# SPECTRAL INVARIANTS OF FOLIATIONS

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One of the problems in foliation theory is to relate the transverse geometry of the foliation to its topological invariants, the exotic classes. In this paper we introduce a spectral invariant related to the transverse geometry for an important class of foliations and relate it to exotic characteristic numbers.

A Lie group acting by isometries with constant orbit dimension generates a Riemannian foliation. In this paper we study the case of $R^n$ acting locally freely by isometries, this being an interesting class of foliations; the study of a much larger class of foliations can also be reduced to that of $R^n$.

Let $R^n$ act by isometries locally freely on a compact oriented $(4k-1)$-manifold $M$. Let $f$ be a symmetric homogeneous polynomial of degree $k$ in $2k$ indeterminates with integral coefficients. For $\theta \neq 0$ in $R^n$ we construct an eta function $\eta_f(s;\theta)$. Eta is constructed from the infinitesimal generator of the $R$ action corresponding to $\theta$ and the transverse signature operator (with coefficients) to the orbits of the $R$ action. We relate the value $\eta_f(s;\theta)$ at $s=0$ to the Simons characteristic number $S_f[M]$ associated to the codimension $4k-n-1$ Riemannian foliation arising from the $R^n$ action and $f$. We assume throughout this paper that our foliations are oriented.

THEOREM 1. *For generic $\theta$, $\eta_f(s;\theta)$ converges absolutely for* $\text{Re}(s)$ *large and extends to a meromorphic function on the $s$ plane with a finite value at $s=0$. $\eta_f(0;\theta)$ is independent of $\theta$ and*

$$\eta_f(0;\theta) = (-1)^k 2^{2k+1} S_f[M] \mod Z[\tfrac{1}{2}].$$

REMARK. Generic is defined in Section 1. Thus $\eta_f(0;\theta)$ is independent of $\theta$ for generic $\theta$.

As a corollary to the method of proof we obtain the following.

THEOREM 2. *Let $R^n$ act by isometries on the compact, oriented $4k$ manifold $W$ with boundary $M$ with the action locally free on the boundary. Let $\eta_f$ be the eta function for the action on $M$, and let $\Gamma$ be the fixed set for the action on $W$. For generic $\theta$,*

$$\eta_f(0;\theta) = (-1)^k 2^{2k+1} \text{Residue}(\theta, f, \Gamma) \mod Z.$$

Here residue is that of [6] and [5].

COROLLARY 2. $\eta_f(0;\theta) = 0 \mod Z[\tfrac{1}{2}]$ *when $n > 2$ for generic $\theta$.*

REMARK. This allows us to regard $\eta_f(0;\theta)$ as an obstruction to extending an isometric locally free $R$ action to an isometric locally free $R^n$ action for $n > 2$.

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For a closed oriented $4k$ manifold $M$, let us consider an isometric $R$ action with fixed set having connected components $\Gamma_j$, and let $\eta_f(s)_j$ be the eta function of the action on the boundary of a tubular neighborhood of $\Gamma_j$. Then we have the following.

COROLLARY 3. $\Sigma\eta_f(s)_j = 0 \bmod Z$.

A principal tool is Theorem 3.10 of [3], which relates the eta function of an operator on the boundary, the index of a certain boundary value problem, and the heat equation asymptotics.

In Section 1 we discuss the transverse signature operator, define our eta function, and present examples. In Sections 2 and 3 we give the proofs. We are grateful to Professor Atiyah for a very helpful conversation at Berkeley in 1983.

**1. Riemannian foliations, transverse signature operator, Simons class and the eta function.** Recall from [13] that a codimension $q$ Riemannian foliation on $M$ is given by a family $\{U_i, f_i, h_{ij}, g_i\}$, where $\{U_i\}$ is a covering, $f_i: U_i \to R^q$ a submersion, $g_i$ a Riemannian metric on $R^q$, and (for each $x \in U_i \cap U_j$) $h_{ij}^x$ is an isometry of a neighborhood of $f_i(x)$ with one of $f_j(x)$ which satisfies $f_i = h_{ij} f_j$. The $\{f^{-1}(p)\}$ are the local leaves. The normal bundle is obtained from $\bigcup_i TR^q$ by identification using the $dh_{ij}$, and the $g_i$ yield a metric $g$ on the normal bundle $\nu$, called an *adapted* metric for the foliation.

Assume that the $h$'s are orientation preserving. Such a Riemannian foliation is called oriented. Orientability of the foliation is equivalent to the normal bundle being oriented. Then $g$ and the orientation give rise to a star operator on $\Omega$, the sections of $\Lambda^*\nu$, and hence a splitting $\Omega = \Omega^+ + \Omega^-$. We can also obtain this splitting by using the star operator in $R^q$ and identifying using $dh_{ij}$. Let $\nu$ be the normal bundle to an oriented Riemannian foliation and $\Omega = C^\infty \Lambda\nu^*$ ($\Lambda$ will always mean the exterior algebra of the complexified bundle). Reinhart [14] introduced a transverse $d$ and $\delta$. We can, following Paul Baum, describe $d$ invariantly. Let $\pi: T^*M \to \nu$ be the map induced by orthogonal projection using a metric on $TM$ compatible with that on $\nu$, and

