THE LOCAL HULL OF HOLOMORPHY OF SEMIRIGID SUBMANIFOLDS OF CODIMENSION TWO

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1. Introduction. There has been considerable literature on determining the local hull of holomorphy for CR submanifolds of C^n. The case for real hypersurfaces was considered by Lewy in [11]. For generic submanifolds of higher codimension, it was shown in [10] that if the excess dimension of the Levi algebra at a point is maximal then the local hull of holomorphy contains an open set in C^n. Since that time, the goal of CR extension research has been (and still is) to more precisely determine the size of the local hull of holomorphy. This was done in [7] and later in [1] where it was shown that the cross section of the local hull of holomorphy "almost fills in" the convex hull of the image of the first Leviform at a point in the submanifold. The case when the first Leviform vanishes at a point, the relationship between the local hull of holomorphy and the second Leviform, has been given in [5], [6], and [9]. Recently in [2] it was shown that the local hull of holomorphy for a semirigid submanifold of higher type (to be defined below) contains a "wedge." This is a considerable improvement over the result in [10] alluded to above, which only guarantees that the local hull contains a cusplike set. However, the relationship between the size of the wedge and the type of the point was still missing. The goal of this paper is to explain this relationship for semirigid submanifolds of real codimension two.

Let us introduce notation. Suppose M is a smooth (C^k for k sufficiently large) CR submanifold of C^n of real codimension d. We let $T^C(M)$ and $H^C(M)$ be the complexified tangent bundle and complexified holomorphic tangent bundle respectively. We introduce the following tower of spaces. For $p \in M$, we let $L_p^0(M) = H_p^C(M)$ and in general we let $L_p^j(M)$ be the vector space spanned by $H_p^C(M)$ and all Lie brackets at p of elements in $H^C(M)$ up through length j. Note that the length of a Lie bracket is the number of vector fields in the bracket, that is, $\Lambda_m = [L_1, [L_2, \dots, [L_{m-1}, L_m], \dots]$ has length m. This notion of length is the same as that in [4] and length = order + 1 where order is the concept used in [8]. Note that

$$H_p^C(M) \subset L_p^j(M) \subset L_p^{j+1}(M) \subset T_p^C(M)$$

for all nonnegative j and in particular $2n-2d \le \dim_C L_p^j(M) \le 2n-d$.

DEFINITION 1.1 [4]. Let M be a generic submanifold of real codimension d ($d < n$) of C^n, and let $p \in M$. We say that p is a point of *type* $l = (l_1, \dots, l_d)$ where $l_1 \le l_2 \le \cdots \le l_d$ are integers if the following holds:

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(i) $\dim_C L_p^j(M) = 2n - 2d$ for all $j < l_1$.
(ii) $\dim_C L_p^j(M) = 2n - 2d + i$ for all j with $l_i \le j < l_{i+1}$, $i = 1, \dots, d-1$.
(iii) $\dim_C L_p^j(M) = 2n - d$ for all $j \ge l_d$.

Let us write C^n as $C^d \times C^m$ with coordinates $z = x + iy$ and $w \in C^m$. If $l = (l_1, \dots, l_d)$ is a d-tuple of positive integers with $l_i \le l_{i+1}$, then (z, w, l) is said to be a weighted coordinate system if each w_i and $\bar{w}_i$ has weight 1 for $1 \le i \le m$ while each z_i and $\bar{z}_i$ has weight l_i for $1 \le i \le d$. A monomial in $z, \bar{z}, w, \bar{w}$ has weight γ if the sum of the weights of the $z_j, \bar{z}_j, w_i, \bar{w}_i$ which occur is γ. A smooth function $f: C^n \to C$ has weight γ if among the monomials in the formal Taylor series expansion of f (about 0) there is one of weight γ but none of lower weight.

The main theorem from [4] is that if $0 \in M$ is a point of type $l = (l_1, \dots, l_d)$, then a weighted coordinate system (z, w, l) and functions

$$h = (h_1, \dots, h_d): R^d \times C^m \to R^d$$

can be chosen so that, near 0, $M = \{x = h(y, w, \bar{w})\}$ and furthermore each h_i can be put in standard form; that is, for $i = 1, \dots, d$,

$$h_i(y, w, \bar{w}) = p_{l_i}(w, \bar{w}, y_1, \dots, y_{i-1}) + E_{l_i+1}(w, \bar{w}, y), \tag{1.2}$$

where each p_{l_i} is a polynomial of weight l_i with no pure terms (that is, no monomials of the form w^α or $\bar{w}^\beta$ or $y^\alpha w^\beta$ or $y^\alpha \bar{w}^\beta$) and E_{l_i+1} is smooth of weight at least $l_i + 1$.

Following [2], if $0 \in M$ is a point of type $(l_1, \dots, l_d)$ then we say that M is *semirigid* if $[\Lambda_{m_1}, \lambda_{m_2}]_0 = 0$ for all $\Lambda_{m_i} = [L_1, [L_2, \dots, [L_{m_i-1}, L_{m_i}], \dots]$, $L_j \in H^C(M)$, with $m_1 + m_2 \le l_1$ and $m_1 \ge 2$, $m_2 \ge 2$. In terms of the normal form given above, this simply means that the polynomials p_{l_i} are homogeneous of degree l_i in w and $\bar{w}$ only (no y dependence) and with no pure terms. The functions E_{l_i+1} may still depend on y.

For $p \in M$, we set $N_p(M)$ to be the set of all vectors in $T_p(C^n)$ which are normal to M. If Ω is the local hull of holomorphy of M, then we are interested in the size of $N_p(M) \cap \Omega$. If $\operatorname{codim}_R M = 2$, then $N_p(M)$ is a real two-dimensional plane. In this case we define the following triangular wedges. For $a, b > 0$ and $v \in N_p(M)$ a unit vector, we set

$$W(v, a, b) = \left\{ x \in N_p(M) : x \cdot v > \frac{a|x|}{\sqrt{a^2 + b^2}},\ 0 \le x \cdot v < a \right\}.$$

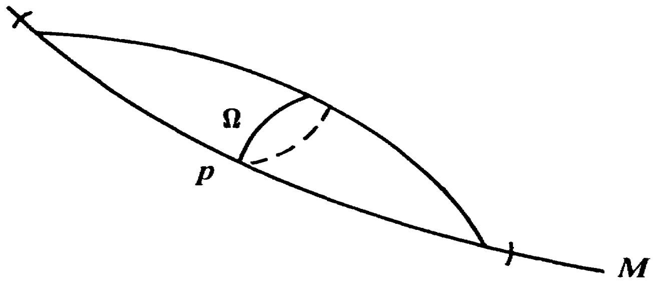
Here $(\cdot)$ denotes the Euclidean inner product on R^{2n}. $W(v, a, b)$ is an isosceles triangle in $N_p(M)$ with altitude a and base $2b$ and the triangle is bisected by v. Finally, we let $B(p, r)$ be the open ball in C^n centered at p of radius r. We now state our theorem.

