THE BEST CONSTANT IN A BMO-INEQUALITY FOR THE BEURLING-AHLFORS TRANSFORM

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Introduction. The Beurling-Ahlfors operator Tf for $f \in L^2(\mathbf{C})$ is defined by the following relation between Fourier transforms:

$$(Tf)^\wedge(\xi) = m(\xi)\hat{f}(\xi),$$

where $m(\xi) = (\xi/|\xi|)^2$ for every $\xi \in \mathbf{C} - \{0\}$.

Clearly T is a unitary operator on $L^2(\mathbf{C})$ commuting with translations and dilations. Another definition of T is the convolution formula

$$(Tf)(z) = -\text{p.v.}\,\frac{1}{\pi}\int \frac{f(t)\,d\sigma(t)}{(z-t)^2},$$

where p.v. means the principal value and $d\sigma(t)$ the Lebesgue measure in $\mathbf{C}$. This operator can be regarded as an analogue of the Hilbert transform in the complex plane with an even kernel.

The importance of the Beurling-Ahlfors operator to the elliptic equations ([3], [4], [13]) as well as to quasiconformal mappings in the plane lies in the fact that it changes the complex derivative $\partial_{\bar{z}}$ into ∂_z: in symbols,

$$T\left(\frac{\partial w}{\partial \bar{z}}\right) = \frac{\partial w}{\partial z} \tag{1}$$

for every w in the Sobolev space $\mathcal{W}_2^1(\mathbf{C})$. We shall appeal to this formula to evaluate Tf for some particular functions f.

As an operator of Calderon-Zygmund type, the Beurling-Ahlfors transform is bounded in $L^p(\mathbf{C})$ for all $1 < p < \infty$. This breaks down for $p = \infty$. However, in this limiting case T extends to a bounded operator from $L^\infty(\mathbf{C})$ into BMO-spaces [12].

Fix $1 \le p < \infty$. A function $f \in L^p_{\text{loc}}(\mathbf{R}^n)$ is said to be of bounded mean oscillation (briefly, BMO_p) if

$$\|f\|_{\text{BMO}_p} = \sup_B \left(⨍_B \left| f(x) - ⨍_B f(y)\,dy \right|^p dx \right)^{1/p} < \infty,$$

where the supremum is taken over all balls B in $\mathbf{R}^n$ and

$$⨍_B f(y)\,dy = \frac{1}{|B|}\int_B f(y)\,dy = f_B$$

is the average of f on B.

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All BMO_p spaces for $1 \leq p < \infty$ are essentially equivalent; namely,

$$\|f\|_{\mathrm{BMO}_p} \leq C(p, q, n) \|f\|_{\mathrm{BMO}_q}$$

for $1 \leq p, q < \infty$. Actually, by Hölder's inequality,

$$\|f\|_{\mathrm{BMO}_p} \leq \|f\|_{\mathrm{BMO}_q} \tag{2}$$

whenever $1 \leq p \leq q < \infty$.

The purpose of this paper is to identify the $L^\infty \to \mathrm{BMO}_2$ norm of the Beurling-Ahlfors operator.

THEOREM 1. *We have*

$$\left(\fint_B |Tf(z) - (Tf)_B|^2 \, d\sigma(z) \right)^{1/2} \leq 3 \tag{3}$$

for every ball $B \subset \mathbf{C}$ and every function $f \in L^2(\mathbf{C})$ such that $|f(z)| \leq 1$, a.e. This inequality is sharp.

Interest in the BMO-inequalities is motivated by their relations with quasiconformal mappings. In this connection we mention the beautiful results of Reimann [11], who discovered that quasiconformal mappings are invariant transformations of independent variables for BMO-functions. We also refer to several deep results due to Jones [9] and Astala and Gehring [1], who characterized domains having the BMO-extension property.