$$\sigma: C^\infty(T^*M \otimes \Lambda\nu^*) \to C^\infty(\Lambda\nu^*)$$

be given by $\sigma(v \otimes w) = \pi(v) \wedge w$. A Riemannian foliation gives rise to a unique Riemannian torsion-free connection on its normal bundle [13]. Let

$$\nabla: C^\infty(\Lambda\nu^*) \to C^\infty(T^*M \otimes \Lambda\nu^*)$$

be the resulting connection on $\Omega$. Then $d = \sigma\nabla$. The metric on $\nu$ and the orientation give rise to a $*$ operator on $\Lambda\nu^*$. Then we can define $\delta$ on $\Omega$ to be

$$(-1)^{q(k+1)+1} * d *$$

on $k$ forms. The involution $t$ on $\Omega$ given by $i^{k(k-1)+q/2}*$ on $k$ forms anti-commutes with $d+\delta$. Hence $\Omega = \Omega^+ + \Omega^-$ where $\Omega^+, \Omega^-$ are the $+1, -1$ eigenforms under $t$. $d+\delta: \Omega^+ \to \Omega^-$ and similarly $d+\delta: \Omega^- \to \Omega^+$, and we denote $d+\delta$ restricted to $\Omega^+$ by $D^+$. We note also that $\delta$ is the adjoint operator of $d$ relative to

$$\langle s, t\rangle = \int_M (s(x), t(x))\, dx,$$

where ( , ) is the metric on $\Lambda\nu^*$ arising from the given metric on $\nu$ and $dx$ is the volume form on $M$. $D^+$ is called the *transverse signature* operator. For $V$ a vector bundle we extend $D^+$, via a connection on $V$, to act on sections of $\Lambda\nu\otimes V$ (denoted by $\Omega(V)$) and on $\Omega^+(V)$ or $\Omega^-(V)$ by the construction of [12, p. 57]. We denote the resulting operator by $D^+\otimes V$. A computation similar to that for the ordinary signature operator yields $\sigma(D^+)_v = \theta(\pi v) - i(\pi v)$, where $\theta$ is wedge product,

$$i(a)(b_1\wedge\cdots\wedge b_n) = \Sigma(-1)^{i+1}(a, b_i) b_1\wedge\cdots i \cdots\wedge b_n),$$

and $\pi$ is an orthogonal projection on $\nu$. $\sigma(D^+\otimes V) = \sigma(D^+)\otimes I$. $D^+$ is transversally elliptic to the leaves of the foliation in the sense of [1].

Exotic characteristic classes for foliations are classes that come from the cohomology of the appropriate classifying space for foliations. A Riemannian foliation has classes that can be obtained from the Simons construction [15] applied to the unique Riemannian torsion-free connection on $\nu$ and appropriate polynomials. Let $f$ be a homogeneous polynomial of degree $k$ in $2k$ indeterminates. Write $f$ as a polynomial in the $\sigma_j$, the elementary symmetric functions. Let $\varphi$ be the corresponding polynomial of degree $2k$ in $4k$ indeterminates obtained by replacing $\sigma_j$ by $\sigma_{2j}$. Recall that for an $n\times n$ matrix $A$, $c_j$ is defined by $\Sigma t^j c_j(A) = \det(I-(t/2\pi)A)$. Then $\varphi(A)$ is defined to be the result of replacing $\sigma_{2j}$ in $\varphi$ by $c_{2j}(A)$. Given a codimension $4k-2$ Riemannian foliation on $M$, the Simons construction [15] applied to $\varphi$ and the Riemannian torsion-free connection yields $S_f$ in $H^{4k-1}(M; R/Z)$.

The same polynomial yields a virtual representation $u_f$ of $SO(4k)$ defined in [4, §I, p. 596], and by restriction of $SO(4k-m)$. $u_f$ has the property that

$$ch(u_f) = f(x_1^2, \ldots, x_{2n}^2) + \text{higher order terms.}$$

$u_f$ can be described directly as follows. Write $f$ as a polynomial in the $\sigma_j$, the elementary symmetric functions of $2k$ indeterminates. Now replace each $\sigma_j$ by the corresponding $\sigma_{2j}$ in the $4k$ indeterminates $t_1-1, \ldots, t_{2k}-1, t_1^{-1}-1, \ldots, t_{2k}^{-1}-1$. This describes $u_f$ as a virtual representation of $SO(4k)$. Let $u_f = \rho_0 - \rho_1$ where the $\rho$ are representations. For an oriented Riemannian vector bundle $E$, $\rho(E)$ is the extension of the principal bundle of $E$ by $\rho$. Thus $u_f(E)$ makes sense and $u_f(E+m) = u_f(E)$.

We need some specific connections on $\nu$ and $TM$. Let $R$ act by isometries on $M$ with no fixed points. Let $X$ be the infinitesimal generator of the action. The normal bundle to the resulting foliation has a unique Riemannian torsion-free connection given on a section $s$ of the normal bundle by

$$\nabla_Y(s) = \omega(Y) L_X(s) + \pi D_{\pi Y}(s),$$

where $\pi$ is orthogonal projection on the complement of $X$, $\omega(Y) = (Y, X)/(X, X)$, and $D$ is the Riemannian connection on $TM$. Analogously, if $R$ acts (possibly with fixed points), an isometry invariant connection on $TM$ is called an *$X$ connection* if away from the singular set $\nabla_Y(s)=\omega(Y)L_X(s)+D_{\pi Y}(s)$. Given an invariant metric on $M$, such a connection can always be constructed. We will be interested in operators $D^+\otimes\rho_j$ where $V$ is either $\nu$ or $TM$ and the connection is one of the above.