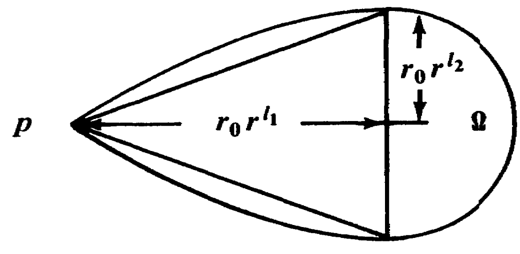
THEOREM 1.3. *Suppose M is a smooth CR generic semirigid submanifold of C^n of real codimension two. Let $p \in M$ be a point of type (l_1, l_2) with $2 \le l_1 < l_2 < \infty$. Then given an open set ω in M with $p \in \omega$, there exists an open set Ω in C^n such that each continuous CR function on ω extends to a unique holomor-*

phic function on Ω. *Moreover, there exists a unit vector* $v \in N_p(M)$ *and there exists an* $r_0 > 0$ *such that if* $0 < r < r_0$ *and if* $B(p, r) \cap M \subset \omega$ *then*

$$W(v, r_0 r^{l_1}, r_0 r^{l_2}) + (M \cap B(p, r_0 r^{l_2 - 1})) \subset \Omega.$$

The theorem conveys the following picture of Ω.





cross section of Ω

REMARKS. 1. The case when $l_1 = l_2$ is not handled in the above theorem since a stronger result about the size of the local hull can be given. This result (for arbitrary codimension) will be forthcoming in a later paper.

2. Except for the proportionality constant r_0, these results are best possible. For example, if $M = \{(z_1, z_2, w) \in C^3 : \mathrm{Re}(z_1) = |w|^2, \mathrm{Re}(z_2) = |w|^2 \mathrm{Re}(w)\}$ (here $p = 0$, $l_1 = 2$, $l_2 = 3$) then the convex hull of $M \cap B(0, r)$ is contained in $M + W(v, r^2, r^3)$, where v is the unit vector along the positive $\mathrm{Re}\, z_1$ axis.

The desired set Ω in the above theorem will be realized as a subset of a family of analytic discs. Let D be the open unit disc in C. An analytic disc is a continuous map $A : \bar{D} \to C^n$ which is holomorphic on D. The boundary of A, denoted bA, is the restriction of A to S^1 (the unit circle in C).

The main contribution of this paper is the following theorem on analytic discs.

THEOREM 1.4. *Under the assumptions given in Theorem* 1.3 *on* M, p, l_1, l_2, *and* ω, *there is an open set* Ω *in* C^n *which satisfies the size description given in Theorem* 1.3 *such that each point in* Ω *lies in the image of an analytic disc whose boundary has image contained in* ω.

Theorem 1.3 can easily be proved as a consequence of Theorem 1.4 and the CR approximation theorem in [3, Theorem and Remark 2.1] and the maximum principle. Details are left to the reader; see also [7, §2].

2. The generalized Bishop's equation. In this section, we discuss the solution of Bishop's equation which is used in the construction of analytic discs alluded to in Section 1. Let us assume the given point p in Theorem 1.3 is the origin and that M is locally graphed over its tangent space at $p = 0$. Let us choose coordinates (z, w) for C^n with $z = x + iy \in C^d$ and $w \in C^m$, where $d + m = n$ and $d = \mathrm{codim}_R(M)$. Later d will be 2, but this section is valid for arbitrary codimension. Since M is generic, we may describe it locally as $M = \{x = h(y, w, \bar{w})\}$, where $h : R^d \times C^m \to R^d$ is smooth with $h(0) = 0$ and $Dh(0) = 0$.

In formulating Bishop's equation, we shall make use of the Poisson kernel which is defined as follows: if $z \in D$ and $u : S^1 \to R^d$ is continuous, then

$$P(z,u) := \frac{1}{2\pi i}\int_{|\zeta|=1} u(\zeta)\left[\frac{1-|z|^2}{|\zeta-z|^2}\right]\frac{d\zeta}{\zeta}.$$

Now suppose $W:\bar{D}\to C^m$ is a given analytic disc and suppose $y\in R^d$, $w\in C^m$, and $z\in D$ are also given. We say that $v: S^1\to R^d$ satisfies the modified Bishop's equation if

$$v(\zeta) = T(H(v+y, W+w, \bar{W}+\bar{w}))(\zeta) - P(z, T(H(v+y, W+w, \bar{W}+\bar{w}))) \tag{2.1}$$

for $\zeta\in S^1$, where T is the Hilbert transform and where we have used the notation

$$H(v+y, W+w, \bar{W}+\bar{w})(\zeta) = h(v(\zeta)+y, W(\zeta)+w, \overline{W(\zeta)}+\bar{w}).$$

Clearly $v(\zeta) = v(W, y, w, z)(\zeta)$ depends on the parameters W, y, w and z. The smoothness of v on S^1 will be specified later.