1. Preliminaries. It clearly suffices to prove (3) when $\mathbf{B}$ is the unit disk in $\mathbf{C}$. From now on we shall assume that

$$\mathbf{B} = \{z \in \mathbf{C};\ |z| < 1\} \quad \text{and} \quad \Omega = \{t \in \mathbf{C};\ |t| \geq 1\}.$$

The following formulas are worth recording. For every integer $k \geq 0$ set

$$\rho_k(z) = \bar{z}^k \chi_{\mathbf{B}}(z). \tag{4}$$

Then

$$T\rho_k(z) = -z^{-k-2} \chi_\Omega(z). \tag{5}$$

In order to see these we apply (1) to the functions

$$w_k(z) = \frac{1}{k+1} [\bar{z}^{k+1} \chi_{\mathbf{B}}(z) + z^{-k-1} \chi_\Omega(z)].$$

Based on formula (1) we also derive that, for

$$\rho(z) = \left(\frac{z}{|z|} \right)^2 \chi_{\mathbf{B}}(z), \tag{6}$$

$$T\rho(z) = (1 + \log|z|^2) \chi_{\mathbf{B}}(z). \tag{7}$$

In this case (1) applies to $w(z) = (z \log|z|^2) \chi_{\mathbf{B}}(z)$. Formulas (4) and (5) immediately imply the following.

COROLLARY 1. *Let* $\mathcal{H}^2(\mathbf{B})$ *denote the space of functions square integrable and analytic in* $\mathbf{B}$. *Then, for every* $h \in \mathcal{H}^2(\mathbf{B})$,

$$T(\chi_{\mathbf{B}}\bar{h})(z) = 0 \tag{8}$$

for $z \in \mathbf{B}$.

2. Certain orthogonality properties of T. The proof of Theorem 1 depends on the orthogonality properties of T. We begin with the following fundamental identities:

$$\int f(z)Tg(z)\,d\sigma(z) = \int g(z)Tf(z)\,d\sigma(z), \tag{9}$$

$$\int Tf(z)\overline{Tg(z)}\,d\sigma(z) = \int f(z)\overline{g(z)}\,d\sigma(z), \tag{10}$$

for every $f, g \in L^2(\mathbf{C})$.

The sets $\mathbf{B}$ and Ω induce naturally an orthogonal decomposition of $L^2(\mathbf{C})$,

$$L^2(\mathbf{C}) = L^2(\mathbf{B}) \oplus L^2(\Omega),$$

obtained by decomposing any function f into the sum $f = \chi_{\mathbf{B}} f + \chi_\Omega f$. This decomposition is not invariant under the Beurling-Ahlfors transform. However, we have the following lemma.

LEMMA 1. *Let* $b \in L^2(\mathbf{B})$ *and* $\omega \in L^2(\Omega)$. *Then the functions* Tb *and* $T\omega$ *are orthogonal in* $L^2(\mathbf{B})$ *and in* $L^2(\Omega)$, *that is,*

$$\int_{\mathbf{B}} Tb(z)\overline{T\omega(z)}\,d\sigma(z) = 0; \tag{11}$$

also,

$$\int_{\Omega} Tb(z)\overline{T\omega(z)}\,d\sigma(z) = 0.$$

Moreover,

$$\int_{\mathbf{B}} Tb(z)\,d\sigma(z) = 0. \tag{12}$$

Proof. In view of identity (9), the integral in (11) takes the form

$$\int (Tb)\overline{(\chi_{\mathbf{B}} T\omega)} = \int bT(\chi_{\mathbf{B}}\overline{T\omega}) = \int_{\mathbf{B}} bT(\chi_{\mathbf{B}}\overline{T\omega}).$$

Observe that $T\omega \in \mathcal{H}^2(\mathbf{B})$ and by Corollary 1 the function $T(\chi_{\mathbf{B}}\overline{T\omega})$ vanishes on $\mathbf{B}$. Hence the equality (11) follows. The second equality follows from (11) together with the fact that T is a unitary operator on $L^2(\mathbf{C})$ and the supports of b and ω are disjoint.

Now replace $T\omega$ in (11) by $\chi_{\mathbf{B}}$, which is obviously in $\mathcal{H}^2(\mathbf{B})$, and repeat the same arguments to obtain (12). □

COROLLARY 2. *Let $f \in L^2(\mathbf{C})$. Then*

$$\fint_{\mathbf{B}} |Tf-(Tf)_{\mathbf{B}}|^2 = \fint_{\mathbf{B}} |Tb|^2 + \fint_{\mathbf{B}} |T\omega-(T\omega)_{\mathbf{B}}|^2, \tag{13}$$

where b and ω are $L^2(\mathbf{B})$ and $L^2(\Omega)$ components of f, respectively.

