DENSE ORBITS ON THE INTERVAL

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Introduction. Let I be a closed interval, and $f: I \to I$ be continuous. Associated with f is the inverse limit space $(I,f) = \{(x_0, x_1, \dots) \mid x_i \in I, \text{ and } f(x_{i+1}) = x_i\}$, and the induced homeomorphism $\hat{f}: (I,f) \to (I,f)$, given by $\hat{f}(x_0, x_1, \dots) = (f(x_0), x_0, x_1, \dots)$. In [2] it is shown that (I,f) can be topologically realized as a global attractor for a homeomorphism of Euclidean space. Indeed, it seems likely that such objects as the "strange attractor" of Henon [6] can be described as inverse limits.

In this paper we explore the relationship between the dynamics of f on I, and $\hat{f}$ on (I,f), with an emphasis on the existence of a dense orbit, and its consequences. It is the existence of a dense orbit on (I,f) which makes the attractor "visible" under the computer generated iteration of a randomly chosen point. For particular examples of (I,f) the reader is referred to [3], [4], and [10].

Suppose now that $f: I \to I$ has a dense orbit; that is, there is an x such that $\{f^n(x) \mid n \geq 0\}$ is dense in I. There are really two distinguishing cases. See [3, Lemma 2].

Case 1: $\{f^{2n}(x) \mid n \geq 0\}$ is dense in I. This is the most natural case, and the one which we will study in detail in this paper.

Case 2: $\{f^{2n}(x) \mid n \geq 0\}$ is not dense in I. See [3, Example 3]. In this case, the interval I splits into two subintervals J and K such that $I = J \cup K$, $J \cap K = \{\text{pt}\}$, $f(J) = K$, and $f(K) = J$. Letting $g = f^2 \mid J$, we have by [3, Lemma 2] that g^2 has a dense orbit, and so we are back in Case 1 for $g: J \to J$. As a consequence of this we adopt the more natural hypothesis that f^2 has a dense orbit.

Definitions and terminology. If a and b are distinct real numbers, we will let $[a,b]$ denote the smallest closed interval containing both a and b, and let (a,b) denote the associated open interval. We will generically let I be a closed interval and will be considering continuous functions $f: I \to I$. All of the functions which we consider are continuous.

If $f: I \to I$, and n is a nonnegative integer, then $f^n: I \to I$ is the n-fold composition of f with itself. If $f: I \to I$, and $x \in I$, then the *orbit* of x under f is $\{f^n(x) \mid n \geq 0\}$. The orbit of x will be denoted $O(x)$. The statement that x *has period* k means that k is a positive integer, $f^k(x) = x$, and if j is an integer $1 \leq j < k$, then $f^j(x) \neq x$. The statement that f *has a dense orbit* means that there is a point $y \in I$ such that $O(y)$ is dense in I.

Associated with $f: I \to I$ is the compact, connected metric space $(I,f) = \{(x_0, x_1, \dots) \mid f(x_i) = x_{i-1}\}$ with metric

$$d((x_0, x_1, \dots), (y_0, y_1, \dots)) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}.$$

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(I, f) is an example of what Bing [4] has called a *snake-like continuum*. The reason for this terminology is that for each $\epsilon > 0$, there is a finite open covering $\{g_1, g_2, \dots, g_n\}$ of (I, f) such that (i) $\operatorname{diam}(g_i) < \epsilon$, and (ii) $g_i \cap g_j \neq \varnothing$ if and only if $|i-j| \leq 1$. We will denote elements of (I, f) by subbarred letters as $\underline{x} = (x_0, x_1, \dots)$. The *projection maps* Π_n, of (I, f) onto I, given by $\Pi_n(\underline{x}) = x_n$, are continuous. If H is a *subcontinuum* (compact, connected subspace) of (I, f) we will let H_n denote $\Pi_n(H)$. Note that H_n is a closed interval or point and that $f(H_{n+1}) = H_n$.

If $f: I \to I$, then f *induces a homeomorphism* $\hat{f}: (I, f) \to (I, f)$ by

$$\hat{f}((x_0, x_1, \dots)) = (f(x_0), x_0, x_1, \dots).$$

Notice that $f \circ \Pi_n = \Pi_n \circ \hat{f}$, $\Pi_n = \Pi_{n+1} \circ \hat{f}$, and $f \circ \Pi_{n+1} = \Pi_n$.

If S is a snakelike continuum, then the intersection of any collection of subcontinua of S is a subcontinuum of S. See [4].

