H is infinite-dimensional) $\lambda_m \to \infty$ as $m \to \infty$. Let $\{u_m\}$ be an orthonormal basis of H of eigenvectors.

(2) The first eigenvalue λ_1 is simple.

(3) (a) The map N is C^k ($k = 1, 2, \dots$ or ∞ or ω) such that $DN(u)$ is nonnegative self-adjoint ($\langle DN(u)\cdot v, v\rangle_H \geq 0$ for every $v \in H$).

(b) If $\langle DN(u)\cdot u_m, u_m\rangle_H = 0$ for some m ($m = 1, 2, \dots$), then $u = 0$. [Statement (b_1) is: $\langle DN(u)\cdot u_1, u_1\rangle_H = 0$ implies $u = 0$.]

(c) $k \geq 2$ and $D^j N(0) = 0$ for $j = 0, 1, 2$. [Statement (c_j) for $j = 0, 1, 2$ is: N is C^j and $D^j N(0) = 0$.]

(d) $k \geq 3$ and $\langle D^3 N(u)(v, v, v), v\rangle_H > 0$ for $0 \neq v \in H$.

(e) $D^4 N(u) \equiv 0$. From Taylor's theorem [32, p. 148, Thm. 4.A] it follows that N is real analytic, and assuming (3)(c), $(3!)N(u) = D^3 N(0)(u, u, u)$, so that $2DN(u)\cdot v = D^3 N(0)(u, u, v)$.

We refer to a map A_λ satisfying (1) and (3)(a) above, and to A defined by $A(u, \lambda) = (A_\lambda(u), \lambda)$, as *abstract* A_λ and A. Often a lemma or theorem will assume abstract A and some of the above conditions; for example, Proposition 1.7 requires abstract A with A a C^2 map and the additional assumption (0.1)(3)(c).

0.2. EXAMPLE [9: (1.3)]. *The standard map A.* Our main example of abstract A is the map A of the first paragraph of this paper; it satisfies all the properties of (0.1) and we call it standard A. Here H is the Sobolev space $W_0^{1,2}(\Omega)$ ([7, §2] or [3, p. 28]), where Ω is a bounded connected open subset of $\mathbf{R}^n$ with $n \leq 4$, and the operators L and N are defined by

$$\langle Lu, \varphi \rangle_H = \int_\Omega u\varphi \quad \text{and} \quad \langle N(u), \varphi \rangle_H = \int_\Omega u^3 \varphi$$

for all $\varphi \in C_0^\infty(\Omega)$, the space of C^∞ real valued functions with compact support in Ω. For more information about standard A, see [9: (1.3)], and for generalization with certain functions $f(u)$ in place of u^3, see [9: (1.4)].

Other examples of (0.1) are given in [9: (1.7), (1.8)]. The von Kármán equations for the buckling of a thin planar elastic plate yield an operator A satisfying most of the properties of (0.1) (see [9, §4, especially (4.6)]).

For related work and announcements see [7], [8], [10], and [15], as well as [9]. The results of the present paper are used in further investigations the authors are pursuing with E. N. Dancer.

0.3. NOTATION. An ordered pair in $X \times Y$ is denoted by (x, y), while the inner product of x and y in a Hilbert space H (resp., in $L^2(\Omega)$) is denoted by $\langle x, y \rangle_H$ (resp., $\langle x, y \rangle_2$). The norm of x in $L^p(\Omega)$ is $\|x\|_p$. Real analytic [32: (8.8), p. 362] is denoted by C^ω. Assume throughout that Ω is a bounded connected open subset of $\mathbf{R}^n$ $(n \leq 4)$.

Since this work is addressed to researchers in singularity theory as well as those purely in partial differential equations, somewhat more detail is given in some places than is usual. The authors are grateful to W. Allegretto and E. N. Dancer for furnishing references and advice. Church thanks Syracuse University for research leave during 1986–87, and the University of Alberta for its hospitality and support during that period.

1. The singular set SA as the union of graphs.

1.1. DEFINITIONS. (a) For abstract A (0.1), $u \in H$ and $\varphi \in H$, let

$$D_u[\varphi] = \langle \varphi, \varphi \rangle_H + \langle DN(u)\varphi, \varphi \rangle_H.$$

Let $\mathfrak{F}_{j-1}$ be the collection of subspaces of H of codimension $j-1$, and let

$$\lambda_j(u) = \sup_{F \in \mathfrak{F}_{j-1}} \inf_{\varphi \in F, \langle L\varphi, \varphi \rangle_H = 1} D_u[\varphi].$$

(b) For abstract A satisfying (0.1)(3)(c)(e), $u \in H$, $\varphi \in H$ and $c \geq 0$, let

$$D_{c,u}[\varphi] = \langle \varphi, \varphi \rangle_H + c\langle D^3N(0)(u, u, \varphi), \varphi \rangle_H$$

and let

$$\lambda_{j,c}(u) = \sup_{F \in \mathfrak{F}_j} \inf_{\varphi \in F, \langle L\varphi, \varphi \rangle_H} D_{c,u}[\varphi].$$

1.2. LEMMA. *Consider abstract A.*

(a) *The function $\lambda_j(u)$ is the jth eigenvalue of $v - \lambda Lv + DN(u)v = 0$.*

(b) *Each $\lambda_j(u) > 0$, $\lambda_j(u) \leq \lambda_{j+1}(u)$, and (if H is infinite dimensional) $\lambda_j(u) \to \infty$ as $j \to \infty$; in particular, each eigenspace has finite dimension.*

(c) *Given any $F \in \mathfrak{F}_{j-1}$, there is $\varphi_{F,u} \in F$ with $\langle L\varphi_{F,u}, \varphi_{F,u} \rangle_H = 1$ such that*

$$D_u[\varphi_{F,u}] = \inf_{\varphi \in F, \langle L\varphi, \varphi \rangle_H = 1} D_u[\varphi].$$

Thus $\lambda_j(u)=\sup\{D_u[\varphi_{F,u}]: F\in\mathfrak{F}_{j-1}\}$. *Moreover this* sup *is achieved at an* F.

(d) *There is* $v_j(u)\in H$ *with* $\langle Lv_j(u), v_j(u)\rangle_H=1$ *such that* $\lambda_j(u)=D_u[v_j(u)]$; *these are the eigenvectors of* $v-\lambda Lv+DN(u)=0$.

(e) *Assume* A *satisfies* (0.1)(3)(c)(e). *The function* $\lambda_{j,c}(u)$ *is the jth eigenvalue of* $v-\lambda Lv+cD^3N(0)(u,u,v)$, *and the analogs of* (b), (c) *and* (d) *hold.*

Proof. For any $u\in H$ define an inner product $\langle\ ,\ \rangle_u$ on H by

$$\langle v,\varphi\rangle_u=\langle v,\varphi\rangle_H+\langle DN(u)\cdot v,\varphi\rangle_H$$

(see (0.1)(3)(a)); let $\|\ \|_u$ be the resulting norm, and let H_u be H with this norm. For every $\varphi\in H$, $\|\varphi\|_H\le\|\varphi\|_u$ and $\langle DN(u)\cdot\varphi,\varphi\rangle_H\le\|DN(u)\|\,\|\varphi\|_H^2$; thus $\|\ \|_H$ and $\|\ \|_u$ are equivalent norms, so H_u is a Hilbert space. Define $M(u)=M: H_u\to H_u$ by $\langle Mv,\varphi\rangle_u=\langle Lv,\varphi\rangle_H$. For every $v, w\in H$,

$$\begin{aligned}\|Mv-Mw\|_u&=\sup_{\|\varphi\|_u=1}\langle Mv-Mw,\varphi\rangle_u\\&=\sup_{\|\varphi\|_u=1}\langle Lv-Lw,\varphi\rangle_H\le\sup_{\|\varphi\|_H=1}\langle Lv-Lw,\varphi\rangle_H=\|Lv-Lw\|_H;\end{aligned}$$

since L is a compact self-adjoint positive linear operator on H, M is a compact self-adjoint positive linear operator on H_u.