DEFINITION OF THE ETA FUNCTION. We will define our eta function in terms of a first-order operator associated to an isometric $R$ action. Then in Theorem 1.6 we will show that this eta function can be constructed in terms of the action of $R$ on the kernel and cokernel of the transverse signature operator, thus bringing it closer to index theory.

Let $R$ act by isometries with no fixed points on $M$. Let $X$ be the infinitesimal generator to the action. We can introduce a new metric $g^0=\|X\|^{-1}g$, where $\|X\|=g(X,X)^{1/2}$. It follows that $X$ is also an infinitesimal isometry relative to $g^0$. We want to use $g^0$ in the following discussion to define the eta function, so we might as well assume $g(X,X)=1$. Let $N=M\times R$ with the product metric on $N$. Let $p:N\to M$ be the projection. Take the orientation on $TN$ given by

$$p^*(\nu)+\{X,d/du\}.$$

From [4, p. 576],

$$\begin{aligned}\Lambda^+(N)&=\Lambda^+(\nu)\otimes\Lambda^+(R^2)+\Lambda^-(\nu)\otimes\Lambda^-(R^2)\\ \Lambda^-(N)&=\Lambda^+(\nu)\otimes\Lambda^-(R^2)+\Lambda^-(\nu)\otimes\Lambda^+(R^2).\end{aligned}\tag{1.1}$$

Here $p^*\nu$ is shortened to $\nu$ and $N\times R^2$ to $R^2$. We consider a representation $\rho$ of $SO(4k)$ and a coefficient bundle of $\rho(TN)$.

$$\Lambda^+(N)\otimes\rho(TN)=\Lambda^+(\nu)\otimes\Lambda^+(R^2)\otimes\rho(TN)+\Lambda^-(\nu)\otimes\Lambda^-(R^2)\otimes\rho(TN),\tag{1.2}$$

and similarly for $\Lambda^-$.

Let $\omega$ be the one form $\omega(Y)=g(Y,X)$. Then $\{\omega,du\}$ is an oriented orthonormal basis for $\Lambda^*(R^2)$. $\Lambda^+(R^2)$ has basis $s_1=1+i\omega\wedge du$, $s_2=du-i\omega$ and $\Lambda^-(R^2)$ has basis $t_1=1-i\omega\wedge du$, $t_2=du+i\omega$. Let $D^+,D^-$ be the transverse signature operators on $\Lambda^+(\nu),\Lambda^-(\nu)$. Let $D^+\otimes\rho$ be the extension to

$$\Lambda^+(\nu)\otimes\Lambda^+(R^2)\otimes\rho(TN)$$

obtained by using the connection arising on $\rho(TN)$ from an $X$ connection on $TN$ and the flat connection on $\Lambda^+(R^2)$ relative to $s_1,s_2$ (similarly for $D^-\otimes\rho$). Now $R$ acts on $M\times R$ (trivially on the $R$ factor) and so the action of $X$ extends to $TN$, and hence to $\Lambda^+(N)\otimes\rho(TN)$. We remark that the action of $X$ on $\Lambda^+(N)\otimes\rho(TN)$ coincides with that of $X\otimes\rho$ obtained by using an $X$ connection. Finally, let $\sigma$ be the symbol of the ordinary signature operator $D_N^+$ at the cotangent $du$. We recall from [3, p. 63] that $D_N^+=\sigma(d/du+B)$, where $B$ is elliptic with symbol given by (4.6) of [3]. Let

$$E_0=\Lambda^+(N)\,|\,M\times\{0\}\quad\text{and}\quad E_1=\Lambda^-(N)\,|\,M\times\{0\}.\tag{1.3}$$

Thus $E_0,E_1$ are bundles on $M$. $E_0$ has the decomposition of (1.2) and similarly for $E_1$. We describe a first-order operator $A$ which is a matrix of first-order operators with components $A_{11}$, $A_{12}$, $A_{21}$, $A_{22}$ on the bundle $E_0$ relative to the above decomposition (1.2). These are given by

$$A_{11}=(-ig(X,X)^{-1/2})X, \qquad A_{12}=-\sigma(D^+\otimes\rho),$$
$$A_{21}=-\sigma(D^-\otimes\rho), \qquad A_{22}=(ig(X,X)^{-1/2})X.$$

(Of course we have chosen $g(X,X)=1$.)