If S is a continuum, the statement that S is *indecomposable* means that S is not the union of two of its proper subcontinua. If S is a continuum, and $p \in S$, then the *composant of S containing p* is the union of all proper subcontinua of S which contain p. The continuum S is indecomposable if and only if S has uncountably many composants and they are mutually disjoint. See [7, pp. 139–141].

If S is a continuum, then a set $A \subset S$ is *residual* if and only if A is dense in S and is the intersection of countably many open sets in S.

We will utilize the following lemma from [1].

LEMMA 0. *Suppose that X is a compact metric space and that $f: X \to X$ is continuous. Then f has a dense orbit if and only if for each nonempty open set $U \subset X$, $\bigcup_{n=1}^{\infty} f^{-n}(U)$ is dense in X.*

Proof. See [1, Lemma 3]. □

In all that follows, I is a closed interval, and $f: I \to I$ is continuous.

LEMMA 1. *Suppose f has a dense orbit. Then there is a residual set C in (I, f) such that if $\underline{x} \in C$, then both $\{\hat{f}^n(\underline{x}) \mid n \geq 0\}$ and $\{\hat{f}^{-n}(\underline{x}) \mid n \geq 0\}$ are dense in (I, f).*

Proof. Let x be a point of I whose orbit is dense under f. Let $x_1, x_2, \dots$ be chosen so that $f(x_1) = x$ and if $i \geq 1$, $f(x_{i+1}) = x_i$. Let $\underline{x} = (x, x_1, x_2, \dots) \in (I, f)$. Then $\hat{f}^n(\underline{x}) = (f^n(x), f^{n-1}(x), \dots, f(x), x, x_1, \dots)$ and it is clear that $\{\hat{f}^n(x) \mid n \geq 0\}$ is dense in (I, f). It follows from [1] that there is a residual set $A \subset (I, f)$ such that if $\underline{y} \in A$ then $\{\hat{f}^n(\underline{y}) \mid n \geq 0\}$ is dense in (I, f). □

Since the homeomorphism $\hat{f}$ has a dense orbit, so does $\hat{f}^{-1}$. Thus, again by [1], there is a residual set $B \subseteq (I, f)$ such that if $\underline{y} \in B$ then $\{\hat{f}^{-n}(\underline{y}) \mid n \geq 0\}$ is dense in (I, f). Let $C = A \cap B$.

LEMMA 2. *Suppose that $x \in I$, and that $\{f^{2n}(x) \mid n \geq 0\}$ is dense in I. Then if j and k are integers, $j \geq 0$, $k \geq 1$, the set $\{f^{kn+j}(x) \mid n \geq 0\}$ is dense in I.*

Proof. This is a portion of Lemma 2 of [3]. □

NOTATION. If X is a compact metric space with a given metric, we let $D(X)$ be the diameter of X.

THEOREM 3. *Suppose that f^2 has a dense orbit. Let H be a proper subcontinuum of (I, f). Then* $\lim_{n\to\infty} D(\hat{f}^{-n}(H))=0$.

Proof. Let H be a proper subcontinuum of (I, f) and for each integer n, $n\geq 0$, let $H_n=\Pi_n(H)$ be the projection of H onto the nth coordinate space. It is clearly enough to show that $\lim_{n\to\infty} D(H_n)=0$.

Suppose to the contrary that $\lim_{n\to\infty} D(H_n)\neq 0$. Then there is a positive number ϵ and an increasing sequence $n_1, n_2, \ldots$ of integers such that for each i, $D(H_{n_i})>\epsilon$. Now, there is a closed interval J which is contained in infinitely many of the intervals $H_{n_1}, H_{n_2}, \ldots$. That is, there is a closed interval J and an increasing sequence $m_1, m_2, \ldots$ such that for each i, $J\subset H_{m_i}$. Now since the periodic points of f are dense ([8] or [3, Corollary to Lemma 2]), there are distinct periodic points p and q in J. We suppose that the period of p is r_1 and the period of q is r_2.

We will next show that H contains two distinct periodic orbits. Now, since at least one of the numbers $(m_i-m_1) \bmod r_1$ must be repeated infinitely many times, there is a subsequence $t_1, t_2, \ldots$ of $m_1, m_2, \ldots$ such that for each i and j, $(t_i-t_j)=0 \bmod r_1$. We now construct, in (I, f) the point

$$\underline{p}=(\ldots, p_{t_1}, \ldots, p_{t_2}, \ldots, p_{t_3}, \ldots)$$

where $p_{t_i}=p$. Then $\underline{p}\in(I, f)$, $\hat{f}^{r_1}(\underline{p})=\underline{p}$, and in fact $\underline{p}\in H$ since $\Pi_n(\underline{p})\in H$ for infinitely many values of n. Similarly, we construct $\underline{q}\in H$, with $\hat{f}^{r_2}(\underline{q})=\underline{q}$, and $\underline{q}\neq\underline{p}$.