The eigenvalues of $M(u)$ are given [18, pp. 349–350] by

$$\mu_j(u)=\min_{F\in\mathfrak{F}_{j-1}}\max_{\varphi\in F}\frac{\langle M(u)\cdot\varphi,\varphi\rangle_u}{\langle\varphi,\varphi\rangle_u}\qquad(j=1,2,\ldots),$$

and $\mu_j(u)\ge\mu_{j+1}(u)>0$ with $\mu_j(u)\to 0$ as $j\to\infty$ (if H is infinite-dimensional). Now λ is an eigenvalue and v is a corresponding eigenvector of $v-\lambda Lv+DN(u)\cdot v=0$ if and only if

$$\langle Lv,\varphi\rangle_H=(1/\lambda)\langle v+DN(u)\cdot v,\varphi\rangle_H$$

for every $\varphi\in H$, if and only if

$$\langle M(u)\cdot v,\varphi\rangle_u=(1/\lambda)\langle v,\varphi\rangle_u$$

for every $\varphi\in H_u$, that is, if and only if $1/\lambda$ is an eigenvalue and v is a corresponding eigenvector of $M(u)$. It follows that the eigenvalues of $v-\lambda Lv+DN(u)v=0$ are $1/\mu_j(u)$, which are the $\lambda_j(u)$ given in (1.1). (That there are really maxima and minima and not merely sup and inf results from the compactness of M—use [3: (1.3.12)(iii), p. 31 (add hypothesis "closed"), and (1.3.31), p. 35].)

To prove (e) replace $\langle v,\varphi\rangle_u$ by $\langle v,\varphi\rangle_{c,u}=\langle v,\varphi\rangle_H+cD^3N(0)(u,u,v),\varphi\rangle_H$. Since $2DN(u)\cdot v=D^3N(0)(u,u,v)$, $\langle\ ,\ \rangle_{c,u}$ is bilinear by (0.1)(3)(a)(c)(e). □

1.3. REMARK. The eigenvectors $v_j(u)$ are an orthogonal basis of H_u (use [18, pp. 349–350, Proposition 27.1, first line of the proof]), and thus they are linearly independent in H. For standard A they are orthonormal in $L^2(\Omega)$, and

$$\lambda_k(u)=\inf\{Du[\varphi]:\|\varphi\|_2=1,\langle\varphi, v_i(u)\rangle_2=0\ (i=1,2,\ldots,k-1)\}.$$

1.4. DEFINITION. Let E and $\bar{E}$ be Banach spaces and let U be open in E. A function $\lambda: U \to \bar{E}$ is called *locally Lipschitzian* [19: (10.4.6), p. 285] if, for every $w \in U$, there are an open neighborhood W of w in U and $C(W) > 0$ such that, for every $u, \bar{u} \in W$,

$$|\lambda(u) - \lambda(\bar{u})| \le C(W) \|u - \bar{u}\|.$$

1.5. THEOREM.
(a) *For abstract A, the singular set SA is the union of the graphs of $\lambda_j : H \to \mathbf{R}$ $(j = 1, 2, \dots)$.*
(b) *Assuming, in addition, that A is C^2 and* (0.1)(3)(c), *then each $\lambda_j : H \to \mathbf{R}$ is locally Lipschitzian; in particular, it is continuous $(j = 1, 2, \dots)$.*
(c) *Assuming, in addition,* (0.1)(3)(c)(e) (e.g. *standard A*), *then for all $u, \bar{u} \in H$*

$$|\lambda_j(u) - \lambda_j(\bar{u})| \le \lambda_j(\bar{u}) \|D^3N(0)\| (\|\bar{u}\|_H \|u - \bar{u}\|_H + \|u - \bar{u}\|_H^2) \quad (j = 1, 2, \dots).$$

(d) *In particular,*

$$\lambda_j(u) \le \lambda_j(1 + \|D^3N(0)\| \|u\|_H^2),$$

that is, the growth of $\lambda_j(u)$ for large u is at most quadratic in u $(j = 1, 2, \dots)$. For standard A $(n \le 4)$, $\|D^3N(0)\| \le 6(K(\Omega))^4$, where $\|v\|_4 \le K(\Omega) \|v\|_H$ for all $v \in H$ [1, p. 97].

Actually the continuity of λ_j follows directly from [22: IV, §3, subsec. 5, p. 213].

Proof. For (a) note that $SA = \bigcup_\lambda SA_\lambda$ [9: (2.5)] and $u \in SA_\lambda$ if and only if $DA_\lambda(u) \cdot v = 0$ for some $v \ne 0$; that is, $v = \lambda Lv + DN(u) \cdot v = 0$ for some $v \ne 0$.

For (b), by (1.2)(c),

$$D_u[\varphi_{F,u}] - \lambda_j(\bar{u}) \le D_u[\varphi_{F,u}] - D_{\bar{u}}[\varphi_{F,\bar{u}}] \le D_u[\varphi_{F,\bar{u}}] - D_{\bar{u}}[\varphi_{F,\bar{u}}]$$
$$= \langle [DN(u) - DN(\bar{u})]\varphi_{F,\bar{u}}, \varphi_{F,\bar{u}} \rangle_H \le \|DN(u) - DN(\bar{u})\| \|\varphi_{F,\bar{u}}\|_H^2.$$

From (1.1) $\|\varphi\|_H^2 \le D_u[\varphi]$ for all $u \in H$ and $\varphi \in H$, and thus

$$\|\varphi_{F,\bar{u}}\|_H^2 \le D_{\bar{u}}[\varphi_{F,\bar{u}}] \le \lambda_j(\bar{u})$$

by (1.2)(c) for all $F \in \mathfrak{F}_{j-1}$.

As a result,

$$D_u[\varphi_{F,u}] - \lambda_j(\bar{u}) \le \lambda_j(\bar{u}) \|DN(u) - DN(\bar{u})\|$$

for all $F \in \mathfrak{F}_{j-1}$, so by (1.2)(c) and the mean value theorem [19: (8.5.4), p. 153]

$$\lambda_j(u) - \lambda_j(\bar{u}) \le \lambda_j(\bar{u}) \|D^2N(\xi)\| \|u - \bar{u}\|_H \tag{1}$$

for some ξ on the line segment joining u and $\bar{u}$. In case $\lambda_j(u) \ge \lambda_j(\bar{u})$, (1) implies that

$$|\lambda_j(u) - \lambda_j(\bar{u})| \le \lambda_j(\bar{u}) \|D^2N(\xi)\| \|u - \bar{u}\|_H. \tag{2}$$

For the case $\lambda_j(u) < \lambda_j(\bar{u})$, we note that (1) is true with the roles of u and $\bar{u}$ reversed; now replace $\lambda_j(u)$ on the right side by $\lambda_j(\bar{u})$ to obtain (2).

Since N is C^2, for any $\epsilon > 0$ there is a $\delta(\bar{u}) > 0$ such that for $\|u-\bar{u}\|_H < \delta(\bar{u})$, $\|D^2N(\xi) - D^2N(\bar{u})\|_H < \epsilon$; as a result

$$(3) \qquad |\lambda_j(u) - \lambda_j(\bar{u})| \leq C_j(\bar{u})\|u-\bar{u}\|_H,$$

where $C_j(\bar{u}) = \lambda_j(\bar{u})(\|D^2N(\bar{u})\|_H + \epsilon)$. In particular, $\lambda_j : H \to \mathbf{R}$ is continuous.

For $\eta > 0$ and $w \in H$, let $B_\eta(w) = \{u \in H : \|u-w\| < \eta\}$. There exists $\eta > 0$ such that for every $\xi \in B_\eta(w)$, $\|D^2N(\xi)\| \leq \|D^2N(w)\| + 1$, and for every $\bar{u} \in B_\eta(w)$, $\lambda_j(\bar{u}) \leq \lambda_j(w) + 1$. From (2), λ_j is locally Lipschitzian at w with $W = B_\eta(w)$ and $C = (\lambda_j(w)+1)\|D^2N(w)\| + 1$.

For (c), use the mean value theorem [19: (8.5.4), p. 153] and (0.1)(3)(c)(e) to note that in (2)

$$\|D^2N(\xi)\| = \|D^3N(\eta)\|\,\|\xi\|_H \leq \|D^3N(0)\|(\|\bar{u}\|_H + \|u-\bar{u}\|_H),$$

where η is on the line segment joining 0 and ξ.