If v satisfies the modified Bishop's equation, then we define

$$V(\zeta) = V(W, y, w, z)(\zeta)$$

by

$$V(W,y,w,z)(\zeta) := P(\zeta, T(H(v+y, W+w, \bar{W}+\bar{w}))) \\ - P(z, T(H(v+y, W+w, \bar{W}+\bar{w}))),$$

for $|\zeta|\le 1$. Clearly V is harmonic in ζ, $|\zeta|\le 1$, and $V(\zeta) = v(\zeta)$ for $|\zeta| = 1$. Moreover, we record for future reference that

$$V(W, y, w, z)(\zeta = z) = 0. \tag{2.2}$$

We also define

$$U(W,y,w,z)(\zeta) := P(\zeta, H(v(W,y,w,z)+y, W+w, \bar{W}+\bar{w})) \quad \text{for } |\zeta|\le 1.$$

Clearly U is harmonic in ζ, $|\zeta|\le 1$. Moreover,

$$U(\zeta)+iV(\zeta) = H(v+y, W+w, \bar{W}+\bar{w})(\zeta) \\ + i[T(H(v+y, W+w, \bar{W}+\bar{w}))(\zeta) \\ - P(z, T(H(v+y, W+w, \bar{W}+\bar{w})))]$$

for $|\zeta| = 1$. Since $P(z, T(H(v+y, W+w, \bar{W}+\bar{w})))$ is constant in ζ, $U(\zeta)+iV(\zeta)$ is analytic in ζ, $|\zeta|<1$, by definition of the Hilbert transform. So, we may define the analytic disc A with parameters W, v, w, z by

$$A(W,y,w,z)(\zeta) := (U(W,y,w,z)+i(V(W,y,w,z)(\zeta)+y), W(\zeta)+w).$$

Note that $A(W,y,w,z): \bar{D}\to C^d\times C^m = C^n$. Since

$$U(\zeta) = H(v+y, W+w, \bar{W}+\bar{w})(\zeta) \quad \text{for } |\zeta| = 1,$$

we must have that the image of bA is contained in M. Therefore, a solution to equation (2.1) generates an analytic disc with boundary in M.

We shall now summarize the construction of the solution to equation (2.1). For j a nonnegative integer and $0\le\alpha\le 1$ and $K\subset R^N$, we let $C^{j,\alpha}(K, R^l)$ be the space of R^l-valued functions on K whose derivatives up through order j are Hölder continuous with exponent α. The norm on $C^{j,\alpha}(K, R^l)$ will be denoted $\|\cdot\|_{C^{j,\alpha}(K)}$. The set K will either be S^1 or D. We let $O^{j,\alpha}(D, C^m)$ be the space of analytic discs $W: \bar{D} \to C^m$ with $W|_{S^1} \in C^{j,\alpha}(S^1, C^m)$.

THEOREM 2.3. *Let $h: R^d \times C^m \to R^d$ be of class $C^{k,\epsilon}$ for $k \geq 3$ and for some $\epsilon > 0$, and suppose $D(h)(0) = 0$. Fix $0 < \alpha < 1$ and let k' and j be integers with $0 \leq j \leq k' \leq k$.*

(a) *There exist neighborhoods $\mathcal{W}^{k,\alpha} \subset O^{k,\alpha}(D, C^m)$, $\mathcal{W} \subset C^m$, and $Y \subset R^d$ containing the origins, and there exist $\delta = \delta(\alpha, \epsilon) > 0$ and a map*

$$v: \mathcal{W}^{k,\alpha} \times Y \times \mathcal{W} \times D \to C^{j,\delta}(S^1, R^d)$$

which is uniformly C^{k-j} in the sense of Banach spaces such that for each

$$(W, y, w, z) \in \mathcal{W}^{k,\alpha} \times Y \times \mathcal{W} \times D,$$

the function $v(W, y, w, z) \in C^{j,\delta}(S^1)$ is the unique solution to equation (2.1).

(b) *There exists a constant $C > 0$ with*

$$\|v(W, y, w, z)\|_{C^{j,\delta}(S^1)} \leq C\|W\|_{C^{k,\alpha}(S^1)} \tag{2.3}$$

for all $W \in \mathcal{W}^{k,\alpha}$, $y \in Y$, $w \in \mathcal{W}$, $z \in D$.

The proof of this theorem can be found in [8, Theorem 4.7]. Part (b) follows from the fact that v is Lipschitz in its dependence on W and because $V \equiv 0$ when $W \equiv 0$. We shall use this theorem with $k = \infty$ and with $k' = j = 1$. In this case δ can be chosen to equal α as shown in the proof of Theorem 4.7 in [8].

LEMMA 2.4. *Let 0 be a point of type $l = (l_1, \ldots, l_d)$ and assume M is presented as $M = \{x = h(y, w, \bar{w})\}$, with $h(y, w, \bar{w})$ in normal form* (1.2) *(semirigidity is not assumed here). Fix $0 < \alpha < 1$. There exist neighborhoods $\mathcal{W}^{1,\alpha} \subset O^{1,\alpha}(D, C^m)$, $\mathcal{W} \subset C^m$, $Y \subset R^d$ containing the origins, and there is a constant $C > 0$ such that for $W \in \mathcal{W}^{1,\alpha}$, $w \in \mathcal{W}$, $y \in Y$, and $z \in D$,*

$$\max\{\|v^j(W, y, w, z)\|_{C^{1,\alpha}(S^1)}, \|V^j(W, y, w, z)\|_{C^{1,\alpha}(D)}\}$$
$$\leq C((|y| + |w|)\|W\|_{C^{1,\alpha}(S^1)} + \|W\|^{l_j}_{C^{1,\alpha}(S^1)}) \quad \text{for } 1 \leq j \leq d.$$

Proof. We shall prove the above estimate for v_j. The estimate for V_j will then follow because $V_j = \bar{P}v_j$ where $\bar{P}: C^{1,\alpha}(S^1) \to C^{1,\alpha}(D)$ is the bounded linear map given by $\bar{P}u(z) = P(z, u)$, $|z| \leq 1$ for $u \in C^{1,\alpha}(S^1)$.

In view of Theorem 2.3, we can for a given $r_0 > 0$ choose $\mathcal{W}^{1,\alpha}, Y, \mathcal{W}$ so that

$$\max\{\|W\|_{C^{1,\alpha}(S^1)}, \|v(W, y, w, z)\|_{C^{1,\alpha}(S^1)}, |y|, |w|\} < r_0$$

for $W \in \mathcal{W}^{1,\alpha}$, $y \in Y$, $w \in \mathcal{W}$, $z \in D$. The number r_0 will be chosen later.

We first show that

$$\begin{aligned} \|v\|_{C^{1,\alpha}(S^1)} &\leq \sum_{j=1}^{d} \|v^j\|_{C^{1,\alpha}(S^1)} \\ &\leq C((|y| + |w|)\|W\|_{C^{1,\alpha}(S^1)} + \|W\|^{l_1}_{C^{1,\alpha}(S^1)}) \end{aligned} \tag{2.5}$$

In view of equation (2.1) we have

$$\|v\|_{C^{1,\alpha}(S^1)} \le C_\alpha \|T(H(v+y, W+w, \bar{W}+\bar{w}))\|_{C^{1,\alpha}(S^1)}$$

with $C_\alpha = (1+\|\bar{P}\|)$, where $\|\bar{P}\|$ is the operator norm of $\bar{P}\colon C^{1,\alpha}(S^1) \to C^{1,\alpha}(D)$.