Now let K be the intersection of all subcontinua of (I, f) which contain both $\underline{p}$ and $\underline{q}$. Because (I, f) is a snakelike continuum [4], K is a subcontinuum of (I, f) and $K\subset H$. Moreover, $\hat{f}^{r_1r_2}(K)=K$. This is because $\underline{p}$ and $\underline{q}$ belong to both $\hat{f}^{r_1r_2}(K)$ and $\hat{f}^{-r_1r_2}(K)$.

Now, since K is proper, there is an integer n such that $\Pi_n(K)=K_n$ is a proper subinterval of I. Using the fact that $f^{r_1r_2}\circ\Pi_n=\Pi_n\circ\hat{f}^{r_1r_2}$, we have that $f^{r_1r_2}(K_n)=f^{r_1r_2}(\Pi_n(K))=\Pi_n(\hat{f}^{r_1r_2}(K))=\Pi_n(K)=K_n$. Thus K_n is a proper subinterval of I which is invariant under $f^{r_1r_2}$. This is impossible since it follows from [3, Lemma 2] that there is a point x such that $\{f^{nr_1r_2}(x)\mid n\geq 0\}$ is dense in I. This establishes Theorem 3. □

THEOREM 4. *Suppose that f^2 has a dense orbit. Suppose that C is a composant of (I, f), k is a positive integer, and $\hat{f}^k(C)=C$. Then there is a unique point $\underline{p}\in C$ such that $\hat{f}^k(\underline{p})=\underline{p}$, and if $\underline{x}\in C$ then $\{\hat{f}^{-kn}(\underline{x})\mid n>0\}$ converges to $\underline{p}$.*

Proof. We first argue that there is a point $\underline{p}\in C$ such that $\hat{f}^k(\underline{p})=\underline{p}$. Let $\underline{y}\in C$. Then since $\hat{f}^k(C)=C$, we have that $\hat{f}^k(\underline{y})\in C$. Let H be a proper subcontinuum of (I, f) which contains both $\underline{y}$ and $\hat{f}^k(\underline{y})$. Let $L=\mathrm{cl}(\bigcup_{n=0}^{\infty}\hat{f}^{nk}(H))$. L is a subcontinuum of (I, f). We distinguish two cases.

Case 1: Assume that $\bigcup_{n=0}^{\infty}\hat{f}^{-nk}(H)$ is closed. Then $L\subset C$, and hence L is proper. Furthermore, $\hat{f}^{-k}(L)\subset L$. Since L has the fixed point property [5], there is a point $\underline{p}\in L$ such that $\hat{f}^{-k}(\underline{p})=\underline{p}$. Then $\underline{p}=\hat{f}^k(\underline{p})$.

Case 2: There is a point $\underline{p} \in L - \Sigma_{n=0}^{\infty} \hat{f}^{-nk}(H)$. Then there is an increasing sequence $n_1, n_2, \ldots$ of integers and a sequence of points $\underline{x}_1, \underline{x}_2, \ldots$ such that $\underline{x}_i \in \hat{f}^{-n_ik}(H)$ and $\{\underline{x}_i \mid i \geq 0\} \to \underline{p}$. Now it follows from Theorem 3 that

$$\lim_{i \to \infty} D(\hat{f}^{-n_ik}(H)) = 0.$$

Since both $\hat{f}^{-n_ik}(\underline{y})$ and $\hat{f}^{-(n_i+1)k}(\underline{y})$ belong to $\hat{f}^{-n_ik}(H)$, we have that

$$(\hat{f}^{-n_ik}(\underline{y}) \mid i \geq 0\} \to \underline{p} \quad \text{and} \quad \{\hat{f}^{-(n_i+1)k}(\underline{y}) \mid i \geq 0\} \to \underline{p}.$$

From this it follows that $\hat{f}^{-k}(\underline{p}) = \underline{p}$, and hence that $\hat{f}^{k}(\underline{p}) = \underline{p}$. It remains to be shown that $\underline{p} \in C$.