THEOREM (1.5). *The operator $A$ on $E_0$ is self-adjoint and elliptic. In fact, $\sigma(A)=\sigma(B)$.*

*Proof.* First we show that each $A_{ij}$ is self-adjoint. This is obvious for $A_{11}$ and $A_{22}$. Now $\sigma^*=\sigma^{-1}=-\sigma$, so

$$(-\sigma(D^+\otimes\rho))^*=-(D^-\otimes\rho)\sigma^*=(D^-\otimes\rho)\sigma.$$

For a section $f$ of $\Lambda^*(\nu)$, let $\deg(f)$ be $+1$ or $-1$ depending on whether $f$ is in $\Lambda^{\text{even}}$ or $\Lambda^{\text{odd}}$. Then $\sigma(f\otimes t_j)=(-1)^{j+1}\deg(f)f\otimes s_{j+1}$ and $\deg(D^+f)=-\deg(f)$, and similarly $D^-$. From these remarks it follows that the remaining two operators are self-adjoint. To prove ellipticity, recall that $\sigma=\sigma_{du}(D_N^+)$ and also $\sigma^{-1}=-\sigma$. For $v$ in $T^*M$, $\sigma_v(B)=\sigma^{-1}\sigma_v(D_N^+)$. Now

$$\sigma_v(X)=v(X), \qquad \sigma_v(D^+)=\theta(\pi v)-i(\pi v).$$

Thus if $v=\omega$ (the dual form to $X$), then $\sigma_v(A)=-i$ on $\Lambda^+(\nu)\otimes\Lambda^+(R^2)$ and $+i$ on $\Lambda^-(\nu)\otimes\Lambda^-(R^2)$. Direct computation on $f\otimes s_j, f\otimes t_j$ shows that the same is true for $\sigma_\omega(B)$. If $v$ is orthogonal to $\omega$ then $v(X)=0$. On $\Lambda^*(\nu)\otimes\Lambda^*(R^2)$ for $*=+,-$ we have

$$\sigma_v(A)=\sigma^{-1}\sigma_v(D^+\otimes\rho)=\sigma^{-1}(\theta(\pi v)-i(\pi v))=\sigma^{-1}\sigma_v=\sigma_v(B). \qquad \square$$

Thus $A$ is elliptic and self-adjoint on $M$. The eta function $\eta_A(s,\rho)$ is defined, at least formally as in [3]:

$$\eta_A(s,\rho)=\sum \operatorname{sign}(\lambda)|\lambda|^{-s},$$

the sum taken over the non-zero eigenvalues $\lambda$ of $A$. The virtual representation $u_f$ is $\rho_0-\rho_1$, so we define

$$\eta_f(s)=\eta_A(s,\rho_0)-\eta_A(s,\rho_1).$$

Now let $R^n$ act locally freely by isometries of $M$. The image of $R^n$ is dense in a torus which acts on $M$. Let $\theta$ be an element of $R^n$ which projects to a generator of the torus. We call such a $\theta$ *generic*. Let $R\to R^n$ by sending 1 to $\theta$. Then $R$ acts on $M$. Define $\eta_f(s;\theta)$ to be:

(1.6) $$\eta_f(s;\theta)=\eta_f(s) \quad \text{for this action of } R.$$

SIMPLE EXAMPLE. Let $R$ act on $S^{4k-1}$ by

$$(z_1,\dots,z_{2k})\to(\exp(i\lambda_1 t)z_1,\dots,\exp(i\lambda_{2k}t)z_{2k}).$$

Then Theorem 1 and the theorem of [9] tell us that

$$\eta_f(0)=(-1)^k 2^{2k+1} f(\lambda_1^2,\ldots,\lambda_{2k}^2)/\lambda_1\ldots\lambda_{2k} \mod Z[\tfrac{1}{2}].$$

If $k=1$ and we take $f=X_1+X_2$, then

$$\eta_f(0)=-8(\lambda_1^2+\lambda_2^2)/\lambda_1\lambda_2 \mod Z[\tfrac{1}{2}].$$

Again let $R$ act by isometries with no fixed points on $M$ ($X$ the infinitesimal generator), and let $\rho=\rho_0$ or $\rho_1$. Consider $\ker(D^+\otimes\rho(TM))$ in $L_2(\Lambda^+(\nu)\otimes\rho(TM))$. The action of $R$ preserves $D^+$ so $D^+$ commutes with $X$. $X$ also acts on the tensor product either via its action on the bundle $\Lambda^+(\nu)\otimes\rho(TM)$ or using the connection on $\rho(TM)$ arising from an $X$ connection, and the actions coincide. Thus $X$ acts on the kernel. Let $\lambda$ be an eigenvalue of $X$ corresponding to an eigenfunction lying in $\ker(D^+\otimes\rho)$. Let

$$K_\lambda=\{u\colon (D^+\otimes\rho)u=0,\ -iXu=\lambda u\}$$

and

$$\ker(D^+\otimes\rho)_\lambda=\{u\colon (D^+\otimes\rho)u=0,\ -X^2u=\lambda^2u\}.$$

Then $\ker(D^+\otimes\rho)_\lambda=K_\lambda+K_{-\lambda}$. $D^+\otimes\rho$ is an invariant transversally elliptic operator, and the constructions of [1] show that $\ker(D^+\otimes\rho)_\lambda$ and hence $K_\lambda$ are finite-dimensional. Let

$$\eta^+(s,\rho)=\sum \operatorname{sign}(\lambda)\dim(K_\lambda)|\lambda|^{-s} \quad \text{for } \lambda\neq 0.$$

Do the same for $D^-$ to obtain $\eta^-(s,\rho)$. For $R^n$ acting locally freely by isometries let $\theta$ correspond to a generic $R$ action, so we have $\eta^+(s,\rho,\theta)$ and $\eta^-(s,\rho,\theta)$.

