SOME KREIN SPACES OF ANALYTIC FUNCTIONS AND AN INVERSE SCATTERING PROBLEM

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1. Introduction. The present paper deals with certain reproducing kernel Krein spaces (RKKS) of analytic functions, and we first recall the definition of a reproducing kernel Hilbert space (RKHS). A Hilbert space $\mathcal{H}$ of $\mathbf{C}_n$-valued functions defined on some set S has reproducing kernel $K_\omega(\lambda)$, where $K_\omega(\lambda)$ is a $\mathbf{C}_{n\times n}$-valued function defined for λ and ω in S if

(1) for any ω in S and c in $\mathbf{C}_n$, the function $K_\omega c\colon \lambda \mapsto K_\omega(\lambda)c$ belongs to $\mathcal{H}$;

(2) for any f in $\mathcal{H}$, ω in S, and c in $\mathbf{C}_n$,

$$\langle f, K_\omega c\rangle = c^* f(\omega), \tag{1.1}$$

where $\langle\,,\rangle$ denotes the inner product in $\mathcal{H}$.

The function $K_\omega(\lambda)$ is easily seen to be unique and symmetric:

$$K_\omega(\lambda) = K_\lambda^*(\omega). \tag{1.2}$$

It is moreover positive: for any r, any choice of r points in S, $\omega_1, \dots, \omega_r$, and any $c_1, \dots, c_r$ in $\mathbf{C}_n$, the $r\times r$ matrix with ij entry

$$c_j^* K_{\omega_i}(\omega_j) c_i \tag{1.3}$$

is positive.

Conversely, by the matrix version of a result of Moore [5], to any $\mathbf{C}_{n\times n}$-valued function $K_\omega(\lambda)$ defined on some set S and positive in the sense just explained one can associate a unique reproducing kernel Hilbert space of $\mathbf{C}_n$-valued functions defined on S with reproducing kernel $K_\omega(\lambda)$.

In [19], Sorjonen relaxed the positivity condition and supposed that the function $K_\omega(\lambda)$ has v negative squares. Under this weaker hypothesis, the result of Moore may be extended. There exists a unique reproducing kernel Pontryagin space of $\mathbf{C}_n$-valued functions defined on S with reproducing kernel $K_\omega(\lambda)$.

The problem of associating to a given $\mathbf{C}_{n\times n}$-valued function $K_\omega(\lambda)$ subject to (1.2) a reproducing kernel Krein space with reproducing kernel $K_\omega(\lambda)$ seems open, and the aim of the present paper is twofold: first, in Section 2, we construct a reproducing kernel Krein space when $K_\omega(\lambda)$ is of the form

$$\frac{X(\lambda)JX^*(\omega)}{\rho_\omega(\lambda)}, \tag{1.4}$$

where X is a $\mathbf{C}_{n\times m}$-valued function of bounded type in Δ_+ (Δ_+ designates either the open unit disk $\mathbf{D}$ or the open upper half plane $\mathbf{C}_+$), where J is a signature matrix, that is, a matrix subject to

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$$J = J^* = J^{-1}, \tag{1.5}$$

and where

$$\rho_\omega(\lambda) = 1 - \lambda\omega^* \quad \text{if } \Delta_+ = \mathbf{D} \quad \text{(circle case)}, \tag{1.6}$$

$$\rho_\omega(\lambda) = -2\pi i(\lambda - \omega^*) \quad \text{if } \Delta_+ = \mathbf{C}_+ \quad \text{(line case)}. \tag{1.7}$$

Using the constructed RKKS we then associate to X an inverse scattering problem which, as will be explained, is a generalization of the lossless inverse scattering problem of network theory. Solutions to this inverse scattering problem will be given in Section 3.

We remark that reproducing kernel Hilbert and Pontryagin spaces with a reproducing kernel of the form (1.4) appear in a wide range of problems, from interpolation theory to the study of close to stationary processes (see [13], [14], [15], [16]) and that de Branges studied such reproducing kernel Hilbert spaces for various choices of X and J (see [10], [11]).