For (d) set $\bar{u} = 0$ in (c). For standard A (0.2), $\langle D^3N(0)(v_1, v_2, v_3), v_4\rangle_H = 6\int_\Omega v_1v_2v_3v_4$, so

$$\|D^3N(0)\| = 6\sup\left\{\int_\Omega v_1v_2v_3v_4 : \|v_i\|_H = 1 \ (i = 1,2,3,4)\right\},$$

and the conclusion results from the Hölder inequality [3: (1.3.3), p. 28] and [1, p. 97, first paragraph and (4)]. □

1.6. LEMMA. *For standard A* (0.2) ($n \leq 4$) *and each $u \in H$, $\lambda_1(u)$* (1.1) *is a simple eigenvalue of $v - \lambda Lv + DN(u)v = 0$, and its eigenspace is spanned by an eigenfunction $v_1(u)$ which is positive a.e. on Ω.*

Proof. The first eigenvalue $\lambda_1(u) = \inf\{D_u[\varphi] : \|\varphi\|_2 = 1\}$, where

$$D_u[\varphi] = \int_\Omega |\nabla\varphi\|^2 + 3\int_\Omega u^2\varphi^2;$$

the conclusion results from [21, p. 214, Thm. 8.38]. For $n \leq 3$ the Harnack inequality used is that given in [21, p. 199, Cor. 8.21] but with the hypotheses of [21, p. 209, paragraph after Thm. 8.31] ($b = c = 0$ and $d = 3u^2 \in L^2(\Omega)$ by the Sobolev imbedding theorem [1, p. 97]). For $n = 4$ use the proof of [21, p. 214, Thm. 8.38] with the Harnack inequality citation replaced by [30].

1.7. PROPOSITION. *For abstract A assuming A is C^2 and* (0.1)(3)(c), *if the graph of λ_j is a C^k submanifold of codimension one in a neighborhood of $(\bar{u}, \lambda_j(\bar{u})) \in H \times \mathbf{R}$, then λ_j is a C^k map in a neighborhood of $\bar{u}$ ($k = 2, 3, \ldots$ or ∞ or ω, $j = 1, 2, \ldots$).*

Proof. Assume that the graph of λ_j, $\Gamma_j = \{(u, \lambda_j(u)) : u \in H\} \subset H \times \mathbf{R}$, is a C^k manifold (of codimension 1 in $H \times \mathbf{R}$) near $(\bar{u}, \lambda_j(\bar{u}))$. Let $\pi : H \times \mathbf{R} \to H$ be projection; if $D = D(\pi \mid \Gamma_j)(\bar{u}, \lambda_j(\bar{u}))$ is an isomorphism of the tangent space $T = T\Gamma_j(\bar{u}, \lambda_j(\bar{u}))$ onto H, it follows from the inverse function theorem [19: (10.2.5), pp. 268–269] that $\pi \mid \Gamma_j$ has a C^k inverse near $\bar{u}$, and thus λ_j is C^k near $\bar{u}$.

Hence we may suppose that D is not an isomorphism. Suppose that $(0,1)\notin T$; then T and $(0,1)$ span $H\times\mathbf{R}$. For each $(u,\lambda)\in H\times\mathbf{R}$ there exist $(w,\zeta)\in T$ and $c_1, c_2\in\mathbf{R}$ such that $(u,\lambda)=c_1(w,\zeta)+c_2(0,1)$; it follows that $c_1 w=u$. Since $(c_1 w, c_1\zeta)\in T$, D is surjective. Let $i:\Gamma_j\to H\times\mathbf{R}$ be inclusion; since

$$D=D\pi(\bar{u},\lambda_j(\bar{u}))\cdot Di(\bar{u},\lambda_j(\bar{u})) \quad \text{and} \quad \ker D\pi(\bar{u},\lambda_j(\bar{u}))$$

is spanned by $(0,1)$, D is injective, and by the open mapping theorem [32, p. 777, (36)] is an isomorphism, contradicting our assumption.

As a result we may suppose that $(0,1)\in T\subset H\times\mathbf{R}$. There exists a C^k curve $\Theta:(-\delta,\delta)\to\Gamma_j$ such that $\delta>0$, $\Theta(0)=(\bar{u},\lambda_j(\bar{u}))$ and $D\Theta(0)\cdot 1=(0,1)$, where $\Theta(t)=(u(t),\Theta(t))$. Since $\Theta(t)=\lambda_j(u(t))$, the locally Lipschitzian property of λ_j (1.5) becomes

$$(1)\qquad |\Theta(t)-\Theta(0)|\le C\|u(t)-u(0)\|_H,$$

where $C>0$ and $|t|<\eta$ for some $\eta<\delta$. We may suppose that $C\ge 1$. There exists $\bar{t}$ $(0<\bar{t}<\eta)$ such that for every t with $0<t<\bar{t}$, $D\Theta(t)\ge\frac{1}{2}$ and $\|Du(t)\|_H\le 1/3C$. By the mean value theorem applied to Θ, there exists ξ $(0<\xi<t)$ such that $\Theta(t)-\Theta(0)=D\Theta(\xi)t$, so that

$$(2)\qquad |\Theta(t)-\Theta(0)|\ge t/2;$$

while from the mean value theorem [3: (2.1.21), p. 70] applied to u,

$$(3)\qquad \|u(t)-u(0)\|_H\le t/3C.$$

Inequalities (1), (2) and (3) are contradictory and the lemma is proved. □

In general the graph of a function may be a smooth submanifold even though the function itself is not smooth, for example, $f:\mathbf{R}\to\mathbf{R}$ defined by $f(t)=t^{1/3}$. Thus the locally Lipschitzian property (or some other condition) is needed.

1.8. LEMMA. *Consider the C^k $(k=2,3,\ldots)$ [resp., C^∞, C^ω] map abstract A, and let $\bar{u}\in H$ with $\lambda_j(\bar{u})$ (1.1) a simple eigenvalue. Then there is an open neighborhood U of $\bar{u}$ in H such that $\lambda_j:U\to\mathbf{R}$ is C^{k-1} [resp., C^∞, C^ω]. Moreover, given an eigenvector $\bar{v}_j$ of $\lambda_j(\bar{u})$ with $\|\bar{v}_j\|_H=1$, there is a C^{k-1} [resp., C^∞, C^ω] function $v_j:U\to H$ such that $v_j(u)$ is an eigenvector of $\lambda_j(u)$ with $\|v_j(u)\|_H=1$ and $v_j(\bar{u})=\bar{v}_j$ $(j=1,2,\ldots)$.*

Proof. Let $\bar{\lambda}=\lambda_j(\bar{u})$, let $\bar{v}$ be an eigenvector of

$$v-\lambda Lv+DN(\bar{u})\cdot v=0$$

with eigenvalue $\bar{\lambda}$, and let H_1 be the orthogonal complement of $\bar{v}$ in H. Define $F:H\times\mathbf{R}\times H_1\to H$ by

$$F(u,\lambda,w)=DA_\lambda(u)\cdot v=v-\lambda Lv+DN(u)\cdot v,$$

where $v=\bar{v}+w$. Now $(\bar{u},\bar{\lambda},0)\in F^{-1}(0)$ and

$$D_{(\lambda,w)}F(\bar{u},\bar{\lambda},0)\cdot(0,\mu,\psi)=-\mu L\bar{v}+DA_{\bar{\lambda}}(\bar{u})\cdot\psi$$

[19: (8.9.1), p. 167]. Since $\bar{v}$ generates ker $DA_{\bar{\lambda}}(\bar{u})$ and since $DA_{\bar{\lambda}}(\bar{u})$ is self-adjoint and Fredholm ([9: (2.2), (2.4)]) of index 0, $DA_{\bar{\lambda}}(\bar{u}): H_1 \to H_1$ is a bijection. By the positivity of L, $L\bar{v} \notin H_1$. It follows that

$$D_{(\lambda, w)} F(\bar{u}, \bar{\lambda}, 0): 0 \times \mathbf{R} \times H_1 \to H$$

is a bijection, and by the open mapping theorem [32, p. 777, (36)] is an isomorphism. By the implicit function theorem [32, p. 150, Thm. 4.B], there is an open neighborhood U of $\bar{u} \in H$ and a unique C^{k-1} [resp., C^∞, C^ω] function $\Gamma: U \to \mathbf{R} \times H_1$ with $\Gamma(u) = (\lambda(u), w(u))$, where the graph of Γ coincides with $F^{-1}(0)$ in a neighborhood of $(\bar{u}, \bar{\lambda}, 0)$ in $H \times \mathbf{R} \times H_1$.