We have

$$\begin{aligned} T(H(v+y, W+w, \bar{W}+\bar{w})) &= T(H(y, W+w, \bar{W}+\bar{w})) \\ &\quad + \int_0^1 T[(D_yH)(y+tv, W+w, \bar{W}+\bar{w})\cdot v]\,dt, \end{aligned} \tag{2.6}$$

where $(D_yH)(\cdot,\cdot,\cdot)$ denotes composition with $D_y h$, which is the differential of h with respect to y. Now we Taylor expand $H(y, W+w, \bar{W}+\bar{w})$ about $(y, w, \bar{w})$ in powers of W and $\bar{W}$. Noting that $T(h(y,w,\bar{w}))=0$ (because $h(y,w,\bar{w})$ is just a constant with respect to ζ on S^1), we obtain

$$\begin{aligned} &T(H(y, W+w, \bar{W}+\bar{w})) \\ &\quad = \sum_{0<|\gamma|+|\beta|<l_1} \frac{1}{\gamma!,\beta!} \frac{\partial^{|\gamma|+|\beta|} h(y,w,\bar{w})}{\partial w^\gamma \partial \bar{w}^\beta} T(W^\gamma \bar{W}^\beta) + T(R(W,\bar{W})), \end{aligned}$$

where R is just the Taylor remainder of order l_1. From the normal form in (1.2), we see that h is of weight at least l_1. Since each w_j and $\bar{w}_j$ is counted with weight one, clearly each derivative of h in the first term on the right vanishes at $y=0$, $w=0$. Therefore

$$\begin{aligned} &\|T(H(y, W+w, \bar{W}+\bar{w}))\|_{C^{1,\alpha}(S^1)} \\ &\quad \le C\|T\|_{C^{1,\alpha}(S^1)}((|y|+|w|)\|W\|_{C^{1,\alpha}(S^1)} + \|W\|^{l_1}_{C^{1,\alpha}(S^1)}), \end{aligned} \tag{2.7}$$

where $\|T\|_{C^{1,\alpha}(S^1)}$ is the operator norm of $T\colon C^{1,\alpha}(S^1) \to C^{1,\alpha}(S^1)$.

By the choice of the neighborhoods $W^{1,\alpha}$, Y, W, and since Dh vanishes at the origin, clearly

$$\|(D_yH)(y+tv, W+w, \bar{W}+\bar{w})\|_{C^{1,\alpha}(S^1)} \le \Lambda(r_0)$$

for $(y,w,W,z) \in Y \times W \times W^{1,\alpha} \times D$, where $\Lambda(r_0) \to 0$ as $r_0 \to 0$. Putting this together with (2.6) and (2.7), we have

$$\begin{aligned} \|v\|_{C^{1,\alpha}(S^1)} &\le C_\alpha[(|y|+|w|)\|W\|_{C^{1,\alpha}(S^1)} + \|W\|^{l_1}_{C^{1,\alpha}(S^1)}] \\ &\quad + \Lambda(r_0)\|T\|_{C^{1,\alpha}(S^1)}\|v\|_{C^{1,\alpha}(S^1)}. \end{aligned}$$

Now choosing r_0 small enough so that $\Lambda(r_0)\|T\|_{C^{1,\alpha}(S^1)} < \frac{1}{2}$ and then absorbing the second term on the right yields (2.5).

The proof will be complete when we show by induction on $j = 1, \dots, d$ that

$$\|v^j\|_{C^{1,\alpha}(S^1)} + \cdots + \|v^d\|_{C^{1,\alpha}(S^1)} \le C((|y|+|w|)\|W\|_{C^{1,\alpha}(S^1)} + \|W\|^{l_j}_{C^{1,\alpha}(S^1)}).$$

The case $j=1$ has been handled in (2.5), so we assume that $j>1$ and that (2.8) holds when j is replaced by k for $1 \le k \le j-1$. We let

$$\begin{aligned} v' &= (v^1, \dots, v^{j-1}); \quad H' = (H^1, \dots, H^{j-1}), \\ v'' &= (v^j, \dots, v^d); \quad H'' = (H^j, \dots, H^d). \end{aligned}$$

We have

$$\begin{aligned}\|v''\|_{C^{1,\alpha}(S^1)} \le C_\alpha \|T(H''(v'+y', y'', W+w, \bar W+\bar w))\|_{C^{1,\alpha}(S^1)} \\ + \int_0^1 \|T((D_{y''}H'')(v'+y', y''+tv'', W+w, \bar W+\bar w)\cdot v'')\|_{C^{1,\alpha}(S^1)}\,dt.\end{aligned} \tag{2.8}$$

where $D_{y''}H''(\cdot,\cdot,\cdot)$ denotes composition with $D_{y''}h''$, which is the differential of h'' with respect to y''. We now Taylor expand $H''(v'+y', y'', W+w, \bar W+\bar w)$ about (y', y'', w) in powers of v^k, W_i, $\bar W_i$ $(1\le k\le j-1, 1\le i\le m)$. We shall assign weight 1 to W_j and $\bar W_j$ and weight l_k to v_k for $1\le k\le j-1$. Since $h''(y, w, \bar w)$ has weight at least l_j, the coefficients of the monomials in v', W, $\bar W$ in this Taylor expansion of weight less than l_j will vanish at the origin. In addition, the induction hypothesis assumes

$$\|v^k\|_{C^{1,\alpha}(S^1)} \le C((|y|+|w|)\|W\|_{C^{1,\alpha}(S^1)} + \|W\|^{l_k}_{C^{1,\alpha}(S^1)}), \quad 1\le k\le j-1.$$

Therefore one easily obtains

$$\begin{aligned}&\|T(H''(v'+y', y'', W+w, \bar W+\bar w))\|_{C^{1,\alpha}(S^1)} \\ &\quad\le C_\alpha((|y|+|w|)\|W\|_{C^{1,\alpha}(S^1)} + \|W\|^{l_j}_{C^{1,\alpha}(S^1)})\end{aligned} \tag{2.9}$$

for some constant C_α which depends only on α. Moreover, an argument analogous to that in the case when $j=1$ shows that the second term on the right side of (2.8) is bounded by $\Lambda(r_0)\|T\|_{C^{1,\alpha}(S^1)}\|v\|_{C^{1,\alpha}(S^1)}$ for $(y, w, W, z)\in Y\times W\times W^{1,\alpha}\times D$, where $\Lambda(r_0)\to 0$ as $r_0\to 0$. Combining this with (2.9) and (2.8) and choosing r_0 small enough so that $\Lambda(r_0)\|T\|_{C^{1,\alpha}(S^1)}<\frac{1}{2}$ yields the desired estimate on $\|v'\|_{C^{1,\alpha}(S^1)}$. This completes the proof. □

Our job ahead is to select a family of analytic discs W so that the images of the corresponding analytic discs A sweep out the desired Ω for Theorem 1.4.