THEOREM (1.6).

$$\eta_f(s,\theta)=-2[(\eta^+(s,\rho_0,\theta)-\eta^-(s,\rho_0,\theta))-(\eta^+(s,\rho_1,\theta)-\eta^-(s,\rho_1,\theta))].$$

REMARK. First note that $\eta^+,\eta^-$ depend on $f$. All of the $\eta$ functions involved will be shown to converge absolutely for Re$(s)$ large, extending to a meromorphic function with a finite value as $s=0$. The proof of (1.6) will be given in Section 4.

**2. Proofs of the main theorems.** Let $M$ be a closed oriented $4k-1$ Riemannian manifold and let $R^n$ act locally freely by isometries. $R^n$ then acts through a torus $T$ contained in the isometry group of $M$. By [10, Theorem 3] (see appendix), some multiple of $M$ by a power of 2 bounds an oriented $T$ manifold $W$. We can assume that $W$ has an invariant metric which is a product near the boundary. It will clearly be enough to prove Theorem 1 when $M=\partial W$. Let $X$ be the infinitesimal generator of the isometric $R$ action on $W$ given by a generic $\theta$ in $R^n$ and $\nu$ the complement of $X$ in $TM$. An orientation on $R$ and $M$ induces one on $\nu$.

Let $U_0$ be a neighborhood of $M$ in $W$ taken so that its closure is contained in a neighborhood $C$ on which the metric is a product. Let $\psi$ be a $C^\infty$ function defined on a neighborhood of $M$ in $W$ with values in $[0,1]$ whose support is contained in $U_0$ and which is 1 on a neighborhood of $M$ and 0 outside $U_0$. We can assume that we have an $X$ connection $\nabla$ on $TW$ so that, in a neighborhood of $C$, $\nabla=\omega\otimes L_X+D_\pi$ as in Section 1.

Let $u$ be the inward normal coordinate at $\partial W$ which we can assume is defined throughout $C$. From [3, §4] the ordinary signature operator $D_W^+$ becomes $\sigma[d/du+B]$ and $D_W^+ \otimes \rho(TW)$ becomes $\sigma[d/du+B\otimes\rho(TW)]$ on $C$. From (1.4) we have the operator $A$ on $\Lambda^+(TW)\otimes\rho(TW)\,|\,M$ and on $C=M\times[0,\epsilon)$ we can take $\sigma[d/du+A]$. Thus on $W$ we can consider the operator

$$(2.1)\qquad \begin{aligned} &D_\rho=\psi\sigma[d/du+A]+(1-\psi)D_W^+\otimes\rho,\\ &D_\rho\colon \Lambda^+(TW)\otimes\rho(TW)\to\Lambda^-(TW)\otimes\rho(TW). \end{aligned}$$

Near $M$, $D_\rho=\sigma[d/du+A]$; away from $M$, $D_\rho=D_W^+\otimes\rho$. From (1.5) we conclude $\sigma(D_\rho)=\sigma(D_W^+)$. Thus from (3.10) of [3] we conclude that $\eta_A(s,\rho)$ converges absolutely for $\mathrm{Re}(s)$ large and extends meromorphically with a finite value at zero. Further, $\eta_A(0,\rho)-2\int_W\alpha(x)$ is an integer, where $\alpha(x)$ is the constant term in the asymptotic expansion coming from the heat kernel of $D_\rho^*D_\rho$ and its adjoint on the double of $W$. We take $\rho=\rho_0,\rho_1$ to obtain

$$\eta_A(0,\rho_j)-2\int_W\alpha_j(x)\,dx \ \text{ is an integer.}$$

PROPOSITION (2.2). $\int_W\alpha_0(x)\,dx-\int_W\alpha_1(x)\,dx=(-1)^k2^{2k}\int_W\varphi(K_\nabla)$ *and thus* $\eta_f(0;\theta)=(-1)^k2^{2k+1}\int_W\varphi(K_\nabla)$ mod $Z[\frac{1}{2}]$.

*Proof.* Here $K_\nabla$ is the curvature of the connection $\nabla$. We wish to apply Theorem II of [2] to conclude that $\alpha_0(x)-\alpha_1(x)$ is a product of Chern forms of the virtual bundle $\rho_0(TW)-\rho_1(TW)$. We proceed in the manner that [2] treats the signature operator (cf. §§5, 6). On the bundle $\Lambda(TW)\otimes\rho(TW)$ we have the metric $g$ on $TW$ and an invariant metric $h$ on $\rho(TW)$ with $\nabla$ constructed so as to preserve $h$. Then we get from this data $\alpha(x)$ satisfying (2.2)–(2.5) of [2]. We consider the change $g\to\lambda^2g$. As in [2, p. 306] introduce the map on forms $\epsilon(\varphi)=\lambda^p\varphi$ on $p$-forms. Relative to the new metric $g_0$ we get a new operator $A_0$ with components

$$(-ig_0(X,X)^{-1/2})X,\quad -\sigma_0(D_0^+\otimes\rho),\quad -\sigma_0(D_0^-\otimes\rho),\quad (ig_0(X,X)^{-1/2})X,$$

and a new $D_{0,W}^+$ and hence a new $D_{0,\rho}$. The $X$ connection does not change. We let $L=(-ig(X,X)^{-1/2})X$. A direct computation shows $\epsilon\sigma_0\epsilon^{-1}=\sigma$, $\epsilon\sigma_0L_0\epsilon^{-1}=\lambda^{-1}\sigma L$, and (from [2]) $\epsilon(D_0^+\otimes\rho)\epsilon^{-1}=\lambda^{-1}D^+\otimes\rho$. Thus $\epsilon D_{0,\rho}\epsilon^{-1}=\lambda^{-1}D_\rho$.