Since $\lambda_j(\bar{u}) = \bar{\lambda}$ is a simple eigenvalue, $\lambda_{j-1}(\bar{u}) < \lambda_j(\bar{u}) < \lambda_{j+1}(\bar{u})$ (if $j=1$, eliminate λ_{j-1}), and we may choose U small enough that $\lambda_{j-1}(u) < \lambda(u) < \lambda_{j+1}(u)$ for all $u \in U$ (by (1.2)). Since $\lambda_m(u) \le \lambda_{m+1}(u)$ for all $u \in H$, it follows that $\lambda(u)$ must be the eigenvalue $\lambda_j(u)$ for all $u \in U$. Since $\zeta: H - \{0\} \to \mathbf{R}$ defined by $\zeta(v) = 1/\|v\|_H$ is real analytic, $v_j(u) = (\bar{v} + w(u))/\|\bar{v} + w(u)\|_H$ is C^{k-1} [resp., C^∞, C^ω] on U. □

Alternatively, we could prove (1.8) using [17, p. 163, Lemma 1.3], but the proof uses the additional hypotheses (0.1)(3)(c)(e).

1.9. THEOREM.

(a) *For standard A (with $n \le 4$) the first eigenvalue map $\lambda_1: H \to \mathbf{R}$ is real analytic. Moreover, there is a real analytic map $v_1: H \to H$ such that $\|v_1(u)\|_H = 1$, $v_1(u)$ is positive a.e. on Ω, and $v_1(u)$ is an eigenvector of $\Delta v + \lambda v - 3u^2 v = 0$ with $\lambda = \lambda_1(u)$.*

(b) *For the C^k ($k = 2, 3, \ldots$) [resp., C^∞, C^ω] map abstract A with (0.1)(2)(3)(c_1), let $U = \{u \in H: \lambda_1(u) < \lambda_2\}$. Then $\lambda_1: U \to \mathbf{R}$ is C^{k-1} [resp., C^∞, C^ω], $\lambda_1(u)$ is simple for $u \in U$, and there is a C^{k-1} [resp., C^∞, C^ω] eigenvector map $v_1: U \to H$ with $\|v_1(u)\|_H = 1$. Thus for $\lambda < \lambda_2$ the singular set SA is the graph of the real analytic map $\lambda_1: U \to \mathbf{R}$.*

Assuming (0.1)(3)(b)(c)(e) in addition, in (2.4) we will prove that U is an open star-shaped region about 0. For standard A there exist $u \in H$ such that $v_1(u): \Omega \to \mathbf{R}$ is not real analytic; however the map $v_1: H \to H$ is real analytic.

Proof. Conclusion (a) follows from (1.6) and (1.8). Now $u \in SA_\lambda$ with

$$DA_\lambda(u) \cdot v = 0 \quad \text{for } v \neq 0$$

if and only if λ is an eigenvalue of $v - \lambda Lv + DN(u) \cdot v = 0$ with eigenvector v. From [9: (2.4)] $\lambda = \lambda_1(u)$ is a simple eigenvalue if $\lambda_1(u) < \lambda_2$. From (0.1)(2)(3)(c_1) $\lambda_1(0) = \lambda_1 < \lambda_2$ so that $0 \in U$. Conclusion (b) results from (1.8).

1.10. PROPOSITION. *In* (a) *and* (b) *consider abstract A with* (0.1)(2) *and suppose that* dim ker $DA_\lambda(\bar{u}) = 1$ *with generator $\bar{v}$.*

(a) *Let $\lambda: U \to \mathbf{R}$ be the function given on a neighborhood of $\bar{u}$ in H by* (1.8). *Then $\bar{u}$ is a fold point* [9: (3.1)] *of A if and only if $D\lambda(\bar{u}) \cdot v \neq 0$, and is a precusp point of A if and only if $D\lambda(\bar{u}) \cdot v = 0$.*

(b) *If* $w: U \to \mathbf{R}$ *is the function given in the proof of* (1.8), $v(u) = \bar{v} + w(u)$, *and* $\bar{u}$ *is a precusp point, then* $Dv(\bar{u}) \cdot \bar{v} = Dw(\bar{u}) \cdot \bar{v} = -y$, *the element defined in the proof of* [9: (3.6)] *chosen to be in* H_1, *the H-orthogonal complement of* $\bar{v} = e$.

(c) *For standard A and* $\bar{u} \in H$ *with* $\dim \ker DA_\lambda(\bar{u}) = 1$, $\bar{u}$ *is a singular point of* λ *if and only if* $\bar{u} = 0$.

(d) *Thus, for standard A the real analytic map* $\lambda_1 : H \to \mathbf{R}$ *has* $\bar{u} = 0$ *as its only singular point.*

(e) *For standard A and for* $0 \neq \bar{u} \in H$ *and* $c \in \mathbf{R}$ *with* $\dim \ker DA_\lambda(c\bar{u}) = 1$, *the function* ξ *defined by* $\xi(c) = \lambda(c\bar{u})$ *has derivative* $D\xi(c) > 0$ *for* $c > 0$, $D\xi(c) < 0$ *for* $c < 0$, *and* $D\xi(0) = 0$.

According to (a) for abstract A and $\bar{u} \in H$ with $0 \neq \bar{v}$ generating $\ker DA_\lambda(\bar{u})$, $\bar{v}$ is transverse [resp., tangent] to the level surface through $\bar{u}$ of the function λ through $\bar{u}$ if and only if $\bar{u}$ is a fold [resp., precusp] point. It is important to realize that $v(u)$ is the e of [19, §3], but it is normalized in a certain way.

Proof. From the proof of (1.8), the implicit function theorem [3, p. 115, Corollary], $w(\bar{u}) = 0$ and $v(\bar{u}) = \bar{v}$,

$$D(\lambda, w)(\bar{u}) \cdot \varphi = -[D_{(\lambda, w)} F(\bar{u}, \lambda(\bar{u}), 0)]^{-1} D_u F(\bar{u}, \lambda(\bar{u}), 0) \cdot \varphi$$

for every $\varphi \in H$. Now $D_u F(\bar{u}, \lambda(\bar{u}), 0) \cdot \varphi = D^2 A_{\lambda(\bar{u})}(\bar{u})(\bar{v}, \varphi)$. By the self-adjointness $DA_{\lambda(\bar{u})}(\bar{u}) : H_1 \to H_1$ is an isomorphism; and $D^2 A_{\lambda(\bar{u})}(\bar{u})(\bar{v}, \varphi) \in H_1$ if and only if

$$\begin{aligned} -[D_{(\lambda, w)} F(\bar{u}, \lambda(\bar{u}), 0)]^{-1} & D^2 A_{\lambda(u)}(\bar{u})(\bar{v}, \varphi) \\ &= (0, -[DA_{\lambda(\bar{u})}(\bar{u})]^{-1} D^2 A_{\lambda(\bar{u})}(\bar{u})(\bar{v}, \varphi)). \end{aligned}$$

Since Range $DA_{\lambda(\bar{u})}(\bar{u}) = H_1$, if we set $\varphi = \bar{v}$, $D^2 A_{\lambda(\bar{u})}(\bar{u})(\bar{v}, \bar{v}) \notin H_1$ if and only if $\bar{u}$ is a fold point [9: (3.1)]. Conclusions (a) and (b) follow from [9: (3.5)].

Suppose that $D\lambda(\bar{u}) = 0$. By the above argument, $D^2 A_{\lambda(\bar{u})}(\bar{u})(\bar{v}, \varphi) \in H_1$ for every $\varphi \in H$. Set $\varphi = \bar{u}$ and use the fact that H_1 is the orthogonal complement of $\bar{v}$ to obtain $6 \int_\Omega \bar{u}^2 \bar{v}^2 = 0$. Since $\bar{v} > 0$ a.e. (1.6), $\bar{u} = 0$, and conclusions (c) and (d) result.