3. The proof. In this section, we complete the proof of Theorem 1.4. We shall assume that M is a semirigid smooth submanifold of C^n of real codimension two and that $0\in M$ is of type (l_1, l_2), with $2\le l_1<l_2<\infty$. From section 1, we have

$$M=\{(z_1, z_2, w)\in C^2\times C^m : \mathrm{Re}(z_j)=x_j=h^j(y, w, \bar w),\ j=1,2\},$$

where

$$h^j(y, w, \bar w)=p_{l_j}(w,\bar w)+E_{l_j+1}(y, w, \bar w), \quad 1\le j\le 2, \tag{3.1}$$

where p_{l_j} is a nonvanishing homogeneous polynomial of degree l_j and E_{l_j+1} has weight at least l_j+1. By a complex linear change of variables in w we may assume that both p_{l_1} and p_{l_2} are nonvanishing on the complex line $w_1\ne 0$, $w_2=w_3=\cdots=w_m=0$. By a linear change of scale in z_1, z_2, and w_1, we will assume that

$$\sum_{q=1}^{l_j-1}\frac{1}{q!(l_j-q)!}\binom{l_j-2}{q-1}\frac{\partial h^j(0)}{\partial w_1^q\partial \bar w_1^{l_j-q}}=1 \quad \text{for } j=1,2. \tag{3.2}$$

For $r>0$, we define the set

$$T_r=\left\{(a,\lambda);\ a\in C,\ 0\le\lambda\le 1, \text{ and } \frac{|a|}{1-\lambda}\le r\right\}.$$

We define the analytic disc $W(a,\lambda): D \to C^m$ by

$$W(a,\lambda)(\zeta) = (W^1(a,\lambda)(\zeta), \dots, W^m(a,\lambda)(\zeta)),$$

$$W^1(a,\lambda)(\zeta) = \frac{a(\zeta+\lambda)}{1-\lambda^2},$$

$$W^j(a,\lambda)(\zeta) = 0 \quad \text{for } 2 \le j \le m.$$

We have

$$\|W(a,\lambda)\|_{C^{1,\alpha}(S^1)} = \|W(a,\lambda)\|_{C^{1,0}(D)} \le 3r \quad \text{for } (a,\lambda) \in T_r. \tag{3.3}$$

Here α is fixed $0<\alpha<1$. Let us also initially choose $r_0>0$ so that if $0<r<r_0$ and if $(a,\lambda)\in T_r$, then $W(a,\lambda)\in W^{1,\alpha}$ and $\{|y|<r\}\subset Y$ and $\{|w|<r\}\subset W$, where $W^{1,\alpha}$, Y, W are the open sets that satisfy the requirements of Theorem 2.3 and Lemma 2.4. Using Theorem 2.3, we obtain a map

$$v: T_r \times \{|y|<r\} \times \{|w|<r\} \times D \to C^{1,\alpha}(S^1)$$

which is at least C^1 and such that

$$v(a,\lambda,y,w,z)(\cdot) := v(W(a,\lambda),y,w,z)(\cdot)$$

is the unique solution to (2.1) with $W = W(a,\lambda)$. We also define

$$V(\zeta) = V(a,\lambda,y,w,z)(\zeta), \quad U(\zeta) = U(a,\lambda,y,w,z)(\zeta)$$
$$\text{and} \quad A(\zeta) = A(a,\lambda,y,w,z)(\zeta)$$

for $|\zeta|\le 1$ as in Section 2. We need the following estimates.

LEMMA 3.4. *Let M and l_1, l_2, v, V, U and W be as above. There exist constants $C>0$ and $r_0>0$ such that for $0<r<r_0$ and $(a,\lambda)\in T_r$, $|y|<r$, $|w|<r$, and $z\in D$ we have*

$$\|W(a,\lambda)\|_{C^{1,\alpha}(D)} \le C\left(\frac{|a|}{1-\lambda}\right) \le Cr, \tag{3.4a}$$

$$\begin{aligned} &\max\{\|v^j(a,\lambda,y,w,z)(\cdot)\|_{C^{1,\alpha}(S^1)}, \|V^j(a,\lambda,y,w,z)(\cdot)\|_{C^{1,\alpha}(D)}\} \\ &\le C\left[(|y|+|w|)\left(\frac{|a|}{1-\lambda}\right) + \left(\frac{|a|}{1-\lambda}\right)^{l_j}\right] \quad \text{for } j=1,2. \end{aligned} \tag{3.4b}$$

$$\|U(a,\lambda,y,w,z)\|_{C^{1,\alpha}(D)} \le Cr^2. \tag{3.4c}$$

Proof. Inequality (3.4a) just restates (3.3). Inequality (3.4b) follows from (3.4a) and Lemma 2.4. Inequality (3.4c) follows from the definition of U and (3.4a, b) and from the fact that $\bar{P}: C^{1,\alpha}(S^1) \to C^{1,\alpha}(D)$ is a bounded linear map. □

From this lemma it is clear (in particular) that $|bA(a,\lambda,y,w,z)(\zeta)| \le Cr$ for $|\zeta|=1$ and $(a,\lambda)\in T_r$, $|y|<r$, $|w|<r$, $z\in D$. Therefore we may further restrict r_0 so that if $0<r<r_0$ then the image of $bA(a,\lambda,y,w,z)(\cdot)$ is contained in ω, where $\omega \subset M$ is the open set in Theorem 1.3.