As in [2, §5], $\alpha_0,\alpha_1$ and $\alpha_0-\alpha_1$ are regular invariants of the metric of weight zero. If we jointly change $g$ to $\lambda^2g$ and the metric $h$ to $\mu^2h$, then $\nabla$ is unaffected and so $D_\rho$ is independent of the change in $h$. Thus $\alpha$ is a joint invariant of $g$ and $h$ of weight zero, and so we conclude (as in [2, pp. 309–310]) that $\alpha$ is a polynomial in the Pontryagin forms of $g$ and the Chern forms of $\rho(TW)$. Then the identical argument of [2, pp. 310–311] enables us to conclude

$$(2.3)\qquad \alpha_0(x)-\alpha_1(x)=\sum(ch_j(\rho_0(TW)-\rho_1(TW))F_v(p_1(g),\ldots)),$$

where the sum is taken over $2j+4v=4k$. However the virtual representation $u_f=\rho_0-\rho_1$ has the property that $ch(u_f)$ starts in degree $4k$ with the term $f(x_1^2,\ldots,x_{2k}^2)$, and this is represented by the form $(-1)^k\varphi(K_\nabla)$. Thus

$$(2.4)\qquad \int_W(\alpha_0(x)-\alpha_1(x))\,dx=(-1)^k\int_WF_0\,\varphi(K_\nabla).$$

Here $K_\nabla$ is the curvature of $\nabla$. However $\varphi(K_\nabla)$ as a form vanishes identically when, in the notation of Section 1, $\nabla = \omega L_X + D_\pi$ (see Remark 3.1A). $\nabla$ was chosen to be $\omega L_X + D_\pi$ on the support of $\psi$ and away from this support $D_\rho = D_W^+ \otimes \rho$. Thus by the locality property (2.3) of [2], $F_0(x)$ is the term coming from the corresponding term for $D_W^+ \otimes \rho(TW)$, and this has been computed [2, p. 311] to be $2^{2k}$. □

Let $F_\theta$ be the codimension $4k-2$ Riemannian foliation given by the $R$ action on $M$. Then we have the following.

PROPOSITION (2.5). $\int_W \varphi(K_\nabla) = S_f(F_\theta)[M] \bmod Z$.

Let $F$ be the codimension $4k-n-1$ foliation of $M$ given by the $R^n$ action on $M$. We remark that the Simons class $S_f(F_\theta)$ mod $Z$ is independent of the choice of adapted metric on the normal bundle to $F_\theta$ (see 3.5).

PROPOSITION (2.6). *For generic* $\theta$, $S_f(F_\theta)[M] = S_f(F)[M] \bmod Z$.

The proofs of (2.5) and (2.6) will be given in Section 3 where we discuss Simons classes. The proof of Theorem 1 follows from these propositions. We have already shown convergence and that $\eta_f(0,\theta) = (-1)^k 2^{2k+1} S_f(F_\theta)[M] \bmod Z[\frac{1}{2}]$. $S_f(F_\theta)$ mod $Z$ is the same for $g$ or $g(X,X)^{-1/2} g$ since both are adapted metrics, and so Theorem 1 follows from (2.6).

Now we consider Theorem 2. We first note that the residue is that of [6] and [5]. We note that when $R$ is a subgroup of $R^n$ which is dense in the boundary-preserving isometry group of $W$, the fixed set of the $R$ action is the same as that of the $R^n$ action and is contained in the interior of $W$. Let $\{\Gamma_j\}$ be the set of components of the fixed set $\Gamma$. Each $\Gamma_j$ is orientable and the boundary of a tubular neighborhood of $\Gamma_j$ inherits an orientation from $W$. The form $\varphi(K_\nabla)$ certainly vanishes outside the union of disjoint tubular neighborhoods by our remarks in the proof of (2.2), and the theory of [5] applied to $W$ (and keeping in mind that $\varphi(K_\nabla)$ vanishes near $\partial W$) shows that

$$\int_W \varphi(K_\nabla) = \sum \operatorname{Res}(X, f, \Gamma_j),$$

where the residue is given by the right-hand side of (2.1) of [5] and also by the left-hand side of Theorem 2 of [6].

Corollary 2 is really a consequence of [8]. In this paper we defined and studied, for certain transversally elliptic operators, an $R/Z$ invariant called virtindex. This invariant came out of studying the transverse index ([1], [16]) and hence is related to $K$ theory. From [8, Theorem 2] we have

$$\operatorname{virtindex}_R(D_\theta^+ \otimes u_f) = -2^{2k-1} S_f(F)[M] \bmod Z[\tfrac{1}{2}],$$

where $D_\theta^+$ is the transverse signature operator to a generic $R$ action, and $u_f$ is the virtual bundle obtained from the virtual representation $u_f$ and the normal bundle to $F_\theta$. Then we show in [8], using $K$ theory, that if $n > 2$, then

$$\operatorname{virtindex}_R(D_\theta^+ \otimes u_f) = 0.$$

Thus $S_f(F)[M]$ and hence $\eta_f(0;\theta)$ is zero mod $Z[\frac{1}{2}]$.