To conclude this introduction, we recall the definition of a Krein space. A vector space V endowed with an Hermitian form $[\ ,\ ]$ will be called a Krein space if it can be written as

$$V = V_+ [\dot{+}] V_-,$$

where V_+ endowed with $[\ ,\ ]$ and V_- endowed with $-[\ ,\ ]$ are both Hilbert spaces and where $[\dot{+}]$ denotes direct orthogonal sum: for any v_+ in V_+ and v_- in V_-, $[v_+, v_-] = 0$ and $V_+ \cap V_- = \{0\}$. If V_+ or V_- is finite-dimensional, V is then called a Pontryagin space.

A word on notation: $\mathbf{C}_{n\times l}$ denotes the space of n-row $\times$ l-column matrices with complex entries, and $\mathbf{C}_{n\times 1}$ will be denoted by $\mathbf{C}_n$. Δ_+ was already introduced in the text and L_n^2 (resp. H_n^2) will denote the Lebesgue space (resp. the Hardy space) of square summable $\mathbf{C}_n$-valued functions associated to the boundary of Δ_+ (i.e., either the unit circle or the real line). $H_{n\times l}^\infty$ stands for the space of n-row $\times$ l-column matrices with entries in H^∞, the Hardy space of functions analytic and bounded in Δ_+. Finally, A^* will designate the adjoint of the operator or of the matrix A: in particular, ω^* will be the conjugate of the complex number ω.

2. The main theorem. In this section we describe a reproducing kernel Krein space with reproducing kernel (1.4); it is convenient to prove first some preliminary lemmas. Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\ ,\ \rangle$ and let Γ be a bounded self-adjoint operator on $\mathcal{H}$. Let $\lambda \to E_\lambda$ be the resolution of the identity associated with Γ, so that

$$\Gamma = \int \lambda \, dE_\lambda, \tag{2.1}$$

where λ is in some compact subset of the real line, and recall that $|\Gamma|$ and $\operatorname{sgn}\Gamma$, the absolute value and the signum of Γ, are defined by

$$|\Gamma| = \int |\lambda| \, dE_\lambda,$$

$$\operatorname{sgn}\Gamma = \int_{\lambda\neq 0} \frac{\lambda}{|\lambda|}\, dE_\lambda.$$

On the range of the operator Γ, $\operatorname{Ran}\Gamma$, we define two Hermitian forms $\langle\,,\,\rangle_\Gamma$ and $[\,,\,]_\Gamma$ by:

$$\langle \Gamma u, \Gamma v\rangle_\Gamma = \langle |\Gamma| u, v\rangle, \tag{2.2}$$

$$[\Gamma u, \Gamma v]_\Gamma = \langle \Gamma u, v\rangle. \tag{2.3}$$

$\langle\,,\,\rangle_\Gamma$ and $[\,,\,]_\Gamma$ are easily seen to be well defined and, moreover, $\operatorname{Ran}\Gamma$ endowed with $\langle\,,\,\rangle_\Gamma$ is a pre-Hilbert space. We will need the following.

LEMMA 1. *Let $\mathcal{K}$ be the completion of* $(\operatorname{Ran}\Gamma, \langle\,,\,\rangle_\Gamma)$ *in the topology induced by $\langle\,,\,\rangle_\Gamma$. Then $[\,,\,]_\Gamma$ is continuously extendable to $\mathcal{K}$ and $(\mathcal{K}, [\,,\,]_\Gamma)$ is a Krein space. Moreover,* $\operatorname{sgn}\Gamma$ *is a bijection from $\mathcal{K}$ onto $\mathcal{K}$ and, on $\mathcal{K}$:*

$$\operatorname{sgn}\Gamma = (\operatorname{sgn}\Gamma)^{-1} = (\operatorname{sgn}\Gamma)^*, \tag{2.4}$$

where $(\operatorname{sgn}\Gamma)^*$ *denotes the adjoint of the operator* $\operatorname{sgn}\Gamma$ *in the Hilbert space $\mathcal{K}$.*

LEMMA 2. *Suppose that the Hilbert space $\mathcal{H}$ where Γ is defined is a reproducing kernel Hilbert space of $\mathbf{C}_{n\times k}$-valued functions defined on $\mathbf{C}$. Then $\mathcal{K}$ is a reproducing kernel Krein space and $\mathcal{K}$ is included in $\mathcal{H}$.*