Now consider the map $f: T_r \times \{|y|<r\} \times \{|w|<r\} \to C^n$ defined by

$$f(a,\lambda,y,w) := A(a,\lambda,y,w,z=-\lambda)(\zeta=-\lambda)$$
$$= \begin{pmatrix} U(a,\lambda,y,w,z=-\lambda)(\zeta=-\lambda)+iy \\ w \end{pmatrix}.$$

The second equation uses the definition of A and the fact $W(a,\lambda)(\zeta=-\lambda)=0$ and $V(a,\lambda y,w,z=-\lambda)(\zeta=-\lambda)=0$ (see (2.2)). Each point in the image of f belongs to the image of an analytic disc with boundary in ω. So we shall complete the proof by showing that the image of f contains the desired Ω in C^n.

Clearly we must analyze $U=(U^1,U^2)$. Now $U^j(a,\lambda,y,w,z=-\lambda)(\zeta)$ is harmonic in ζ, $|\zeta|\le 1$, for $j=1,2$. So

$$\begin{aligned} &U^j(a,\lambda,y,w,z=-\lambda)(\zeta=-\lambda) \\ (3.5)\qquad &= P(-\lambda, U^j(a,\lambda,y,w,z=-\lambda)(\cdot)) \\ &= P(-\lambda, H^j(v(a,\lambda,y,w,z=-\lambda)+y, W(a,\lambda)+w, \overline{W(a,\lambda)}+\bar w)) \end{aligned}$$

for $j=1,2$, where we have used that $U(\zeta)=H(v+y, W+w, \bar W+\bar w)(\zeta)$ for $|\zeta|=1$; that is, $bA\subset M$ (cf. the definition of U in §2).

We now wish to use the Taylor expansion of $H^j(v+y, W+w, \bar W+\bar w)$ about the point (y,w) in the right side of (3.5). We shall use the following facts:

$$P(-\lambda, v(a,\lambda,y,w,z=-\lambda))=0,$$
$$P\left(-\lambda, \operatorname{Re}\left\{\frac{\partial^{|\alpha|}h^j(y,w)}{\partial w^\alpha}[W(a,\lambda)]^\alpha\right\}\right)=0 \quad \text{for } 0<|\alpha|<\infty,$$
$$P(-\lambda, h(y,w)) = h(y,w).$$

The first fact follows from (2.2). The second fact follows because $W(a,\lambda)$ is holomorphic in ζ and vanishes at $\zeta=-\lambda$, and the third fact follows because $h(y,w)$ is a constant (with respect to $\zeta\in D$). We also keep in mind that $W_j\equiv 0$ for $2\le j\le m$. Therefore, for $j=1,2$,

(3.6)
$$U^j(a,\lambda,y,w,z=-\lambda)(\zeta=-\lambda)$$
$$= h^j(y,w,\bar w) + \sum_{q=1}^{l_j-1}\frac{1}{q!(l_j-q)!}\frac{\partial^{l_j}h(y,w,\bar w)}{\partial w_1^q\partial\bar w_1^{l_j-q}}P(-\lambda, W_1^q\bar W_1^{l_j-q}) + P(-\lambda, E_j),$$

where

(3.7)
$$E_j(W_1,\bar W_1, v_1, v_2, y, w, \bar w)$$
$$= \sum_{2\le p_1+p_2<l_j}\frac{1}{p_1!p_2!}\frac{\partial^{p_1+p_2}h(y,w,\bar w)}{\partial w_1^{p_1}\partial\bar w_1^{p_2}}W_1^{p_1}\bar W_1^{p_2}$$
$$+ \sum_{\substack{2\le p_1+p_2+p_3+p_4\le l_j\\ 0<p_1+p_2\le l_j}}\frac{1}{p_1!p_2!p_3!p_4!}\frac{\partial^{p_1+p_2+p_3+p_4}h(y,w,\bar w)}{\partial w_1^{p_1}\partial\bar w_1^{p_2}\partial y_1^{p_3}\partial y_2^{p_4}}W_1^{p_1}\bar W_1^{p_2}v_1^{p_3}v_2^{p_4}$$
$$+ R_{l_j+1}(W_1,\bar W_1, v_1, v_2, y, w, \bar w),$$

where R_{l_j+1} is the Taylor remainder of order l_j+1 in $W_1, \bar{W}_1, v_1, v_2$. Note that we have organized the terms in this expansion as follows. The sum on the right of (3.6) contains all the terms of order l_j in W_1 and $\bar{W}_1$. The first sum on the right of (3.7) contains all the monomials of order less than l_j in W_1 and $\bar{W}_1$. The second sum on the right of (3.7) contains all terms of order less than or equal to l_j which also contain at least one v_1 or v_2.

In the next lemma, we shall estimate the error term $P(-\lambda, E_j)$ and then we shall explicitly compute the first sum appearing on the right side of (3.6).

LEMMA 3.8. *There exist constants $C>0$ and $r_0>0$ such that for $0<r<r_0$ and $(a,\lambda)\in T_r$ and $|y|<r$, $|w|<r$, $z\in D$, we have*

$$\text{(3.8a)} \quad |P(-\lambda, E_j)| \le C\left((|y|+|w|)\left(\frac{|a|^2}{1-\lambda^2}\right)+\frac{|a|^{l_j+1}}{(1-\lambda^2)^{l_j}}\right) \quad \text{for } j=1,2.$$

Proof. We first show the following estimates for $j=1,2$:

$$\text{(3.9a)} \qquad |W(a,\lambda)(\zeta)| \le \frac{|a|}{1-\lambda^2}\,|\zeta+\lambda|,$$

$$\text{(3.9b)} \qquad |v_j(a,\lambda,y,w,z=-\lambda)(\zeta)| \le \left[(|y|+|w|)\left(\frac{|a|}{1-\lambda}\right)+\left(\frac{|a|}{1-\lambda}\right)^{l_j}\right]|\zeta+\lambda|.$$

The first estimate is clear from the definition of W. The second estimate is proved as follows. For $|\zeta|=1$,

$$\begin{aligned}
&|v_j(a,\lambda,y,w,z=-\lambda)(\zeta)| \\
&\quad = |v_j(a,\lambda,y,w,z=-\lambda)(\zeta)-V_j(a,\lambda,y,w,z=-\lambda)(\zeta=-\lambda)| \quad \text{by (2.2)} \\
&\quad \le \|V_j\|_{C^{1,0}(\bar{D})}\,|\zeta+\lambda| \\
&\quad \le C\left[(|y|+|w|)\left(\frac{|a|}{1-\lambda}\right)+\left(\frac{|a|}{1-\lambda}\right)^{l_j}\right)|\zeta+\lambda|,
\end{aligned}$$

where the last inequality uses (3.4b). We have also used that $1/(1-\lambda^2)\approx 1/(1-\lambda)$ for $(a,\lambda)\in T_r$.