If $\underline{p} \notin C$, then L is not a proper subcontinuum of (I, f) and hence $L = (I, f)$. The previous argument shows that if $\underline{z} \in L - \bigcup_{n=0}^{\infty} \hat{f}^{-kn}(H)$, then $\hat{f}^{k}(\underline{z}) = \underline{z}$. Since $(I, f) - C$ is dense in (I, f), it follows that $\hat{f}^{k}$ is the identity, and hence that f^k is the identity. Then every point of I is periodic, and this contradicts the fact that f^2 has a dense orbit. Thus $\underline{p} \in C$.

The uniqueness of $\underline{p}$, and the fact that if $\underline{x} \in C$, then $\{\hat{f}^{-kn}(\underline{x}) \mid n \geq 0\} \to \underline{p}$, both follow immediately from Theorem 3. This establishes Theorem 4. □

NOTATION. *If* $\underline{y} \in (I, f)$ we let $C(\underline{y})$ denote the composant of (I, f) which contains $\underline{y}$. In other words, $C(\underline{y})$ is the union of all proper subcontinua which contain $\underline{y}$.

COROLLARY 5. *Suppose that f^2 has a dense orbit. Let $\underline{z}$ be a point of (I, f) such that $\{\hat{f}^{-n}(\underline{z}) \mid n \geq 0\}$ is dense in (I, f). Then if $\underline{y} \in C(\underline{z})$, $\{\hat{f}^{-n}(\underline{y}) \mid n \geq 0\}$ is dense in (I, f).*

Proof. Let $\underline{x} \in (I, f)$ and let $\epsilon > 0$. Let H be a proper subcontinuum of (I, f) which contains $\underline{y}$ and $\underline{z}$. Now, by Theorem 3 and the fact that $\{\hat{f}^{-n}(\underline{z}) \mid n \geq 0\}$ is dense, there is a positive integer j such that (i) $d(\hat{f}^{-j}(\underline{z}), \underline{x}) < \epsilon/2$, and (ii) $D(\hat{f}^{-j}(H)) < \epsilon/2$. Then $d(\hat{f}^{-j}(\underline{y}), \underline{x}) < \epsilon$, and so $\{\hat{f}^{-n}(\underline{y}) \mid \geq 0\}$ is dense in (I, f). □

THEOREM 6. *f^2 has a dense orbit if and only if, for each subinterval J of I and each pair $c, d \in \operatorname{int} I$, there is an integer N such that if $n > N$ then $[c, d] \subset f^n(J)$.*

Proof. We first assume that f^2 has a dense orbit. Let J be a subinterval of I. Since the periodic points of f are dense ([8] or [3, Corollary to Lemma 2]), there is a periodic point $p \in \operatorname{int} J$. Suppose that p has period k. Let $g = f^k$. Let $L = \operatorname{cl}(\bigcup_{n=0}^{\infty} g^n(J))$. Then L is a closed interval. Now let $y \in \operatorname{int} J$ such that $\{f^{2n}(y) \mid n \geq 0\}$ is dense in I. Then from Lemma 2 it follows that $\{g^n(y) \mid n \geq 0\}$ is dense in I. Hence $L = I$.

We will next show that if x is a periodic point such that $O(x) \subset \operatorname{int} I$, then there is an integer M such that $O(x) \subset f^M(J)$. To this end, suppose that x is periodic, the period of x is t, and $O(x) \subset \operatorname{int} I$. Let x_1 and x_2 be, respectively, the smallest and largest elements in $O(x)$. We may assume that $x_1 \neq p$. Now, since $\bigcup_{n=0}^{\infty} (g^n(J)) = \operatorname{int} I$, there is an integer r such that $[x_1, p] \subset g^r(J)$. Let $h = g^r$

and notice that $h^t(x_1)=x_1$, $h^t(p)=p$. Now it follows from two applications of Lemma 2 that there is a point $z\in\operatorname{int}[x_1,p]$ such that $\{h^{tn}(z)\mid n\geq 0\}$ is dense in I. Therefore there is an integer m such that $h^{tm}(z)>x_2$. Then we have $h^{tm}(p)=p$, $h^{tm}(x_1)=x_1$, and $h^{tm}(z)>x_2$. It follows that $h^{tm}(J)\supset[x_1,x_2]\supset O(x)$. Thus, for $m=t\cdot m\cdot r\cdot k$ we have $f^M(J)\supset O(x)$. Then if $m\geq M$, $f^m(J)\supset O(x)$.