Notice that $|b| \le \chi_{\mathbf{B}}$ and $|\omega| \le \chi_{\Omega}$, whenever $|f| \le 1$.

Proof. Identity (13) is a simple consequence of (11) and (12). □

Corollary 2 leads naturally to two independent extremal problems. We begin by considering the simpler one.

3. The estimate in $L^2(\mathbf{B}) \cap L^\infty(\mathbf{C})$.

LEMMA 2. *Let b be measurable and such that $|b(z)| \le \chi_{\mathbf{B}}(z)$, a.e. Then*

$$\fint_{\mathbf{B}} |Tb(z)|^2 \, d\sigma(z) \le 1. \tag{14}$$

The inequality is sharp.

Proof. Inequality (14) is an immediate consequence of the fact that T is an isometry in $L^2(\mathbf{C})$:

$$\fint_{\mathbf{B}} |Tb|^2 \le \frac{1}{|\mathbf{B}|} \int |Tb|^2 = \frac{1}{|\mathbf{B}|} \int |b|^2 = \fint_{\mathbf{B}} |b|^2 \le 1.$$

This argument also shows how to achieve equality in (14). For this we require (i) $\chi_\Omega(z) Tb(z) = 0$ and (ii) $|b(z)| = \chi_{\mathbf{B}}(z)$. Certainly the function ρ defined by (6) satisfies these two requirements. This ends the proof of Lemma 2. □

4. The estimate in $L^2(\Omega) \cap L^\infty(\mathbf{C})$.

LEMMA 3. *Let $\omega \in L^2(\mathbf{C})$ be such that $|\omega(z)| \le \chi_\Omega(z)$, a.e. Then*

$$\fint_{\mathbf{B}} |T\omega(z)-(T\omega)_{\mathbf{B}}|^2 \, d\sigma(z) \le 8. \tag{15}$$

The inequality is sharp.

For the extremal function see the remark after the proof.

Proof. Clearly $\chi_{\mathbf{B}} T\omega \in \mathcal{H}^2(\mathbf{B})$, and for $z \in \mathbf{B}$ we have the following Taylor expansion:

$$T\omega(z) = -\frac{1}{\pi} \int_\Omega \frac{\omega(t)\, d\sigma(t)}{(z-t)^2} = -\frac{1}{\pi} \sum_{k=0}^{\infty} (k+1) z^k \int t^{-k-2} \omega(t)\, d\sigma(t)$$
$$= \sum_{k=0}^{\infty} (k+1) c_k z^k,$$

where

$$c_k = -\frac{1}{\pi} \int t^{-k-2} \omega(t)\, d\sigma(t) \tag{16}$$

for $k = 0, 1, 2, \ldots$. Since $c_0 = T\omega(0) = (T\omega)_{\mathbf{B}}$,

$$T\omega(z) - (T\omega)_{\mathbf{B}} = \sum_{k=1}^{\infty} (k+1) c_k z^k \tag{17}$$

for $z \in \mathbf{B}$. This last series is also well defined when we drop the assumption $\omega \in L^2(\mathbf{C})$, because of the following uniform estimate:

$$|c_k| \le \frac{1}{\pi} \int_{\Omega} |t|^{-k-2} \, d\sigma(t) \le \frac{1}{\pi} \int_{\Omega} |t|^{-3} \, d\sigma(t) = 2$$

for $k = 1, 2, \ldots$. Here we have

$$\int_{\Omega} |t|^{-3} \, d\sigma(t) = 2\pi \int_1^{\infty} r^{-2} \, dr = 2\pi. \tag{18}$$

Notice that

$$\int_{\mathbf{B}} |z|^{2k} \, d\sigma(z) = 2 \int_0^1 r^{2k+1} \, dr = \frac{1}{k+1}$$

for $k = 1, 2, 3, \ldots$. This, together with the mutual orthogonality of the functions $z^k \chi_{\mathbf{B}}(z)$ and (17), gives

$$\int_{\mathbf{B}} |T\omega(z) - (T\omega)_{\mathbf{B}}|^2 \, d\sigma(z) = \sum_{k=1}^{\infty} (k+1) |c_k|^2, \tag{19}$$

where c_k are defined by (16).