Proof of Lemma 1. We first prove (2.4). Let $F=\Gamma u$ and $G=\Gamma v$ be two elements of $\operatorname{Ran}\Gamma$: clearly $\operatorname{sgn}\Gamma F$ and $\operatorname{sgn}\Gamma G$ belong to $\operatorname{Ran}\Gamma$, and

$$\langle F, G\rangle_\Gamma = \langle |\Gamma| u, v\rangle = \langle |\Gamma| \operatorname{sgn}\Gamma u, \operatorname{sgn}\Gamma v\rangle = \langle \operatorname{sgn}\Gamma F, \operatorname{sgn}\Gamma G\rangle_\Gamma.$$

Thus $\operatorname{sgn}\Gamma$ is an isometry on $\mathcal{K}$, and in particular is continuous. Moreover, $\operatorname{sgn}\Gamma$ is self-adjoint, since

$$\langle \operatorname{sgn}\Gamma F, G\rangle_\Gamma = \langle u, \Gamma v\rangle = \langle F, \operatorname{sgn}\Gamma G\rangle_\Gamma.$$

A self-adjoint isometry is unitary, and so $\operatorname{sgn}\Gamma$ is invertible in $\mathcal{K}$ and (2.4) holds.

The other claims of the lemma follow from the identity

$$[F, G]_\Gamma = \langle F, \operatorname{sgn}\Gamma G\rangle_\Gamma, \tag{2.5}$$

which holds first for F, G in $\operatorname{Ran}\Gamma$ and by continuity for any F, G, since

$$|\langle F, \operatorname{sgn}\Gamma G\rangle_\Gamma|^2 \leq \langle F, F\rangle_\Gamma \cdot \langle G, G\rangle_\Gamma. \qquad \square$$

Proof of Lemma 2. Let $k_\omega(\lambda)$ denote the reproducing kernel of $\mathcal{H}$. Then, for c in $\mathbf{C}_n$, ω in $\mathbf{D}$, and $F=\Gamma u$ in $\operatorname{Ran}\Gamma$,

$$c^*(\Gamma u)(\omega) = \langle \Gamma u, k_\omega c\rangle = \langle |\Gamma| u, \operatorname{sgn}\Gamma k_\omega c\rangle$$

and so

$$c^*(\Gamma u)(\omega) = \langle \Gamma u, \operatorname{sgn}\Gamma\Gamma k_\omega c\rangle_\Gamma, \tag{2.6}$$

which shows that $F \to c^*F(\omega)$ is continuous in the Hilbert space $(\mathcal{K}, \langle\,,\,\rangle_\Gamma)$ and so that $(\mathcal{K}, \langle\,,\,\rangle_\Gamma)$ is a reproducing kernel Hilbert space; thus $(\mathcal{K}, [\,,\,]_\Gamma)$ is a reproducing kernel Krein space and

$$c^*(\Gamma u)(\omega) = [\Gamma u, \Gamma k_\omega c]_\Gamma.$$

We now show that $\mathcal{K} \subset \mathcal{K}$. Let (Γu_p) be a converging sequence in $\mathcal{K}$ with limit G, and let F be its limit in $\mathcal{K}$, which exists since, as is easily shown,

$$\langle F, F\rangle \leq \| |\Gamma|^{1/2} \|^2 \langle F, F\rangle_\Gamma$$

for F in $\mathcal{K}$ ($\| |\Gamma|^{1/2} \|$ denotes the operator $|\Gamma|^{1/2}$ in $\mathcal{K}$). Then, with $F_p = \Gamma u_p$,

$$\begin{aligned} c^*F(\omega) &= \lim c^*F_p(\omega) \\ &= \lim\langle F_p, k_\omega c\rangle \\ &= \lim\langle \Gamma u_p, k_\omega c\rangle \\ &= \lim\langle \Gamma u_p, \operatorname{sgn}\Gamma\Gamma k_\omega c\rangle_\Gamma \\ &= c^*G(\omega) \end{aligned}$$

since, by (2.6), $\operatorname{sgn}\Gamma\Gamma k_\omega$ is the reproducing kernel at ω in $(\mathcal{K}, \langle\,,\,\rangle_\Gamma)$. This concludes the proof of Lemma 2. □

To describe the reproducing kernel Krein space with reproducing kernel (1.4) it is convenient to suppose first that the function X belongs to $H^\infty_{n\times m}$. Let $\underline{\underline{p}}$ denote the orthogonal projection from L^2_m onto H^2_m. The operator Γ defined by

$$(\Gamma u)(\lambda) = X(\lambda)J(\underline{\underline{p}}X^*u)(\lambda) \tag{2.7}$$

is a bounded self-adjoint operator from H^2_n into itself, to which the preceding lemmas are applicable. This allows us to build the desired reproducing kernel Krein space.