From the previous paragraph, for $\bar{u} \neq 0$, $D^2 A_{\lambda(c\bar{u})}(c\bar{u})(\bar{v}, \bar{u}) \notin H_1$ if $c \neq 0$, so that $D\lambda(c\bar{u}) \cdot \bar{u} \neq 0$. By the chain rule $D\xi(c) = D\lambda(c\bar{u}) \cdot \bar{u}$, and since each $\lambda_j(cu)$ is a strictly increasing function of $c \geq 0$ (2.3), $D\xi(c) > 0$ for $c > 0$. Since $\lambda(c\bar{u}) = \lambda(-c\bar{u})$ (see (1.1)), the rest of conclusion (e) results. □

1.11. PROPOSITION. *Consider standard A* ($n \leq 4$) *with the boundary of* Ω *a compact* C^∞ *manifold* $\partial\Omega$.

(i) *For any* j ($j = 1, 2, \ldots$) *there is an open dense subset* V_j *of H such that* $\lambda_j(u)$ *is a simple eigenvalue of every* $u \in V_j$. *Thus* $\lambda_j \mid V_j : V_j \to \mathbf{R}$ *is real analytic (1.8).*

(ii) *Moreover, there is a subset* $\Gamma \subset \bigcap_j V_j$ *dense in H such that, for every* $u \in \Gamma$ *and every* j ($j = 1, 2, \ldots$), (a) $\lambda_j(u)$ *is a simple eigenvalue,* (b) $v_j(u) \mid \operatorname{int} \Omega$ (1.8) *is a Morse function,* (c) 0 *is not a critical value of* $v_j(u) \mid \operatorname{int} \Omega$, *and* (d) *the normal derivative of* $v_j(u)$ *on* $\partial\Omega$ *has* 0 *as a regular value.*

Proof. Since $C_0^\infty(\Omega)$ is dense in H, it suffices to prove that there is a dense subset Γ of $C_0^k(\Omega)$, where $k>n+2$, satisfying (ii). We modify the argument of [31, p. 1074, Thm. 7], with $L=\Delta$, $U=\Omega$, $M=\bar{\Omega}$, $B=C_0^k(\Omega)-\{0\}$ $(k>n+2)$, and $L+b$ replaced by $\Delta-3u^2$ (our u is different from the u in [31, p. 1074]!). Here (1) $\varphi(v,\lambda,u)=\Delta v-3u^2v+\lambda v=0$, and $D_2\varphi(v,\lambda,u)\cdot(0,0,z)=-6uvz$. Since v is continuous by the Sobolev imbedding theorem [1, p. 97], u, v, and z are all continuous. We use [31, p. 1067, Prop. (2.10)]: Suppose that there is a

$$w\in L^1(\bar{\Omega})\cap C^2(\bar{\Omega}-y)$$

such that (2) $\int_\Omega 6uvzw=0$ for all $z\in C_0^k(\Omega)$; it suffices to prove that (3) $w\equiv 0$ on some open set of Ω.

(4) Suppose that $uv\equiv 0$ on Ω. From (1) v is an eigenvector of $-\Delta$; since v is real analytic [11, p. 136, pp. 207–210], its set of zeros has measure 0, and from (4), $u\equiv 0$ on Ω. Since $\{0\}\notin B$, $u\not\equiv 0$ and so $uv\not\equiv 0$ on Ω.

Thus there are an open ball $X\subset\Omega-y$ and $\epsilon>0$ such that $|(uv)(x)|\geq\epsilon$ for all $x\in X$. If $w\,|\,X\equiv 0$, our desired conclusion (3) results; thus we may suppose that there are an open ball in $Y\subset X$ and $\delta>0$ such that $|w(x)|\geq\delta$ for all $x\in Y$. Let Z be an open ball in Y, and let $z:\mathbf{R}^n\to\mathbf{R}$ be C^∞ such that $z(x)=1$ for $x\in Z$, $z(x)=0$ for $x\in\mathbf{R}^n-Y$, and $z(x)\geq 0$ for $x\in\mathbf{R}^n$. Thus $|\int_\Omega 6uvzw|\geq 6\epsilon\delta\mu(Z)>0$, contradicting (2), so our supposition that $w\,|\,X\not\equiv 0$ was false, and conclusion (3) results. □

1.12. COROLLARY. *Consider standard A $(n\leq 4)$ with $\partial\Omega$ a compact C^∞ manifold. Then there is an open dense subset W of the singular set SA such that for every $(u,\lambda)\in W$,* $\dim\ker DA_\lambda(u)=1$.

Proof. Let $\lambda_0=0$. Since $SA=\bigcup_{j=1}^\infty \operatorname{graph}\lambda_j$ (1.5), by (1.11)(i)

$$W_k=\bigcup_{j=1}^{k}\{(u,\lambda_j(u)): u\in V_j \text{ and } \lambda_{k-1}<\lambda_j(u)<\lambda_{k+1}\}$$

is the desired open dense subset of

$$\{(u,\lambda): u\in SA_\lambda \text{ and } \lambda_{k-1}<\lambda<\lambda_{k+1}\}\quad (k=1,2,\dots).$$

Let $W=\bigcup_{k=1}^\infty W_k$. □

2. The curves $\lambda_j(cu)$ as functions of c.

2.1. LEMMA. *For abstract A assuming* (0.1)(3)(c)(e) *and $u\in H$, each $\lambda_j(cu)$ [and $\lambda_{j,c}(u)$* (1.2)(e)*] is an increasing function of $c\geq 0$, that is, $0\leq a\leq b$ implies $\lambda_j(au)\leq\lambda_j(bu)$ $(j=1,2,\dots)$.*

Proof. By (0.1)(3)(c)(e) and Taylor's theorem [32, p. 148, Thm. 4.A], $3!N(u)=D^3N(0)(u,u,u)$, so that $2DN(u)\cdot v=D^3N(0)(u,u,v)$. By (1.1),

$$(1)\qquad \lambda_j(cu)=\sup_{F\in\mathfrak{F}_{j-1}}\ \inf_{\delta\in F,\ \langle L\varphi,\varphi\rangle_H=1}\{\|\varphi\|_H^2+c^2\langle DN(u)\cdot\varphi,\varphi\rangle_H\}.$$

Now $\langle DN(u)\cdot\varphi,\varphi\rangle_H\geq 0$ by (0.1)(3)(a), and there exists (1.2)(c)(d) $F=F(au)\in\mathfrak{F}_{j-1}$ and $\varphi_{F,au}=v_j(au)\in F$ such that $\lambda_j(au)=D_{au}[\varphi_{F,au}]$. By (1),

$$\lambda_j(au) = D_{au}[\varphi_{F,au}] \le D_{au}[\varphi_{F,bu}] \le D_{bu}[\varphi_{F,bu}] \le D_{bu}[\varphi_{F(bu),bu}] = \lambda_j(bu).$$

2.2. THEOREM. *Consider abstract A with* (0.1)(3)(b)(c)(e). *Then, for each $0 \ne u \in H$ and each j ($j = 1, 2, \ldots$), $\lambda_j(cu)$ [and $\lambda_{j,c}(u)$(1.2)(e)] is a strictly increasing function of c; that is, $0 \le a < b$ implies $\lambda_j \le \lambda_j(au) < \lambda_j(bu)$.*

Proof. By (2.1), $\lambda_j(au) \le \lambda_j(bu)$. Suppose that strict inequality is not true, so that $\lambda_j(cu)$ is constant for $a \le c \le b$. If we define

$$D_u[v, \varphi] = \langle v, \varphi \rangle_H + \langle DN(u)v, \varphi \rangle_H$$

for all $v, \varphi \in H$, then

(1) $$D_{cu}[v_j(cu), \varphi] = \lambda_j(cu)\langle Lv_j(cu), \varphi \rangle_H$$

for all $\varphi \in H$ (see (1.2) and its proof). From the self-adjointness of L and $DN(u)$, (0.1), and the first line of the proof of (1.2), for $a \le c$, $d \le b$,

$$\begin{aligned} 0 &= (\lambda_j(cu) - \lambda_j(du))\langle Lv_j(cu), v_j(du) \rangle_H \\ &= D_{cu}[v_j(cu), v_j(du)] - D_{du}[v_j(cu), v_j(du)] \\ &= (\tfrac{1}{2})(c^2 - d^2)\langle DN(u) \cdot v_j(cu), v_j(du) \rangle_H \end{aligned}$$

(see the first line of the proof of (2.1)). Thus for $c \ne d$,

(2) $$\langle DN(u) \cdot v_j(cu), v_j(du) \rangle_H = 0.$$

Define a bounded bilinear form B on H by $B(v, w) = \langle DN(u) \cdot v, w \rangle_H$ for all $v, w \in H$ and let $\|v\|_B = B(v, w)$. Then B satisfies all the properties of an inner product except that $\|v\|_B = 0$ need not imply $v = 0$. In particular, (3) $|B(v, w)| \le \|v\|_B \|w\|_B$ for every $v, w \in H$ [20, p. 248].