Now in the expansion given in (3.7) we assign weight 1 to W_1 and $\bar{W}_1$ and weight l_j to v_j ($j=1,2$). The weight of a monomial in $W_1, \bar{W}_1, v_1, v_2$ is defined to be the minimum weight of the monomials appearing in the Taylor expansion of E in $W_1, \bar{W}_1, v_1, v_2$ about $W_1=0$, $v_1=v_2=0$. We are interested in determining which coefficients of the monomials in the expansion in (3.7) vanish at $y=0$, $w=0$. In view of the normal form for semirigid submanifolds given in (3.1) at the beginning of this section, the coefficients of all monomials of weight less than l_j in $W_1, \bar{W}_1, v_1, v_2$ must vanish at $y=0$, $w=0$. Furthermore, the coefficients of all monomials of weight equal to l_j which involve at least one v_1 or v_2 must also vanish at $y=0$, $w=0$. Therefore, using (3.7) and (3.4a, b), we obtain

$$|E_j| \le C\left[(|y|+|w|)\frac{|a|^2}{(1-\lambda)^2}+\left(\frac{|a|}{1-\lambda}\right)^{l_j+1}\right]|\zeta+\lambda|^2.$$

Finally, we have

$$|P(-\lambda, E_j)| = \frac{1}{2\pi} \int_{|\zeta|=1} \frac{1-\lambda^2}{|\zeta+\lambda|^2} |E_j| \, |d\zeta|$$
$$\leq C\left[(|y|+|w|)\frac{|a|^2}{1-\lambda} + \frac{|a|^{l_j+1}}{(1-\lambda)^{l_j}}\right]$$

as desired. □

We also need to compute the second term on the right side of (3.6).

LEMMA 3.10. *For p, m positive integers, we have*

$$P(-\lambda, W_1^p \bar{W}_1^m) = \binom{p+m-2}{p-1} \frac{a^p \bar{a}^m}{(1-\lambda^2)^{p+m-1}} + E(a, \lambda),$$

where for some positive C which is independent of a and λ,

$$|E(a, \lambda)| \leq C \frac{|a|^{p+m}(1-\lambda)}{(1-\lambda^2)^{p+m-1}}.$$

Proof. We compute

$$P(-\lambda, W_1^p \bar{W}_1^m) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{a^p \bar{a}^m (\zeta+\lambda)^{p-1}(\bar{\zeta}+\lambda)^{m-1}}{(1-\lambda^2)^{p+m-1}} \cdot \frac{d\zeta}{\zeta}$$
$$= \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{a^p \bar{a}^m (\zeta+\lambda)^{p-1}(1+\lambda\zeta)^{m-1}}{(1-\lambda^2)^{p+m-1}\zeta^m} \, d\zeta \quad (\text{since } \bar{\zeta} = 1/\zeta).$$

By setting $\lambda = 1$ in the numerator of the integrand, we obtain

$$P(-\lambda, W_1^p \bar{W}_1^m) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{a^p \bar{a}^m (\zeta+1)^{p+m-2}}{(1-\lambda^2)^{p+m-1}\zeta^m} \, d\zeta + E(a, \lambda),$$

where $E(a, \lambda)$ clearly has the estimate stated in the lemma. The last integral can be evaluated using residue theory. □

REMARK. Semirigidity is essential for the previous two lemmas because it guarantees that the monomials of weight l_j in the expansion of U do not involve v_1 or v_2, and therefore their Poisson integrals can be easily computed.

Now let us require $a \geq 0$. We apply Lemma 3.10 with $p = q$ and $m = l_j - q$, and we see that the second term on the right side of (3.6) equals the left side of (3.2) multiplied by the expression $a^{l_j}/(1-\lambda^2)^{l_j-1}$ plus an error term (which arises from the fact that the derivatives of h^k in (3.2) are evaluated at 0 instead of $(y, w, \bar{w})$). We may summarize this as follows. Using the definition of f, (3.6), Lemma 3.10, Lemma 3.8, and (3.2), we obtain

$$f(a, \lambda, y, w) = \begin{pmatrix} h(y, w, \bar{w}) + iy \\ w \end{pmatrix} + \begin{pmatrix} \chi(a, \lambda, y, w) \\ 0 \end{pmatrix} \in C^2 \times C^m, \tag{3.11}$$

where $\chi = (\chi^1, \chi^2)$ and

$$\chi^j(a, \lambda, y, w) = \frac{a^{l_j}}{(1-\lambda^2)^{l_j-1}} + E_j(a, \lambda, y, w) \tag{3.12}$$

with

$$|E^j(a,\lambda,y,w)| \le C\left[(|y|+|w|)\left(\frac{|a|^2}{1-\lambda^2}\right)+\left(\frac{|a|}{1-\lambda}+(1-\lambda)\right)\frac{|a|^{l_j}}{(1-\lambda^2)^{l_j-1}}\right]$$

for $j=1,2$ and for $(a,\lambda)\in T_r$, $r<1$.

Now we further restrict our parameters. For $0<r<r_0$ let

$$\tilde{T}_r := \left\{(a,\lambda);\ a\ge 0,\ 0\le\lambda\le 1,\ \frac{r}{2}\le\frac{a}{1-\lambda}\le r,\ 1-\lambda\le r_0\right\}.$$

We shall also require $\max\{|y|,|w|\}<r_0 r^{l_2-1}$. With these restrictions we obtain, for $j=1,2$,

$$(3.13)\qquad |E^j(a,\lambda,y,w)|\le Cr_0\frac{a^{l_j}}{(1-\lambda^2)^{l_j-1}}\quad\text{for } (a,\lambda)\in\tilde{T}_r,\ r<r_0,$$

where C is some uniform positive constant.

The map

$$\phi: \{\max\{|y|,|w|\}<r_0 r^{l_2-1}\}\to C^n,$$

$$\phi(y,w)=(h(y,w,\bar{w})+iy,w)$$

appearing on the right side of (3.11) parameterizes a piece of M which contains $\{M\cap B(0,r_0 r^{l_2-1})\}$. Therefore, to prove Theorem 1.4, it suffices to show that for fixed y, w with $\max\{|y|,|w|\}<r_0 r^{l_2-1}$ the image of the map

$$(a,\lambda)\to(\chi_1(a,\lambda,y,w),\chi_2(a,\lambda,y,w))\in R^2\quad\text{for } (a,\lambda)\in\tilde{T}_r$$

contains a wedge $W(v, r_0' r^{l_1}, r_0' r^{l_2})$ for some unit vector $v\in R^2$ and an appropriately chosen $r_0'>0$.