Finally, the last corollary follows from Theorem 2 by taking $W$ as the union of disjoint tubular neighborhoods of the $\Gamma_j$; we do not need $\frac{1}{2}$ since $M$ is actually $\partial W$.

**3. Simons classes and the proofs of (2.5), (2.6).** Recall the $\lambda$ construction from [7, p. 64]. Given connections $\nabla^0, \nabla^1$ and a polynomial $\varphi$, $\lambda(\nabla^1, \nabla^0)(\varphi)$ is a differential form satisfying $d\lambda(\nabla^1, \nabla^0)(\varphi) = \varphi(K_1) - \varphi(K_0)$, where $K_i$ is the curvature of $\nabla^i$. Let $\nabla^L$ be the Riemannian torsion-free connection on the normal bundle to $F_\theta$, $\nabla^{\text{fl}}$ the connection on the line bundle $(X)$ which is globally flat relative to $X/|X|$ and $\nabla^1 = \omega \otimes L_X + D^1_\pi$ an $X$ connection on $TM$ ($D^1$ is the Riemannian connection on $TM$). Then $\nabla^1$ and $\nabla^{\text{fl}} + \nabla^L$ are connections on $TM$.

LEMMA (3.1). *$\lambda(\nabla^1, \nabla^{\text{fl}} + \nabla^L)(\varphi) = 0$ as a form if $\varphi$ has degree $2k$.*

*Proof.* Choose local coordinates $\{x, y_1, \ldots, y_q\} = U$, where $q = 4k-2$ with $X = \partial/\partial x$. Let us consider the local framing $\{X/|X|, \partial/\partial y_1, \ldots, \partial/\partial y_q\}$ and $h: U \to R^q$ given by $h(x,y) = y$. Let $\theta_1$ and $\theta_0$ be the local connection matrices of $\nabla^1$ and $\nabla^{\text{fl}} + \nabla^L$. We show that $\theta_1$ and $\theta_0$ lie in $h^*\Omega(TR^{4k-2})$. Then a direct computation shows $\lambda(\nabla^1, \nabla^{\text{fl}} + \nabla^L)(\varphi)$ is a sum of terms $\varphi(\sigma^{2i+1} \wedge (d\sigma + [\sigma, \theta_0])^j \wedge \Omega_0^v)$, where $\sigma = \theta_1 - \theta_0$ is in $h^*\Omega^1(R^q)$, $\Omega_0 = d\theta_0 + [\theta_0, \theta_0]$ and $d\sigma + [\sigma, \theta_0]$ are in $h^*\Omega^2(R^q)$, and $i + j + v = 2k-1$. Thus each summand is of degree at least $4k-1$, hence zero.

To show that $\theta_1$ is in the given ideal use the definition of $\nabla^1$ and the invariance of this connection. Then a direct computation shows $i(X)$ and $L_X$ annihilate $\theta_1$. For $\theta_0$, introduce $p$, the orthogonal projection on $X$. Then

$$(\nabla^{\text{fl}} + \nabla^L)s = (p\nabla^{\text{fl}} ps, \pi\nabla^L \pi s).$$

A direct computation—together with the facts that $\pi\nabla^1 = \nabla^L$ on $\nu$ and that $\nabla^L$ is locally pulled back, via $h$, from $R^q$—will yield the result. The computations are similar to those in [9].

REMARK (3.1A). Since the connection matrix $\theta_1$ lies in the ideal of forms $I = h^*\Omega(TR^{4k-2})$, the curvature $K_1$ lies in $I^2$ and hence $\varphi(K) = 0$ as a form (since $\varphi$ is a polynomial of deg $2k$ and $\varphi(K) \in I^{4k}$). This same remark shows that $\varphi(K_\nabla) = 0$ for an $X$ connection on $W$ outside the set where $\nabla \neq \omega \otimes L_X + D_\pi$.

LEMMA (3.2). *$S_\varphi(\nabla^L) = S_\varphi(\nabla^{\text{fl}} + \nabla^L)$.*

*Proof.* This is (1.2) of [9]. □

REMARK. We will use the notation $S_f$ and $S_\varphi$ to mean the same thing when $f$ and $\varphi$ are related as in Section 1.

LEMMA (3.3). *$S_\varphi(\nabla^1) = S_\varphi(\nabla^{\text{fl}} + \nabla^L)$.*

*Proof.* According to [15, p. 31], $S_\varphi(\nabla^1) - S_\varphi(\nabla^{\text{fl}} + \nabla^L)$ as a function on $4k-1$ cycles is given by integrating (over the cycle) the expression

$$(2k) \int_{[0,1]} \varphi(\sigma \wedge \Omega_t^{2k-1})\, dt,$$

where $\sigma = \theta_1 - \theta_0$, $\Omega_t = t^2\sigma^2 + t(d\sigma + [\sigma, \theta_0]) + \Omega_0$. It is then easily seen that this $4k-1$ form is just $\lambda(\nabla^1, \nabla^{\text{fl}} + \nabla^L)(\varphi)$, which is zero. □

Now let $\nabla$ be an $X$ connection *on* $TW$ and let $D$ be the Riemannian connection on $TW$. On $TW$ restricted to $M$ we have the following.