We will prove that (4) for some $c > 0$, $a \le c \le b$, $\|v_j(cu)\|_B = 0$. If $v_j(cu) = v_j(du)$ for some $c \ne d$, then (4) results from (2). Thus we may suppose that the vectors $v_j(cu)$ for $a \le c \le b$ are all distinct with $\|v_j(cu)\|_B \ne 0$. Let $w_c = k_c v_j(cu)$, where $k_c > 0$ is so chosen that $\|w_c\|_B = 1$; by (2), $\{w_c\}$ are B-orthonormal. For $c \ne d$,

$$2 = \|w_c\|_B^2 + \|w_d\|_B^2 = \|w_c - w_d\|_B^2 \le C\|w_c - w_d\|_H^2$$

by the boundedness of B. Since $\|w_c - w_d\|_H \ge [2/C]^{1/2}$, H fails to satisfy the Lindelöf property, contradicting by [20, pp. 12, 21] the separability of H (0.1)(1), and (4) results.

Now consider a $c > 0$ such that $\|v_j(cu)\|_B = 0$; $B(v_j(cu), \varphi) = 0$ for all $\varphi \in H$ by (3) and

$$\langle v_j(cu), \varphi \rangle_H = \lambda_j(cu)\langle Lv_j(cu), \varphi \rangle_H$$

for all $\varphi \in H$ by (1). Thus $\lambda_j(cu)$ is an eigenvalue λ_i of $w = \lambda Lw$ and $v_j(cu)$ is a corresponding eigenvector u_i. By (4), $\langle DN(u)u_i, u_i \rangle_H = 0$ and by hypothesis (0.1)(3)(b), $cu = 0$; since $u \ne 0$, $c = 0$, contradicting its choice.

For the $\lambda_{j,c}(u)$ version (1.2)(e), $2DN(u) \cdot v = D^3N(0)(u, u, v)$ and the proof is identical except for multiplying the parameter c by a factor of $2^{1/2}$. For example, (2) becomes

(2′) $$\langle D^3N(0)(u,u,v_{j,c}(u)), v_{j,d}(u)\rangle = 0.$$

2.3. COROLLARY. *For standard A and* $0 \neq u \in H$, $\lambda_j(cu)$ *is a strictly increasing function of* $c \geq 0$ $(j=1,2,\dots)$.

Proof. Use (0.2) and (2.1). □

2.4. COROLLARY. *For abstract A with* (0.1)(2)(3)(b)(c)(e), *the set*

$$U = \{u \in H : \lambda_1(u) < \lambda_2\}$$

of (1.9)(b) *is an open star-shaped region about* 0.

The following result is a generalization of [9: (3.11)].

2.5. COROLLARY. *Consider abstract A satisfying* (0.1)(3)(b)(c)(e).
(a) *If* $0 \neq u \in H$ *and* $\lambda < \lambda_{j+1}$, *then*

$$\sum_{c \geq 0} \dim(\ker DA_\lambda(cu)) \leq j;$$

(b) *in particular, the ray* $\{cu : c \geq 0\}$ *meets* SA_λ *in at most j points.*

Proof. The singular set $SA = \bigcup_\lambda SA_\lambda$ [9: (2.5)] is the union of the graphs of the $\lambda_m : H \to \mathbf{R}$ $(m=1,2,\dots)$ by (1.5), and the ray $\{cu : c \geq 0\}$ meets SA_λ at cu if and only if $\dim\{\ker DA_\lambda(cu)) \geq 1$, that is, if and only if $\lambda = \lambda_m(cu)$ for some m. For each m the map $c \to \lambda_m(cu)$ is strictly increasing for $c \geq 0$ by (2.1), so the ray meets the graph of λ_m in at most one point. Since $\lambda < \lambda_{j+1} = \lambda_{j+1}(0) \leq \lambda_{j+1}(cu) \leq \lambda_m(cu)$ for $j+1 \leq m$, the ray cannot meet the graph of λ_m for $m > j$, and conclusion (b) results.

If $\dim(\ker DA_\lambda(cu)) = r > 0$, then for some least natural number s the eigenvalue $\lambda = \lambda_s(cu)$ has eigenspace of dimension r. Thus $\lambda = \lambda_s(cu) = \lambda_{s+1}(cu) = \cdots = \lambda_{s+r-1}(cu) < \lambda_{j+1}$, and conclusion (a) results. □

For $\lambda < \lambda_2$ the ray $\{cu : c \geq 0\}$ meets SA_λ in at most one point, as shown in [9: (3.11)]. We now note that "at most one" cannot be replaced by "precisely one."

2.6. EXAMPLE. *Consider standard A with* $\partial\Omega$ *a (compact)* C^∞ *manifold. Given any* $\lambda \in \mathbf{R}$, *there is a* $0 \neq u \in W_0^{1,2}(\Omega)$ *such that the line* $\{cu : c \in \mathbf{R}\}$ *is disjoint from the singular set* SA_λ *if* $\lambda \neq \lambda_j$, *and for* $\lambda = \lambda_j$ *meets it only in* $(0, \lambda_j)$ $(j=1,2,\dots)$.

Proof. If $\lambda \leq \lambda_1$, the conclusion for any $u \neq 0$ results from [9: (2.3) and (2.7)(i)], so we may suppose that $\lambda_1 < \lambda$. Let m $(m=1,2,\dots)$ be the largest number such that $\lambda_m < \lambda$. Pick $\Omega' \neq \varnothing$ a domain with $\bar{\Omega}' \subset \Omega$ and $\partial\Omega'$ a C^∞ manifold close enough to $\partial\Omega$ that $\lambda_m(\Omega') < \lambda$ also [16, p. 421, Thm. 10]. Pick U an open n-ball with $\bar{U} \subset \Omega - \bar{\Omega}'$, and let u be any eigenfunction of minus the Laplacian $-\Delta$ on U with null boundary conditions; thus u is C^∞ on $\bar{U}$ [21, p. 187, Thm. 8.13], and we may extend it to Ω to be 0 on $\Omega - U$, so $u \in W_0^{1,2}(\Omega)$. Any $v \in W_0^{1,2}(\Omega')$ can also be extended to Ω to be 0 on $\Omega - \Omega'$, so that $\int_\Omega u^2v^2 = 0$; thus from (1.1),

(1) $$D_{cu}[v] = \int_\Omega [|\nabla v|^2 + 3c^2u^2v^2] = \int_{\Omega'} |\nabla v|^2$$

for every $v \in W_0^{1,2}(\Omega')$.

We now prove: If $\mathfrak{F}_i(\Omega)$ is the set of codimension i hyperspaces in $W_0^{1,2}(\Omega)$, then

(2) $$\{F \cap W_0^{1,2}(\Omega') : F \in \mathfrak{F}_{j-1}(\Omega)\} \subset \bigcup_{i=1}^{j} \mathfrak{F}_{i-1}(\Omega') \quad (j = 1, 2, \ldots).$$

Suppose, on the contrary, that (*) $F \in \mathfrak{F}_{j-1}(\Omega)$, while $F \cap W_0^{1,2}(\Omega')$ has codimension at least j; that is, $W_0^{1,2}(\Omega')$ has a subspace Γ such that $\dim \Gamma \geq j$ and $\Gamma \cap (F \cap W_0^{1,2}(\Omega')) = \{0\}$. Since $\Gamma \cap (F \cap W_0^{1,2}(\Omega')) = (\Gamma \cap W_0^{1,2}(\Omega')) \cap F = \Gamma \cap F$, $\Gamma \cap F = \{0\}$, contradicting (*); statement (2) results.