THEOREM 1. *Let $\mathcal{K}$ be the closure of the range of the operator Γ defined by* (2.7) *in the norm $\langle\,,\,\rangle_\Gamma$; then $\mathcal{K}$, endowed with the form $[\,,\,]_\Gamma$ defined in* (2.3), *is a reproducing kernel Krein space of $\mathbf{C}_n$-valued functions analytic in Δ_+ with reproducing kernel $X(\lambda)JX^*(\omega)/\rho_\omega(\lambda)$.*

Proof. Lemmas 1 and 2 already ensure that $\mathcal{K}$ is a RKKS and that $\mathcal{K} \subset H^2_n$. It remains to show that $X(\lambda)JX^*(\omega)/\rho_\omega(\lambda)$ is the reproducing kernel of $\mathcal{K}$. For any c in $\mathbf{C}_n$ and any ω in Δ_+, $\lambda \to c/\rho_\omega(\lambda)$ belongs to H^2_n and

$$\frac{X(\lambda)JX^*(\omega)}{\rho_\omega(\lambda)}c = \left(\Gamma\frac{c}{\rho_\omega}\right)(\lambda). \tag{2.8}$$

Thus the function $\lambda \to (X(\lambda)JX^*(\omega)/\rho_\omega(\lambda))c$ belongs to $\mathcal{K}$ for any ω in Δ_+ and any c in $\mathbf{C}_n$.

Moreover, for $F = \Gamma u$, $u \in H^2_n$, we have

$$\left[F, \Gamma\frac{c}{\rho_\omega}\right]_\Gamma = \left[\Gamma u, \Gamma\frac{c}{\rho_\omega}\right]_\Gamma = \left\langle\Gamma u, \frac{c}{\rho_\omega}\right\rangle_{H^2_n}.$$

Hence

$$\left[F, \Gamma\frac{c}{\rho_\omega}\right] = c^*F(\omega). \tag{2.9}$$

By continuity, (2.9) will hold for any F in $\mathcal{K}$, and (2.9) exhibits (1.4) as the reproducing kernel of $\mathcal{K}$. The theorem is proved. □

The case where X is of bounded type in Δ_+ is an easy consequence of the case where X belongs to $H^\infty_{n\times m}$. Write

$$X = Y_0^{-1} X_0, \tag{2.10}$$

where X_0 is in $H^\infty_{n\times m}$ and Y_0 is in $H^\infty_{n\times n}$, and let $\mathcal{K}_0$ denote the reproducing kernel Krein space defined in Theorem 1, with reproducing kernel $X_0(\lambda) J X_0^*(\omega)/\rho_\omega(\lambda)$.

THEOREM 2. *The set of functions*

$$\mathcal{K} = \{G = Y_0^{-1} F,\ F \in \mathcal{K}_0\}$$

endowed with the inner product

$$[G, G]_{\mathcal{K}} = [F, F]_{\mathcal{K}_0}$$

is a reproducing kernel Krein space of $\mathbf{C}_n$*-valued functions with reproducing kernel* (1.4). *The elements of* $\mathcal{K}$ *are of bounded type in* Δ_+.

The proof of Theorem 2 will be omitted.

We will denote by $\mathcal{B}(X)$ the spaces built in Theorems 1 and 2. To ease the notation, the matrix J is not referred to and will always be understood from the context. For arbitrary functions X of bounded type in Δ_+, we do not know whether $\mathcal{B}(X)$ is the only RKKS with the reproducing kernel (1.4). When (1.4) has a finite number of negative squares then $\mathcal{B}(X)$ is the only such space, by the analysis of Sorjonen [19].