Now suppose that $c,d\in\operatorname{int} I$ and that $c<d$. Let α and β be periodic points such that (i) $[c,d]\subset[\alpha,\beta]$, and (ii) $O(\alpha)\cup O(\beta)\subset\operatorname{int} I$. Then there are positive integers M_1 and M_2 such that $O(\alpha)\subset f^{M_1}(J)$ and $O(\beta)\subset f^{M_2}(J)$. Let $N=\max\{M_1,M_2\}$. Then if $n>N$, $[c,d]\subset[\alpha,\beta]\subset f^n(J)$. This concludes the first half of the argument.

Next, suppose that for each subinterval J of I, and for each pair $c,d\in\operatorname{int} I$, there is a positive integer N such that if $n>N$ then $[c,d]\subset f^n(J)$. We will first argue that f has a dense orbit. Let U be an open interval in I, and let $x\in U$. We will show that $\bigcup_{n=0}^{\infty} f^{-n}(x)$ is dense in I. If not, there is a closed subinterval J of I such that $J\cap(\bigcup_{n=0}^{\infty} f^{-n}(x))=\varnothing$. But by the condition, there is a positive integer N such that $x\in f^N(J)$. Hence, there is a point $y\in J$ such that $f^N(y)=x$. Then $y\in J\cap(\bigcup_{n=0}^{\infty} f^{-n}(x))$. Therefore, $\bigcup_{n=0}^{\infty} f^{-n}(x)$ is dense in I, and so $\bigcup_{n=0}^{\infty} f^{-n}(U)$ is dense in I. It follows from [1] that f has a dense orbit.

Then there is a point $z\in I$ such that $\{f^n(z)\mid n\geq 0\}$ is dense in I. If $\{f^{2n}(z)\mid n\geq 0\}$ is not dense in I, then it follows from [3, Lemma 2] that there are closed intervals C_1 and C_2 in I such that $C_1\cup C_2=I$, $C_1\cap C_2=\{\text{pt}\}$, $f(C_1)=C_2$, and $f(C_2)=C_1$. Now let $c\in\operatorname{int} C_1$, $d\in\operatorname{int} C_2$ and let J be C_1. Then, for each positive integer n, $[c,d]\not\subset f^n(J)$. This is a contradiction, and hence f^2 has a dense orbit. This establishes Theorem 6. □

As an interesting corollary to the argument for Theorem 6, we have the following.

COROLLARY 7. *If f^2 has a dense orbit, and $x\in\operatorname{int} I$, then $\bigcup f^{-n}(x)$ is dense in I.*

THEOREM 8. *Suppose that f^2 has a dense orbit. Suppose that H is a nondegenerate subcontinuum of (I,f), and that $\epsilon>0$. Then there is a positive integer N such that if $n>N$ and $\underline{x}\in(I,f)$, then $d(\underline{x},\hat{f}^n(H))<\epsilon$. In particular, $\bigcup_{n=0}^{\infty}\hat{f}^n(H)$ is dense in (I,f).*

Proof. Let H be a nondegenerate subcontinuum of (I,f) and let $\epsilon>0$. Now there is a positive integer j and a $\delta>0$ such that if $\underline{x},\underline{y}\in(I,f)$ and $|x_j-y_j|<\delta$, then $d(\underline{x},\underline{y})<\epsilon$. Since H is nondegenerate there is an integer m such that $H_m=\Pi_m(H)$ is a closed interval. Since f has a dense orbit, the image of a closed interval is nondegenerate, and it follows that $H_0=\Pi_0(H)$ is nondegenerate.

Now it follows from Theorem 6 that there is a positive integer N, $N>j$, such that if $n>N$ and $x\in I$ then $d(x,f^{n-j}(H_0))<\delta$. Now, let $\underline{x}=(x_0,x_1,\dots)\in(I,f)$ and suppose that $n>N$. Then $\Pi_j(\hat{f}^n(H))=f^{n-j}(H_0)$ and then there is a point $\underline{y}=(y_0,y_1,\dots)\in f^{n-j}(H_0)$ such that $|x_j-y_j|<\delta$. It follows that $d(\underline{x},\underline{y})<\epsilon$, and hence that $d(\underline{x},\hat{f}^n(H))<\epsilon$. This establishes Theorem 8. □

THEOREM 9. *Suppose that f^2 has a dense orbit. Let A be an infinite subset of the positive integers. Then there is a residual set X in I such that if $x \in X$ then $\{f^j(x) \mid j \in A\}$ is dense in I.*