From (1), (2) and (1.1),

$$\begin{aligned}
\lambda_{j,\Omega}(cu) &= \sup_{F \in \mathfrak{F}_{j-1}(\Omega)} \inf_{v \in F} \{D_{cu}[v] : \|v\|_{2,\Omega} = 1\} \\
&\leq \sup_{F \in \mathfrak{F}_{j-1}(\Omega)} \inf_{v \in F \cap W_0^{1,2}(\Omega')} \{D_{cu}[v] : \|v\|_{2,\Omega'} = 1\} \\
&\leq \sup_{F' \in \bigcup_{i=1}^{j} \mathfrak{F}_{i-1}(\Omega')} \inf_{v \in F'} \{D_{cu}[v] : \|v\|_{2,\Omega'} = 1\} \\
&= \sup_{1 \leq i \leq j} \sup_{F' \in \mathfrak{F}_{i-1}(\Omega')} \inf_{v \in F'} \left\{ \int_{\Omega'} |\nabla v|^2 : \|v\|_{2,\Omega'} = 1 \right\} \\
&= \sup_{1 \leq i \leq j} \lambda_i(\Omega') = \lambda_j(\Omega') < \lambda
\end{aligned}$$

$(j = 1, 2, \ldots, m)$. Thus $\lambda_{j,\Omega}(cu) = \lambda_j(cu) < \lambda$, so that the line $\{(cu, \lambda) : c \in \mathbf{R}\}$ is disjoint from the graphs of $\lambda_j : H \to \mathbf{R}$ for $j = 1, 2, \ldots, m$.

If $j \geq m+1$, then $\lambda \leq \lambda_j < \lambda_j(cu)$ for $c \neq 0$ by (2.1). Thus the line is disjoint from the graph of λ_j if $\lambda \neq \lambda_j$, and if $\lambda = \lambda_j$ they meet only in $(0, \lambda_j)$. Since SA is the union of the graphs of $\lambda_j : H \to \mathbf{R}$ $(j = 1, 2, \ldots)$ by (1.5)(a), the conclusion results. □

2.7. REMARK. If the line $\{(cu, \lambda) : c \in \mathbf{R}\}$ meets the graph of λ_j (i.e., if there exists $\bar{c} \geq 0$ such that $\lambda_j(\bar{c}u) = \lambda$), then for any $i \geq j$ with $\lambda_i < \lambda$, the line also meets the graph of λ_i (since by (2.1), $\lambda_i(\bar{c}u) \geq \lambda_j(\bar{c}u) = \lambda > \lambda_i = \lambda_i(0 \cdot u)$ and $\lambda_i : H \to \mathbf{R}$ is continuous (1.5)).

For each fixed $u \in W_0^{1,2}(\Omega)$ the map $c \to \lambda_j(cu)$ for $c \geq 0$ is strictly increasing (2.1). Example (2.6) shows that, for every λ with $\lambda_m < \lambda$, there exists a direction $u \neq 0$ in H such that $\lambda_j(cu) < \lambda$ for all $c \in \mathbf{R}$ and $j = 1, 2, \ldots, m$. On the other hand, there are directions $u \neq 0$ such that $\lambda_m(cu) \to \infty$ as $c \to \infty$ for all m $(m = 1, 2, \ldots)$ by (2.8) below (although the growth is at most quadratic (1.5)(d)).

2.8. EXAMPLE. *Let u_j be the jth eigenfunction of $-\Delta$ on Ω with null boundary conditions. For standard A (with $n \leq 4$), $\lambda_m(cu_j) \to \infty$ as $|c| \to \infty$ $(j, m = 1, 2, \ldots)$.*

Proof. Fix j and m. Since $\lambda_m(cu_j)$ is a strictly increasing function of $c \geq 0$ (2.1) and $\lambda_m(-cu_j) = \lambda_m(cu_j)$, it suffices to prove that there is no bound $M = M(j, m) > 0$ with $\lambda_m(cu_j) \leq M$ for all $c \geq 0$. Suppose the contrary. Since (1.2)

(1) $$\lambda_m(cu_j) = D_{cu_j}[v_m(cu_j)] = \|v_m(cu_j)\|_H^2 + 3c^2\int_\Omega u_j^2(v_m(cu_j))^2,$$

$\{v_m(cu_j)\}$ is bounded in $H = W_0^{1,2}(\Omega)$. By Rellich's lemma [1, p. 144] there exist $c(k)\to\infty$ as $k\to\infty$ such that $v_m(c(k)u_j)$ converges in $L^2(\Omega)$ to some $v_0 = v_0(j,m)$ with $\|v_0\|_2 = 1$. Since u_j is bounded on Ω [21, p. 189, Thm. 8.15], $\int_\Omega u_j^2 v_0^2$ exists and

(2) $$\left|\int_\Omega u_j^2(v_m(c(k)u_j))^2 - \int_\Omega u_j^2 v_0^2\right| \le \left(\max_\Omega u_j^2\right)[2\|v_0\|_2 + C]\,\|v_m(c(k)u_j) - v_0\|_2 \to 0$$

as $k\to\infty$. Since

$$\int_\Omega u_j^2(v_m(c(k)u_j))^2 \le \frac{M}{3(c(k))^2} \to 0$$

as $k\to\infty$ by (1), $\int_\Omega u_j^2 v_0^2 = 0$ by (2). Now u_j is real analytic on Ω [11, pp. 136, 207–210] so that its zeros have measure 0; thus $v_0 = 0$ a.e., contradicting $\|v_0\|_2 = 1$.

More generally, this argument will work for u_j replaced by any

$$u \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$$

whose set of zeros has measure 0. Alternatively, we might assume that $n\le 3$ and use the Rellich–Krondrachov embedding $H\to L^4$ [1, p. 144] and the Hölder inequality [3: (1.3.3), p. 28] to conclude that the result is true for $n\le 3$ and any $u\in W_0^{1,2}(\Omega)$ whose set of zeros has measure 0. □

2.9. PROPOSITION. *Consider standard A, $0\ne u\in W_0^{1,2}(\Omega)$, and $\lambda\in\mathbf{R}$.*

(a) *Suppose that the ray $\{cu: c\ge 0\}$ meets the singular set SA in a point $(\bar u,\lambda)$ with* dim ker $DA_\lambda(\bar u) = 1$. *Then the intersection is transverse, except at the points $(0,\lambda_j)$ $(j = 1,2,\dots)$.*

(b) (i) *In particular, if the ray meets the graph $\{(w,\lambda_1(w)): w\in H\}$, then they meet transversely, except at $(0,\lambda_1)$.* (ii) *Thus for $\lambda<\lambda_2$, if the ray meets SA then it meets transversely, except at $(0,\lambda_1)$.*

In some neighborhood of $(\bar u,\lambda)$ in $H\times\mathbf{R}$, SA is the graph $\{(w,\lambda_j(w))\}$ for some j $(j=1,2,\dots)$ by (1.5) and (1.2)(b), and thus (1.8) is a real analytic submanifold of codimension one. Hence the proposition is meaningful.

Proof. We assume that $(\bar u,\lambda)\ne(0,\lambda_j)$ $(j=1,2,\dots)$. By (1.5), $\lambda=\lambda_j(\bar u)$ is a simple eigenvalue of (1) $\Delta v - \lambda v + 3u^2 v = 0$ with eigenvector $v_j(\bar u)$.

We will prove that (2) $\int_\Omega \bar u^2(v_j(\bar u))^2 > 0$. Suppose the integral is 0 so that $\bar u v_j(\bar u) = 0$ a.e. on Ω. From (1), $\lambda = \lambda_i$ $(i=1,2,\dots)$ and $v_j(\bar u)$ is a corresponding eigenvector of minus the Laplacian $-\Delta$ with null boundary conditions. Since $v_j(\bar u):\Omega\to\mathbf{R}$ is real analytic [11, pp. 136, 207–210], its zeros constitute a set of measure 0; since the integral is assumed 0, $\bar u = 0$ contradicting our assumption that $(\bar u,\lambda)\ne(0,\lambda_j)$ $(j=1,2,\dots)$, and (2) results.