Our goal now is to maximize the above series subject to the conditions $|\omega(z)| \le \chi_{\Omega}(z)$, $\omega \in L^2(\mathbf{C})$. Variational arguments suggest consideration of the following auxilliary function, analytic in Ω:

$$\mathfrak{A}(t) = -\frac{1}{\pi} \sum_{k=1}^{\infty} (k+1) \overline{c_k} \, t^{-k-2}. \tag{20}$$

This function appears in the corresponding Lagrange-Euler equation. It turns out that the local maxima must take the form

$$\omega(t) = \chi_{\Omega}(t) \exp[-i \arg \mathfrak{A}(t)]$$

(cf. [5]). But, in order to avoid a delicate question on existence of the extremals, we do not exploit this extra information.

From the definition of c_k it follows that

$$\begin{aligned} \sum_{k=1}^{\infty} (k+1) |c_k|^2 &= \sum_{k=1}^{\infty} (k+1) \overline{c_k} \left[-\frac{1}{\pi} \int t^{-k-2} \omega(t) \, d\sigma(t) \right] \\ &= \int \omega(t) \mathfrak{A}(t) \, d\sigma(t) \le \int_{\Omega} |\mathfrak{A}(t)| \, d\sigma(t). \end{aligned} \tag{21}$$

Hence, by Hölder's inequality,

$$\sum_{k=1}^{\infty} (k+1) |c_k|^2 \le \left[\int_{\Omega} |t|^3 |\mathfrak{A}(t)|^2 \, d\sigma(t) \right]^{1/2} \left[\int_{\Omega} |t|^{-3} \, d\sigma(t) \right]^{1/2}. \tag{22}$$

We appeal now to the fact that the functions $t^{-k-2}|t|^{3/2}$, $k=1,2,\dots$, are mutually orthogonal in $L^2(\Omega)$ and that

$$\int_\Omega |t|^{-2k-1}\,d\sigma(t)=2\pi\int_1^\infty \frac{dr}{r^{2k}}=\frac{2\pi}{2k-1}$$

for $k=1,2,3,\dots$.

This gives

$$\int_\Omega |t|^3|\mathfrak{A}(t)|^2\,d\sigma(t)=\frac{1}{\pi^2}\int_\Omega \left|\sum_{k=1}^\infty (k+1)\overline{c_k}\,t^{-k-2}|t|^{3/2}\right|^2 d\sigma(t)$$

(23)
$$=\frac{1}{\pi^2}\sum_{k=1}^\infty (k+1)^2|c_k|^2\int_\Omega |t|^{-2k-1}\,d\sigma(t)$$

$$=\frac{2}{\pi}\sum_{k=1}^\infty \frac{(k+1)^2}{2k-1}|c_k|^2\le\frac{4}{\pi}\sum_{k=1}^\infty (k+1)|c_k|^2.$$

This, together with (22) and (18), implies

$$\sum_{k=1}^\infty (k+1)|c_k|^2\le\sqrt{\frac{4}{\pi}}\,\sqrt{2\pi}\left(\sum_{k=1}^\infty (k+1)|c_k|^2\right)^{1/2}.$$

Hence

(24)
$$\sum_{k=1}^\infty (k+1)|c_k|^2\le 8.$$

Finally, in view of (19) the proof of inequality (15) is complete.

To achieve equality in (24) ω must have equality in (21), (22), and (23), which can occur (respectively) under the following conditions:

(i) $\omega(t)=\overline{\mathfrak{A}(t)}/|\mathfrak{A}(t)|$, a.e. in Ω;

(ii) $|t|^3|\mathfrak{A}(t)|^2=\lambda|t|^{-3}$, a.e. in Ω with $\lambda>0$;

(iii) $c_k=0$ for $k=2,3,4,\dots$.