Proof. Let U be an open set in (I, f). Since every open set contains a non-degenerate subcontinuum, it follows from Theorem 8 that $\bigcup \{\hat{f}^j(U) \mid j \in A\}$ is dense in (I, f). Let $\{B_1, B_2, \ldots\}$ be a countable basis for the topology of (I, f), and for each positive integer i let $W_i = \bigcup \{\hat{f}^j(B_i) \mid j \in A\}$. Then W_i is an open dense set in (I, f). Let $Z = \bigcap_{i=1}^{\infty} W_i$. It follows from the Baire Category Theorem that Z is a residual set in (I, f). Now if $\underline{x} = (x_0, x_1, \ldots) \in Z$ then, for each i, $\underline{x} \in W_i$ and so there is an integer $j \in A$ such that $\hat{f}^{-j}(\underline{x}) \in B_i$. Thus, if $\underline{x} \in Z$, we have that $\{\hat{f}^{-j}(x) \mid j \in A\}$ is dense in (I, f).

Now, if U is open in (I, f), let $\underline{x} \in Z \cap U$. It follows that $\bigcup \{f^{-j}(U) \mid j \in A\}$ is open in (I, f). Using the same argument as before, there is a residual set Y in (I, f) such that if $\underline{y} \in Y$ then $\{\hat{f}^j(\underline{y}) \mid j \in A\}$ is dense in (I, f).

Now let $X = \Pi_0(Y)$. Then X is a residual in I, and if $x \in X$ then $\{f^j(x) \mid j \in A\}$ is dense in I. This establishes Theorem 9. □

DEFINITION. Suppose that y is a fixed point of f. The statement that *x is homoclinic to the fixed point y* means that $x \neq y$, and there is a choice of inverse images $f^{-1}(x), f^{-2}(x), \ldots$ such that both $f^n(x) \to y$ and $f^{-n}(x) \to y$. If y is a periodic point of f with period s, then the statement that *x is homoclinic to y* means that x is homoclinic to the fixed point y under f^s.

THEOREM 10. *Suppose that f^2 has a dense orbit. Let y be a point of* int I *which is periodic. Then there is a point x which is homoclinic to y.*

Proof. Suppose that y has period s. Let $g = f^s$. Then $g(y) = y$, and it follows that g^2 has a dense orbit.

Suppose now that $g^{-1}(y) \cap \operatorname{int} I = y$. Since $y \in \operatorname{int} I$ there are subintervals I_1 and I_2 of I such that $I_1 \cup I_2 = I$, $I_1 \cap I_2 = \{y\}$, $g^2(I_1) = I_1$, and $g^2(I_2) = I_2$. This is impossible since g^2 has a dense orbit and hence cannot have an invariant interval. Thus, there is a point $x \neq y$ such that $x \in \operatorname{int} I$ and $g(x) = y$.

In (I, g) let $\underline{y} = (y, y, y, \ldots)$ and let C be the composant of (I, f) containing $\underline{y}$. Since C is a dense, connected set in (I, f) ([7, pp. 139–141]), we have that int $I \subset \Pi_0(C)$. Then there is a point $\underline{x} = (x_0, x_1, \ldots)$ in C with $x = x_0$. From Theorem 3 we have that $\{\hat{g}^{-n}(\underline{x}) \mid n \geq 0\}$ converges to $\underline{y}$, and it follows that $\{x_n \mid n \geq 0\}$ converges to y. Then x is homoclinic to y. □

COROLLARY 11. *If f^2 has a dense orbit, then $\{x \mid$ there is a $y \in I$, and x is homoclinic to $y\}$ is dense in I.*

DEFINITION. If X is a compact metric space, $h: X \to X$ is a homeomorphism, and $x, y, z \in X$, then the statement that *z is heteroclinic from x to y* means that both $d(h^{-n}(z), h^{-n}(x)) \to 0$ and $d(h^n(z), h^n(y)) \to 0$.

THEOREM 12. *Suppose that f^2 has a dense orbit. Suppose that $\underline{x}, \underline{y} \in (I, f)$ and, for some m, $\Pi_m(\underline{y}) \in \operatorname{int} I$. Then there is a point $\underline{z} \in (I, f)$ which is heteroclinic from $\underline{x}$ to $\underline{y}$.*

Proof. Let C be the composant of $\underset{\sim}{x}$. Since int $I \subset \Pi_m(C)$, there is a point $\underset{\sim}{z} \in C$ such that $\Pi_m(\underset{\sim}{z}) = \Pi_m(\underset{\sim}{y})$. We then have that $d(\hat{f}^n(\underset{\sim}{z}), \hat{f}^n(\underset{\sim}{y})) \to 0$. Since $\underset{\sim}{z}$ and $\underset{\sim}{x}$ are in the same composant, it follows from Theorem 3 that

$$d(\hat{f}^{-n}(\underset{\sim}{z}), \hat{f}^{-n}(\underset{\sim}{x})) \to 0. \qquad \square$$

THEOREM 13. *If f^2 has a dense orbit, then f has a point of odd period.*

Proof. Let J and K be disjoint subintervals of I. It follows from Theorem 6 that there is a positive integer N such that if $n > N$ then $(J \cup K) \subset f^n(J) \cap f^n(K)$.