3. An inverse scattering problem. In this section, we define for a function of the form (1.4) a problem which generalizes the lossless inverse scattering problem of network theory, and which we will call inverse scattering problem (ISP).

We first recall that a passive time invariant causal network with p inputs and q outputs can be characterized by a Schur function, that is, a $\mathbf{C}_{p\times q}$-valued function, analytic and bounded by 1 in modulus in Δ_+; this function is called the scattering function of the network ($\Delta_+ = \mathbf{D}$ if the network is a digital network while $\Delta_+ = \mathbf{C}_+$ if we consider an electrical network). The problem of finding representations of the scattering function S of a network as

$$S = (AW + B)(CW + D)^{-1}, \tag{3.1}$$

where

$$\Theta \stackrel{\text{def}}{=} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is J_0 inner,

$$J_0 = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix},$$

and where W is a $\mathbf{C}_{p\times q}$ scattering function, is called the lossless inverse scattering problem associated to S and has an interpretation as the plugging of a network with scattering function W into the lossless network with $p+q$ inputs, $p+q$ outputs, and chain scattering function Θ (see [2], [13]).

Now, as is well known (see e.g. [2]), (3.1) holds if and only if the map

$$F \to [I_p \quad -S] F \tag{3.2}$$

is a contraction from the reproducing kernel Hilbert space with reproducing kernel

$$\frac{J_0-\Theta(\lambda)J_0\Theta^*(\omega)}{\rho_\omega(\lambda)} \tag{3.3}$$

into the reproducing kernel Hilbert space with reproducing kernel

$$\frac{I_p-S(\lambda)S^*(\omega)}{\rho_\omega(\lambda)}. \tag{3.4}$$

(3.4) can be written as

$$\frac{X(\lambda)JX^*(\omega)}{\rho_\omega(\lambda)},$$

with $J=J_0$ and $X=[I_p\ \ S]$.

The equivalence between (3.1) and the properties of the map (3.2) enables us to define an ISP problem for any function of the form (1.4): let Θ be a $\mathbf{C}_{m\times m}$-valued function of bounded type in Δ_+; we denote by $\mathcal{K}(\Theta)$ the space $\mathcal{B}(X)$ corresponding to $X=[I_m\ \ \Theta]$ and signature matrix $\begin{pmatrix} J & 0\\ 0 & -J\end{pmatrix}$.

DEFINITION. The inverse scattering problem associated to the function $X(\lambda)JX^*(\omega)/\rho_\omega(\lambda)$ (or by abuse of language, to X) consists in finding functions Θ which are $\mathbf{C}_{m\times m}$-valued, of bounded type in Δ_+, and such that the map $F\to XJF$ is a contraction from $\mathcal{K}(\Theta)$ into $\mathcal{B}(X)$: for any F in $\mathcal{K}(\Theta)$, XJF belongs to $\mathcal{B}(X)$ and

$$[XJF, XJF]_{\mathcal{B}(X)}\le [F,F]_{\mathcal{K}(\Theta)}.$$

The inverse scattering problem defined here is closely related to interpolation theory; some of these links were discussed in [3] for $X=[I_p\ \ S]$, $J=J_0$, and $X=[I_p\ \ \phi]$;

$$J=\begin{pmatrix} 0 & I_p\\ I_p & 0\end{pmatrix}$$

in the Hilbert space case.

From [7], we see that any interpolation problem satisfied by the Schur function S (or the Caratheodory function ϕ) will generate a solution to the present ISP associated to $[I_p\ \ S]$ (or $[I_p\ \ \phi]$), when a certain Pick matrix is invertible.

It may be worthwhile to mention the following special case of the ISP:

$$\Theta=\begin{pmatrix} A & 0\\ 0 & I_p\end{pmatrix},\quad J=I_{2p}.$$

In the Hilbert space case, we thus look for decompositions of S as AW with the condition that $H(A)$, the RKHS with reproducing kernel $(I_p-A(\lambda)A^*(\omega))/\rho_\omega(\lambda)$, is contractively included in $H(S)$. Equivalence between this inclusion and the decomposition of S is a consequence of a theorem of Rosenblum [17]. It is also equivalent to the requirement that the function

$$-\frac{S(\lambda)S^*(\omega)-A(\lambda)A^*(\omega)}{\rho_\omega(\lambda)}$$

is positive for λ and ω in Δ_+.