Now $v_j(\bar u)$ generates ker $DA_\lambda(\bar u)$, and from (0.2) and (2) above,

$$\langle D^2A_\lambda(\bar u)(v_j(\bar u),\bar u), v_j(\bar u)\rangle_H = 6\int_\Omega \bar u^2(v_j(\bar u))^2 > 0;$$

from the self-adjointness of $DA_\lambda(\bar{u})$, the orthogonal complement of its range is spanned by $v_j(\bar{u})$. Thus

$$D^2A_\lambda(\bar{u})(v_j(\bar{u}),\bar{u}) \notin \text{range } DA_\lambda(\bar{u}). \tag{3}$$

Suppose $v_j(\bar{u})$ is not a multiple of $\bar{u}$; then by (3) $\bar{u}$ is a good point of A_λ; that is, it satisfies [8: (1.6)(0)(1)(3)]. If we replace "precusp" by "good" in [8: (3.3)] and omit conclusions (ii) and (b), then that lemma is still true with the same proof. Thus there is a C^∞ germ of a diffeomorphism $\beta: H \to H$ at $\bar{x} = A_\lambda(\bar{u})$ such that $\beta(\bar{x}) = \bar{u}$, $H = \mathbf{R}^2 \times E$, $D\beta(\bar{x})\cdot(1,0,0) = v(\bar{u}_j)$, $D\beta(\bar{x})\cdot(0,1,0) = \bar{u}$,

$$A_\lambda\beta: \mathbf{R}^2\times E \to \mathbf{R}^2 \times E = H,\ (t,y,v) \to (h(t,y,v),y,v);$$

(a) $(\partial h/\partial t)(\bar{x}) = 0$ and (b) $(\partial^2 h/\partial t\,\partial y)(\bar{x}) \neq 0$. From the implicit function theorem [3, p. 115], the set of zeros of $\partial h/\partial t$ at $\bar{x}$, that is, $S(A_\lambda\beta)$, is the graph of a map germ $y = y(t,v)$. Since the unit vector $(0,1,0)$ in the y-direction is transverse to this graph, and $D\beta(\bar{x})(0,1,0) = \bar{u}$, conclusion (a) results.

Now suppose that $v_j(\bar{u})$ is a multiple of $\bar{u}$; then by (3) $(\bar{u},\lambda)$ is a fold point [8: (1.4)] of A, and by [8: (1.5)] $v_j(\bar{u})$ is transverse to SA at $(\bar{u},\lambda)$. Again, conclusion (a) results.

Conclusion (b)(i) follows from (1.6), and conclusion (b)(ii) follows from [9: (2.4)]. □

2.10. REMARK. Consider standard A and the real analytic map (1.9) $\lambda_1: H \to \mathbf{R}$ whose only singular point is its absolute minimum $\bar{u} = 0$ (1.10)(d). Define $\zeta: H \to \mathbf{R}$ by $\zeta(u) = \lambda_1(u) - \lambda_1$, so that $\zeta(0) = 0$. Suppose that λ_1 satisfies the Palais–Smale condition (C) [26, p. 300] and 0 is a nondegenerate singular (critical) point [26, p. 301]. According to the Morse lemma [26, p. 301] there would be a C^∞ diffeomorphism $\psi: U \to V$ on neighborhoods of 0 in H such that $\zeta(\psi^{-1}(u)) = \|Pu\|^2 - \|(I-P)u\|^2$, where P is an orthogonal projection in H. Since 0 is the absolute minimum of ζ, $\zeta(\psi^{-1}(u)) = \|u\|^2$. We now prove that this conclusion is not true (so that λ_1 either fails to satisfy (C) or is degenerate).

Suppose that ζ satisfies the conclusion of the Morse lemma at $u = 0$. If $D_\epsilon = \{u \in H: \|u\|_H < \epsilon\}$, then $D_\epsilon \subset \psi(D_\delta)$ for some $\delta > 0$ and $\epsilon > 0$. Thus for every $u \in H$ with $\zeta(u) < \epsilon$, that is, $\lambda_1(u) < \lambda_1 + \epsilon$, $u \in D_\delta$. Choose λ, $\lambda_1 < \lambda < \lambda_1 + \epsilon$ and let $0 \neq u \in H$ be as given in (2.6) for λ; then for every $c \geq 0$, $\lambda_1(cu) < \lambda$ by the continuity of $\lambda_1: H \to \mathbf{R}$ (1.5) and $\lambda_1(0) = \lambda_1$. A contradiction of $u \in D_\delta$ results. If λ_j is a simple eigenvalue of $-\Delta$ with null boundary conditions, then $\lambda_j: H \to \mathbf{R}$ is real analytic on a neighborhood of 0 by (1.8), and the analogous conclusion holds.

2.11. REMARK. Condition (C) [26, p. 300], as applied here, states: If X is any subset of H on which $\lambda_1: H \to \mathbf{R}$ is bounded but on which $\|D\lambda_1\|$ is not bounded away from zero, then there is a singular point adherent to X. Let $0 \neq u \in H$ be as given in (2.6) for some $\lambda > \lambda_1$ and define $\xi: \mathbf{R} \to \mathbf{R}$ by $\xi(c) = \lambda_1(cu)$. Since $D\xi(c) > 0$ for all $c > 0$ (1.10)(e) and $\xi(c) = \lambda_1(cu) < \lambda$ for all $c \in \mathbf{R}$ ((2.6) and (2.7)), there is a sequence $c_k \to \infty$ such that $D\xi(c_k) \to 0$. Since $D\xi(c) = D\lambda_1(cu)\cdot u$ by the chain rule, this is consistent with (C) failing. (Using the Ekeland variational principle, Szulkin [29] has shown that λ_1 fails to satisfy condition (C) at $u = 0$.)

3. Further results.

3.1. PROPOSITION. *Consider abstract* A *with* (0.1)(3)(b)(c)(e).

(i) *If* $u \in H$, $\lambda \leq \lambda_k$, $A_\lambda(u) = 0$, *and* u *is in the singular set* SA_λ, *then*

$$(u, \lambda) \in \left(\bigcup_{i=1}^{k-2} \operatorname{graph} \lambda_i \right) \cup \{(0, \lambda_{k-1}), (0, \lambda_k)\}.$$

(ii) *In particular, if* $\lambda \leq \lambda_2$, *then* $u \in SA_\lambda \cap A_\lambda^{-1}(0)$ *if and only if* $(u, \lambda) = (0, \lambda_1)$ *or* $(0, \lambda_2)$.

For $\lambda_1 < \lambda < \lambda_2$ and a similar A, Berger [4, p. 692, Thm. 2] noted that

$$SA_\lambda \cap A_\lambda^{-1}(0) = \varnothing.$$

Proof. By [9: (2.6)], $0 \in SA_\lambda$ if and only $\lambda = \lambda_j$ $(j = 1, 2, \ldots)$; since $\lambda_j(0) = \lambda_j$ (see (1.1) and (0.1)(3)(c)), we may assume that $u \neq 0$. By (0.1)(3)(e) $3!N(u) = D^3N(0)(u, u, u)$, so that $2DN(u) \cdot v = D^3N(0)(u, u, v)$. Since $u \in SA_\lambda$, $\lambda = \lambda_i(u)$ for some i $(i = 1, 2, \ldots)$; that is, λ is the ith eigenvalue of $v - \lambda Lv + DN(u) \cdot v = 0$, equivalently, of $v - \lambda Lv + (\frac{1}{2}) D^3N(0)(u, u, v) = 0$. Since

$$0 = A_\lambda(u) = u - \lambda Lu + N(u),$$

u is an eigenvector and λ is the jth eigenvalue $\mu_j(u)$ (for some $j = 1, 2, \ldots$) of $v - \lambda Lv + (\frac{1}{6}) D^3N(0)(u, u, v) = 0$. By (2.2) $\lambda_j(u) = \lambda_{j,1/2}(u) > \lambda_{j,1/6}(u) = \mu_j(u) > \lambda_{j,0}(u) = \lambda_j$, so that $i < j < k$, and $i \leq k-2$ as desired. □

3.2. REMARK. For $n \leq 3$ standard A is proper [9: (2.8)], so $A(\operatorname{graph} \lambda_1)$ is closed in $H \times \mathbf{R}$. Since $(0, \lambda_2) \notin A(\operatorname{graph} \lambda_1)$ (by (3.1)) there is $\epsilon > 0$ such that for all $\lambda < \lambda_2 + \epsilon$, $\lambda \neq \lambda_1$, $(0, \lambda) \notin A(\operatorname{graph} \lambda_1)$. In case λ_2 is simple this is a special case of a result of Ambrosetti and Mancini [2].

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