Let p be a positive integer which is prime and larger than $2N+2$. Let $r = (p-1)/2$, $s = (p+1)/2$. Then $r > N$, $s > N$, and $r + s = p$.

Now, since $K \subset f^r(J)$, there is a subinterval J_1 of J such that $f^r(J_1) = K$. Then $J_1 \subset f^s(K) = f^{r+s}(J_1) = f^p(J_1)$. Then there is a point $x \in J_1$ such that $f^p(x) = x$. Since $x \in J_1$, $f^r(x) \in K$, and $J \cup K = \varnothing$, it follows that $f(x) \neq x$. Since p is a prime we have that the period of x is p. This establishes Theorem 13. $\square$

Notice that the argument shows that every subinterval of I contains a point of odd period. Actually, it can be shown that if f^2 has a dense orbit, and J is a subinterval of I, then there is an integer N such that if $n > N$ then J contains a point whose period is n. We will deal with this, and related questions, elsewhere.

We conclude with some examples.

EXAMPLE 1. Let

$$f(t) = \begin{cases} 2t & \text{if } 0 \le t \le \frac{1}{2} \\ 2-2t & \text{if } \frac{1}{2} \le t \le 1 \end{cases}.$$

Using Theorem 6, it can be verified that f^2 has a dense orbit. In fact, if J is a subinterval and J contains $[k/2^m, (k+1)/2^m]$, then $f^m(J) = [0,1]$. A description of (I, f) can be found in [3].

EXAMPLE 2. Suppose that n is an odd positive integer, $n = 2k+1$. Define f: $[0,1] \to [0,1]$ by

$$f(0) = \frac{k}{2k}, \quad f\left(\frac{k}{2k}\right) = \frac{k+1}{2k}, \quad f\left(\frac{k+i}{2k}\right) = \frac{k-i}{2k}, \quad f\left(\frac{k-i}{2k}\right) = \frac{k+i+1}{2k},$$

for $1 \le i \le k-1$, and $f(2k/2k) = f(1) = 0$, and f is linear on the intervals complementary to these points. f is the standard example of a function having period n, but no smaller period in the Sarkovskii sequence. See [9]. It can be shown, using Theorem 6, that f^2 has a dense orbit.

EXAMPLE 3. In $[0,1]$ let

$$\cdots < p_{-2} < p_{-1} < p_0 < p_1 < p_2 < \cdots$$

be such that $\{p_n\} \to 1$ and $\{p_{-n}\} \to 0$. For each integer n let $I_n = [p_n, p_{n+1}]$. Define $f_n: I_n \to I_{n-1} \cup I_n \cup I_{n+1}$ by $f_n(p_n) = p_n$,

$$f_n(p_{n+1}) = p_{n+1}, \quad f_n\left(\frac{2p_n + p_{n+1}}{3}\right) = p_{n+2}, \quad f_n\left(\frac{p_n + 2p_{n+1}}{3}\right) = p_{n-1},$$

and f_n is linear on the intervals complementary to these points.

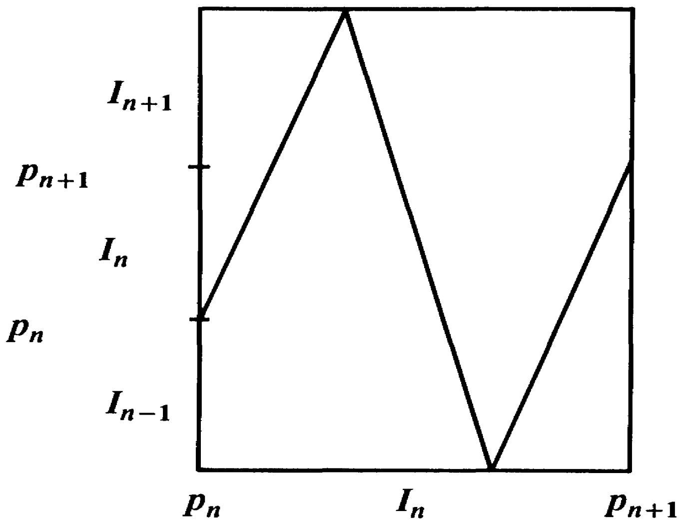

Figure 1

Now, define $f[0,1] \to [0,1]$ by $f(0)=0$, $f(1)=1$, and $f(x)=f_n(x)$ if $x \in I_n$.