This point is exploited in [1], where the case $X=[A \quad B]$, $J=J_0$, was considered, with A, B $\mathbf{C}_{k\times p}$- and $\mathbf{C}_{k\times q}$-valued of bounded type in $\mathbf{D}$, $\mathcal{B}(X)$ being a Hilbert space. Finally, let us mention that the ISP in the Hilbert space case was formulated in [11] for the case $X=[I \quad S]$, in the study of pairs of self-adjoint operators.

We now exhibit a class of solutions to the ISP associated to X for X in $H^{\infty}_{n\times m}$. We first make some definitions: H^2_J will be the space H^2_m endowed with the inner product

$$[F,G]_{H^2_J}=\langle F, JG\rangle_{H^2_m},$$

and for Ξ a $\mathbf{C}_{n\times n}$-valued inner function, P_Ξ will denote the orthogonal projection from L^2_n onto $H^2_n\ominus \Xi H^2_n$ and Γ_Ξ will denote the operator

$$\Gamma_\Xi = P_\Xi \Gamma \mid_{H^2_n\ominus \Xi H^2_n}, \tag{3.5}$$

where Γ is defined in (2.7).

We can now state the following.

THEOREM 3. *Let X be an element of $H^{\infty}_{n\times m}$, let J be some signature matrix, and let Ξ be a $\mathbf{C}_{n\times n}$-valued inner function analytic at some point of the boundary of Δ_+ and such that the operator Γ_Ξ defined by* (3.5) *is invertible. Then the subspace $\mathfrak{M}$ of H^2_J of functions*

$$\underline{\underline{p}}X^*u, \quad u\in H^2_n\ominus \Xi H^2_n \tag{3.6}$$

is a reproducing kernel Krein space in the H^2_J norm. Its reproducing kernel is of the form (3.3) *with a function Θ solution to the inverse scattering problem associated to X.*

For comparison, it is useful to note that in case $X=[I_p \quad S]$ and $\Delta_+=\mathbf{D}$, S being a Schur function, the subspace $\mathfrak{M}$ of Theorem 3 reduces to the J-orthogonal of the modeling space of Ball–Helton [7] and $X\mathfrak{M}$ to the model space of Sarason [18].

In order to prove Theorem 3, we need a characterization of Krein spaces with a reproducing kernel of the form (3.3). In the Hilbert case, and for $\rho_\omega(\lambda)=-2\pi i(\lambda-\omega^*)$, this characterization was first obtained by de Branges [10]; it was then worked out in the circle case by Ball in [6]. We first need two definitions.

DEFINITION. A vector space of $\mathbf{C}_m$-valued functions analytic in some open set U is said to be resolvent invariant if it is closed under the operators R_α, $\alpha\in U$ defined by

$$(R_\alpha f)(\lambda)=\frac{f(\lambda)-f(\alpha)}{\lambda-\alpha}.$$

DEFINITION. A subset U of the complex numbers is symmetric with respect to the unit circle (resp. with respect to the real line) if it is closed under the transformation $\omega\to 1/\omega^*$ (resp. $\omega\to\omega^*$).

We can now state the following.

THEOREM 4. *Let $\mathcal{K}$ be a reproducing kernel Krein space of $\mathbf{C}_m$-valued functions analytic in some open set U, which is symmetric with respect to the unit* *circle. Suppose that U intersects the unit circle, that $\mathcal{K}$ is resolvent invariant, and that for any α, β in U and F, G in $\mathcal{K}$,*

$$[F,G]+\alpha[R_\alpha F,G]+\beta^*[F,R_\beta G]-(1-\alpha\beta^*)[R_\alpha F,R_\beta G]=G^*(\beta)JF(\alpha) \tag{3.7}$$

for some preassigned signature matrix J. Then the reproducing kernel of $\mathcal{K}$ is of the form

$$\frac{J-\Theta(\lambda)J\Theta^*(\omega)}{1-\lambda\omega^*}$$

for some $\mathbf{C}_{m\times m}$-valued function Θ analytic in U.