For $0<t<r_0 r^{l_1}/2^{2l_1-1}$ define

$$S_t := \left\{(a,\lambda)\in\tilde{T}_r;\ a\ge 0,\ \frac{a^{l_1}}{(1-\lambda^2)^{l_1-1}}=t,\ \frac{r}{2}\le\frac{a}{1-\lambda}\le r\right\}.$$

We shall examine the image of $\chi=(\chi_1,\chi_2)$ on the connected path S_t and then let t vary from 0 to $r_0 r^{l_1}/2^{2l_1-1}$.

If $0<t<r_0 r^{l_1}/2^{2l_1-1}$ and if $(a,\lambda)\in S_t$ then one easily sees that $1-\lambda\le r_0$ and so $S_t\subset\tilde{T}_r$. In view of (3.12) and (3.13) with $j=1$, we have

$$(3.14)\qquad t\left(\frac{15}{16}\right)^{l_2-l_1}\le\chi_1(a,\lambda,y,w)\le t\left(\frac{17}{16}\right)^{l_2-l_1}\quad\text{for } (a,\lambda)\in S_t,$$

provided r_0 is chosen suitably small. We also note from (3.12) that

$$\chi_2(a,\lambda,y,w)=t\left(\frac{a}{1-\lambda}\right)^{l_2-l_1}\left(\frac{1}{1+\lambda}\right)^{l_2-l_1}+E_2\quad\text{for } (a,\lambda)\in S_t.$$

Now we use the fact that on S_t the expression $a/(1-\lambda)$ varies between $r/2$ and r and $1/2\le 1/(1+\lambda)\le 1/(2-r_0)$. Thus, χ "almost" varies between $t(r/4)^{l_2-l_1}$ and $t(r/2)^{l_2-l_1}$. More precisely, we may use (3.13) with $j=2$ to obtain

(3.15a) $$\min_{(a,\lambda)\in S_t} \chi_2(a,\lambda) \le t\left(\frac{5r}{16}\right)^{l_2-l_1},$$

(3.15b) $$\max_{(a,\lambda)\in S_t} \chi_2(a,\lambda) \ge t\left(\frac{7r}{16}\right)^{l_2-l_1},$$

provided r_0 is chosen suitably small. Therefore using (3.15a) and (3.15b) and (3.14), we have

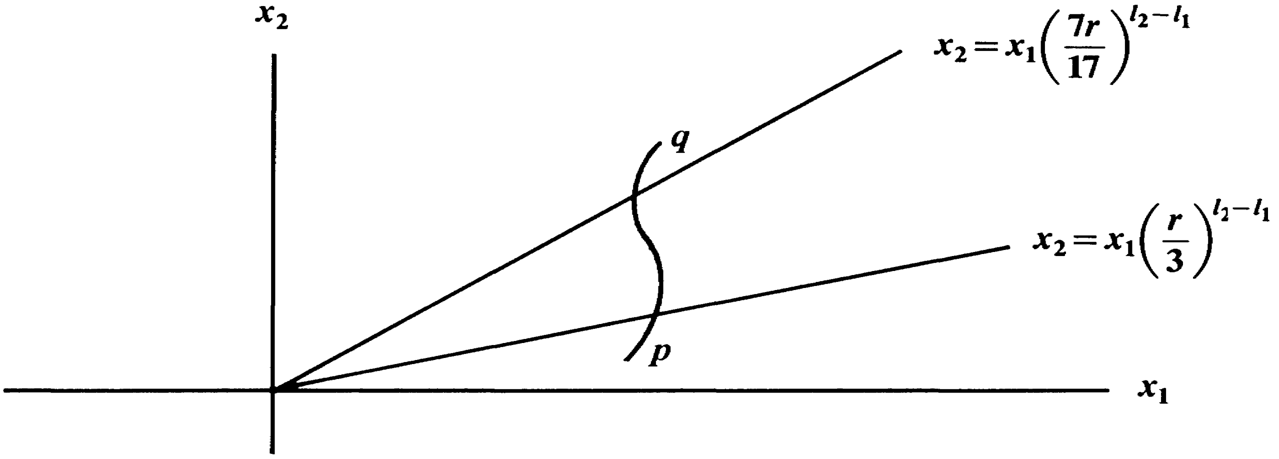
$$\min_{(a,\lambda)\in S_t}\left[\frac{\chi_2(a,\lambda)}{\chi_1(a,\lambda)}\right] \le \left(\frac{r}{3}\right)^{l_2-l_1}$$

$$\max_{(a,\lambda)\in S_t}\left[\frac{\chi_2(a,\lambda)}{\chi_1(a,\lambda)}\right] \ge \left(\frac{7r}{17}\right)^{l_2-l_1}.$$

In addition, S_t is connected and compact and so the image of the map

$$(a,\lambda)\to(\chi_1(a,\lambda,y,w),\chi_2(a,\lambda,y,w))\in R^2 \quad \text{for } (a,\lambda)\in S_t$$

is a connected path with endpoints $p=(\xi_1,\eta_1)$ and $q=(\xi_2,\eta_2)$ such that p lies below the line $x_2=x_1(r/3)^{l_2-l_1}$ and q lies above the line $x_2=x_1(7r/17)^{l_2-l_1}$.



Clearly, as t ranges from 0 to $r_0r^{l_1}/2^{2l_1-1}$, the path from p to q (the image of S_t) sweeps out the wedge

$$\left\{(x_1,x_2)\in R^2;\ 0\le x_1\le \frac{r_0r^{l_1}}{2^{2l_1-1}}\left(\frac{15}{16}\right)^{l_2-l_1},\ x_1\left(\frac{r}{3}\right)^{l_2-l_1}\le x_2\le x_1\left(\frac{7r}{17}\right)^{l_2-l_1}\right\}.$$

Clearly, this wedge contains the wedge $W(v, r_0'r^{l_1}, r_0'r^{l_2})$ for an appropriately chosen r_0' and unit vector $v\in R^2$.

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