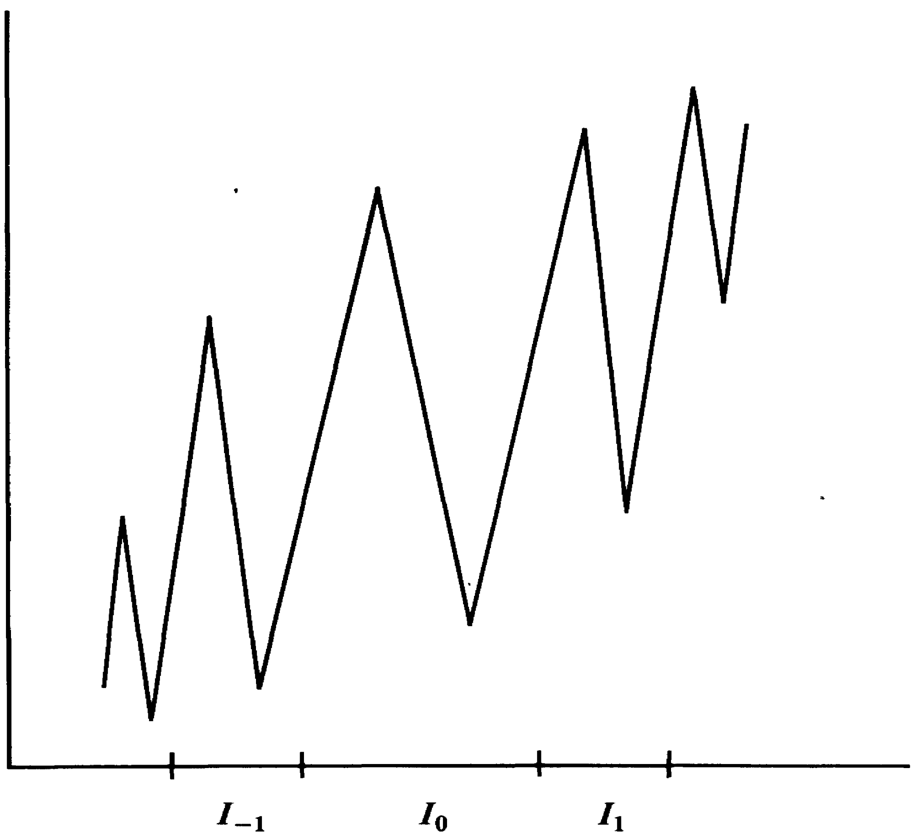

Figure 2

Using Theorem 6, it can be shown that f^2 has a dense orbit. Since $f^{-1}(0)=0$ and $f^{-1}(1)=1$, it can be seen that the hypothesis that $x \in \text{int } I$ is necessary in Corollary 7, Theorem 10, and Theorem 12.

REFERENCES

1. J. Auslander and J. Yorke, *Interval maps, factors of maps, and chaos,* Tôhoku Math. J. (2) 32 (1980), 177–188.
2. M. Barge and J. Martin, *The construction of global attractors,* preprint.
3. ———, *Chaos, periodicity, and snakelike continua,* Trans. Amer. Math. Soc. 289 (1985), 355–365.
4. R. H. Bing, *Snakelike continua,* Duke Math. J. (3) 18 (1951), 653–663.
5. O. H. Hamilton, *A fixed point theorem for pseudo-arcs and certain other metric continua,* Proc. Amer. Math. Soc. 2 (1951), 173–174.
6. M. Henon, *A two dimensional mapping with a strange attractor,* Comm. Math. Phys. 50 (1976), 69–77.
7. J. Hocking and G. Young, *Topology,* Addison-Wesley, Reading, Mass., 1961.
8. Z. Nitecki, *Topological dynamics on the interval.* Ergodic theory and dynamical systems, II (College Park, Md., 1979/80), 1–73, Birkhäuser, Boston, Mass., 1982.
9. P. Stephan, *A theorem of Sarkovskii on the coexistence of periodic orbits of continuous endomorphisms of the real line,* Comm. Math. Phys. 54 (1977), 237–248.
10. W. T. Watkins, *Homeomorphic classification of certain inverse limit spaces with open bonding maps,* Pacific J. Math. 103 (1982), 589-601.

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