THEOREM 5. *Let $\mathcal{K}$ be a reproducing kernel Krein space of $\mathbf{C}_m$-valued functions analytic in an open set U, which is symmetric with respect to the real line. Suppose that U intersects the real line, that $\mathcal{K}$ is resolvent invariant, and that for any α, β in U and F, G in $\mathcal{K}$,*

$$[R_\alpha F,G]-[F,R_\beta G]-(\alpha-\beta^*)[R_\alpha F,R_\beta G]=2\pi i G^*(\beta)JF(\alpha) \tag{3.8}$$

for some preassigned signature matrix J. Then the reproducing kernel of $\mathcal{K}$ is of the form

$$\frac{J-\Theta(\lambda)J\Theta^*(\omega)}{-2\pi i(\lambda-\omega^*)}$$

for some $\mathbf{C}_{m\times m}$-valued function Θ analytic in U.

The proofs of Theorems 4 and 5 follow exactly the Hilbert space case and will not be recalled (see [6], [10]; we also refer to [2] where, in a Hilbert space context, both theorems are given a unified formulation).

We will apply these two theorems for spaces $\mathcal{K}$ isometrically included in H_J^2. In this case, as noted in [4], equations (3.7) and (3.8) are automatically satisfied. Moreover, Theorems 4 and 5 may then be proved using the Ball–Helton theory [7], as we now explain. Let us take $\mathcal{K}$ as in Theorem 5, and suppose $\mathcal{K}$ is isometrically included in H_J^2. Its orthogonal in H_J^2 is thus of the form $\Theta H_{J'}^2$, where J' is some other signature matrix and $\Theta(g)$ is (J, J') isometric for $|g|=1$. Thus $\mathcal{K}=H_J^2\ominus\Theta H_{J'}^2$, and the hypothesis that elements of $\mathcal{K}$ are analytic at some point of the unit circle leads to $J=J'$. This argument shows in particular that $\mathcal{K}(\Theta)$ is uniquely defined for Θ analytic in $\mathbf{D}$ and J-unitary on the unit circle.

We now turn to the proof of Theorem 3, and proceed in a number of steps.

Proof of Theorem 3.

Step 1: Let $\mathfrak{M}$ denote the set of functions of the form (3.6). We first remark that an element of $\mathfrak{M}$ is uniquely defined by its associated element u in $H_n^2\ominus\Xi H_n^2$. Indeed, if $\underline{\underline{p}}X^*u=0$ then $\Gamma_\Xi u=P_\Xi XJ\underline{\underline{p}}X^*u=0$ and thus, by the presumed invertibility of Γ_Ξ, $u=0$. This observation allows us to show that $\mathfrak{M}$ is a RKKS. Indeed, let

$$F=\underline{\underline{p}}X^*u \quad\text{and}\quad G=\underline{\underline{p}}X^*v.$$

Then

$$[F,G]_{H_J^2}=\langle\Gamma_\Xi u,v\rangle_{H_n^2}.$$

We deduce from this equality that $\mathfrak{M}$ is a Krein space. Indeed, let σ be the operator from $\mathfrak{M}$ to $\mathfrak{M}$ defined by

$$\sigma(\underline{\underline{p}}X^*u)=\underline{\underline{p}}X^*(\operatorname{sgn}\Gamma_\Xi)u,$$

where $\operatorname{sgn}\Gamma_\Xi$ denotes the signum of the self-adjoint operator Γ_Ξ. Clearly $\sigma^2=I$, the identity in $\mathfrak{M}$, and for any F, G in $\mathfrak{M}$

$$[F,\sigma G]_{H_J^2}=[\sigma F,G]_{H_J^2}=\langle|\Gamma_\Xi|u,v\rangle_{H_n^2}.$$

Hence, $\mathfrak{M}$ endowed with the inner product

$$\langle F,G\rangle_\sigma=[F,\sigma G]_{H_J^2} \tag{3.9}$$

is a Hilbert space. Indeed, since $|\Gamma_\Xi|$ is bounded and boundedly invertible, there are two constants k_1 and k_2 such that

$$k_2\langle u,u\rangle_{H_p^2}\le\langle F,F\rangle_\sigma\le k_1\langle u,u\rangle_{H_p^2}$$

and hence $\mathfrak{M}$ is readily seen to be complete in the norm induced by (3.9). To conclude that $\mathfrak{M}$ is a Krein space, it suffices to check that the operator σ is a signature operator in the Hilbert space $(\mathfrak{M},\langle\,,\,\rangle_\sigma)$, that is, satisfies $\sigma=\sigma^{-1}=\sigma^*$. But this is self-evident. Thus the relationship

$$[F,G]_{H_J^2}=\langle F,\sigma G\rangle_\sigma$$

expresses the fact that $\mathfrak{M}$ is a Krein space when endowed with the inner product $[\,,\,]_{H_J^2}$.