Since $t^3\mathfrak{A}(t)$ is analytic in Ω, condition (ii) constrains that $\mathfrak{A}(t)=at^{-3}$, $a\in\mathbf{C}$ for $t\in\Omega$. Then (i) becomes

(25)
$$\omega(t)=e^{i\theta}\left(\frac{t}{|t|}\right)^3\chi_\Omega(t),\quad 0\le\theta<2\pi.$$

Finally, we require (iii). Here luck is with us, since for ω defined by (25) the coefficients c_k with $k\ge 2$ vanish. The proof of Lemma 3 is complete. □

REMARK. It is a classical result that the norm of a linear bounded operator in a Banach space is attained (if at all) on the extremal points of the unit ball. For $L^\infty(\Omega)$ the only extremal points are the unimodular functions. Obviously they are not in $L^2(\mathbf{C})$. That is why we could not expect the extremals for inequality (15) to be in the class $L^2(\Omega)\cap L^\infty(\mathbf{C})$. On the other hand, inequality (15), which was proved originally for $\omega\in L^2(\Omega)\cap L^\infty(\mathbf{C})$, extends now to all $\omega\in L^\infty(\mathbf{C})$. The equality in (15) is attained, in this extension, only for the function (25).

Theorem 1 follows readily from Corollary 2 and Lemmas 2 and 3.

EPILOGUE. By (2) we also have

$$\|Tf\|_{\mathrm{BMO}_1} \le 3\|f\|_\infty. \tag{26}$$

However, this inequality is not sharp. The John–Nirenberg Lemma [8] implies the following.

There exists $\mu > 0$ *such that*

$$\fint_B \exp[\mu |Tb - (Tb)_{\mathbf{B}}|] < \infty \tag{27}$$

for every b, *with* $|b(z)| \le \chi_{\mathbf{B}}(z)$, *a.e.*

In order to estimate μ consider the following example:

$$\rho(z) = \left(\frac{z}{|z|}\right)^2 \chi_{\mathbf{B}}(z).$$

By (7), $T\rho(z) = (1 + \log|z|^2)\chi_{\mathbf{B}}(z)$. A computation shows that

$$\fint_B \exp[\mu|T\rho - (T\rho)_{\mathbf{B}}|] = \frac{2\mu}{(1-\mu^2)e} + \frac{e^\mu}{1+\mu} < \infty$$

for $|\mu| < 1$. We believe that (27) holds for every b and $|\mu| < 1$. If this happens to be true, then several interesting L^p-estimates for T and consequently for the derivatives of a quasiconformal mapping would follow ([2], [6], [7], [10]).

REFERENCES

1. K. Astala and F. W. Gehring, *Injectivity, the BMO norm and the universal Teichmüller space,* to appear.
2. A. Baernstein and J. J. Manfredi, *Topics in quasiconformal mapping*. Topics in modern harmonic analysis Vol. I, II (Turin/Milan, 1982), 819–862, Ist. Naz. Alta Mat. Francesco Severi, Rome, 1983.
3. B. Bojarski, *Homeomorphic solutions of Beltrami systems* (Russian), Dokl. Akad. Nauk SSSR. 102 (1955), 661–664.
4. ———, *Generalized solutions of a system of differential equations of first order and of elliptic type with discontinuous coefficients* (Russian), Math. Sb. 43(85) (1957), 451–503.
5. T. Figiel, T. Iwaniec, and A. Pelczyński, *Computing norms and critical exponents of some operators in L^p-spaces.* Studia Math. 79 (1984), 227–274.
6. F. W. Gehring and E. Reich, *Area distortion under quasi-conformal mappings,* Ann. Acad. Sci. Fenn. Ser. AI No. 388 (1966), 1–15.
7. T. Iwaniec, *Extremal inequalities in Sobolev spaces and quasi-conformal mappings,* Z. Anal. Anwendungen 1 (1982), 1–16.
8. F. John and L. Nirenberg, *On functions of bounded mean oscillation,* Comm. Pure Appl. Math. 14 (1961), 415–426.
9. P. Jones, *Extension theorem for BMO,* Indiana J. Math 29 (1980), 41–66.
10. O. Lehto, *Quasiconformal mappings and singular integrals.* Symposia Mathematica, Vol XVIII (INDAM, Rome, 1974), 429–453, Academic Press, London, 1976.

11. H. M. Reimann, *Functions of bounded mean oscillation and quasiconformal mappings,* Comm. Math. Helv. 49 (1974), 260–276.
12. E. M. Stein, *Singular integrals, harmonic functions, and differentiability properties of functions of several variables.* (Proceedings Sympos. Pure Math., Chicago, Ill., 1966), 316–335, Amer. Math. Soc., Providence, R.I., 1967.
13. I. N. Vekua, *Generalized analytic functions,* Pergamon Press, London, 1962.

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