Step 2 consists of computing the reproducing kernel of $\mathfrak{M}$, and showing that $\mathfrak{M}$ is a $\mathcal{K}(\Theta)$ space. By the hypothesis on Ξ, the elements of $H_n^2\ominus\Xi H_n^2$ are analytic at some point of the boundary of Δ_+, and we extend the elements of $\mathfrak{M}$ to the domain of analyticity of Ξ via:

$$F(\lambda)=X^*(0)f(\lambda)+\frac{1}{2\pi i}\int_0^{2\pi}(X^*(e^{it})-X^*(0))\frac{f(e^{it})-f(\lambda)}{e^{it}-\lambda}e^{it}\,dt\quad\text{if }\Delta_+=\mathbf{D}$$

and

$$F(\lambda)=X^*(i)f(\lambda)+\frac{1}{2\pi i}\int_{-\infty}^{\infty}(X^*(t)-X^*(i))\left(\frac{f(t)-+f(\lambda)}{t-\lambda}-\frac{f(\lambda)t}{t^2+1}\right)dt$$
$$\text{if }\Delta_+=\mathbf{C}_+.$$

F and $\underline{\underline{p}}X^*f$ coincide in Δ_+. Rather than $\mathfrak{M}$, we first consider $\mathfrak{M}^0$, the set of functions F with the inner product $[F,F]_{\mathfrak{M}^0}=[\underline{\underline{p}}X^*f,\underline{\underline{p}}X^*f]$. $\mathfrak{M}^0$ is a resolvent invariant Krein subspace of H_J^2, and an application of Theorem 5 if $\Delta_+=\mathbf{D}$ or Theorem 6 if $\Delta_+=\mathbf{C}$ permits us to conclude that its reproducing kernel is of the form (3.3). (One has to check that point evaluations are bounded in $\mathfrak{M}^0$; this is left to the reader.)

The reproducing kernel of $\mathfrak{M}$ is then easily identified as the reproducing kernel of $\mathfrak{M}^0$ restricted to Δ_+, that is, of the form $(J-\Theta(\lambda)J\Theta^*(\omega))/\rho_\omega(\lambda)$, ω,λ in Δ_+ and Θ $\mathbf{C}_{m\times m}$-valued and analytic in Δ_+.

To conclude the proof of the theorem, it remains to show that Θ is a solution to the ISP. This is done in step 3.

Step 3: Let F be in $\mathfrak{M}$. Then $XJF=(XJ\underline{\underline{p}}X^*)f$ and thus, by the description of the space $\mathcal{B}(X)$, XJF belongs to $\mathcal{B}(X)$ and

$$[XJF, XJF]_{\mathcal{B}(X)}=[F,F]_{H_J^2}=\langle \Gamma_{\Xi} f, f\rangle_{H_n^2}.$$

Hence the mapping $F \to XJF$ is an isometry between the spaces $\mathfrak{M}$ and $\mathcal{B}(X)$, and hence Θ is a solution of the ISP associated to X, which concludes the proof of the theorem. □

To conclude, we remark that Theorem 3 deals with functions X in $H^{\infty}_{n\times m}$, but the case where X is of bounded type in Δ_+ is easily adapted. The solutions Θ exhibited in Theorem 3 are such that $\mathcal{K}(\Theta)$ is isometrically included in H_J^2, and such that the map $F \to XJF$ is an isometry. Theorem 3.10 of [7] will show that the present method exhausts all solutions with these properties; details are omitted. Moreover, there may exist solutions which do not satisfy $\mathcal{K}(\Theta)\subset H_J^2$ (see [3], [8], [13] for $X=[I_n \quad S]$). These are not covered by Theorem 3.

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