LEMMA (3.4). $S_\varphi(\nabla)=S_\varphi(\nabla^1)$ *in* $H^{4k-1}(M;R/Z)$.

*Proof.* On a collar of $\partial W$, $\nabla=\omega\otimes L_X+D_\pi$, the metric is a product, and $D=D^1+D^{\mathrm{fl}}$ where $D^{\mathrm{fl}}$ is flat relative to $\partial/\partial u$. Then $\nabla=\nabla^1+D^{\mathrm{fl}}$ and so the lemma follows from (1.2) of [9]. □

Thus we have the following.

PROPOSITION (3.5). $S_\varphi(F_\theta)=S_\varphi(\nabla)$. $S_\varphi(F_\theta)$ *is independent of the choice of adapted metric.*

*Proof of* (2.5). By (3.14) of [15],

$$S_\varphi(\nabla)[M]-S_\varphi(D)[M]=\int_M\lambda(\nabla,D)(\varphi)=\int_W d\lambda(\nabla,D)(\varphi)=\int_W\varphi(\nabla)-\int_W\varphi(D).$$

By theorem (5.15) of [15] we have $S_\varphi(D^1)[M]=\int_W\varphi(D)$. Since $D=D^1+D^{\mathrm{fl}}$, $S_\varphi(D^1)=S_\varphi(D)$. Thus $S_\varphi(F_\theta)[M]=S_\varphi(\nabla)[M]=\int_W\varphi(\nabla)$. (Note: we have used the notation $\varphi(\nabla)$ in place of $\varphi(K_\nabla)$.)

Now we prove independence of the metric. We remark that it is true for $S_f(F)$ for any codimension $q$ Riemannian foliation $F$. For the remainder of this proof let $\nabla$ and $\nabla^1$ be the Riemannian torsion-free connections on the normal bundle of $F$ relative to two adapted metrics. By [15],

$$S_\varphi(\nabla)(\sigma)-S_\varphi(\nabla^1)(\sigma)=\int_\sigma\lambda(\nabla,\nabla^1)(\varphi)$$

for $\sigma$ a smooth simplex. Let $\{x_1,\dots,x_p,y_1,\dots,y_q\}$ be local coordinates for which $y_1=\cdots=y_q=$ constant define the local leaves of the foliation. The connection matrices $\theta,\theta^1$ of $\nabla,\nabla^1$ relative to the local framing $\{\partial/\partial y_1,\dots,\partial/\partial y_q\}$ both lie in the ideal of forms $I=\{dy_1,\dots,dy_q\}$ and then, by direct computation, $\lambda(\nabla,\nabla^1)(\varphi)$ lies in $I^{2v-1}$ where $v$ is the degree of $\varphi$. In our case $v=2k$, $q=4k-2$, and $2v-1=4k-1$, so that $\lambda(\nabla,\nabla^1)(\varphi)=0$ as a form. □

*Proof of* (2.6). Again $X$ is the generator of the $R$ action on $M$. Take

$$X=X_1,X_2,\dots,X_n$$

to be the commuting vector fields which, at each point, generate $F$. Let $\nu$ and $\nu_\theta$ be the normal bundles to $F$ and $F_\theta$ and $\nabla$ and $\nabla^\theta$ the Riemannian torsion-free connections on $\nu$ and $\nu_\theta$. $\nu_\theta=\nu+\{X_2,\dots,X_n\}$ and we let $\nabla^{\mathrm{fl}}$ be flat relative to $X_2,\dots,X_n$. Then, by (4.2)–(4.5),

$$S_\varphi(F)=S_\varphi(\nabla),\quad S_\varphi(F_\theta)=S_\varphi(\nabla^\theta),\quad S_\varphi(\nabla+\nabla^{\mathrm{fl}})=S_\varphi(\nabla),$$

$$S_\varphi(\nabla^\theta)[M]-S_\varphi(\nabla)[M]=\int_M\lambda(\nabla^\theta,\nabla+\nabla^{\mathrm{fl}})(\varphi).$$

Thus it will be sufficient to show $\lambda=0$.

We choose local coordinates $U=\{x_1,\ldots,x_n,y_1,\ldots,y_q\}$ such that $X_j=\partial/\partial x_j$ and $y_j=$ constant describe the local leaves of $F$. Then

$$x_2=\cdots=x_n=y_1=\cdots=y_q=\text{constant}$$

describe the local leaves of $F_\theta$. Let $f\colon U\to R^q$ and $g\colon U\to R^{4k-2}$ be given by

$$f(x,y)=y, \qquad g(x,y)=(x_2,\ldots,x_n,y_1,\ldots,y_q).$$

Let $\omega$ be the local connection matrix of $\nabla$ relative to $\{\partial/\partial y_1,\ldots,\partial/\partial y_q\}$ and $\omega_\theta$ of $\nabla^\theta$ relative to $\{\partial/\partial x_2,\ldots,\partial/\partial x_n,\partial/\partial y_1,\ldots,\partial/\partial y_q\}$. Then $\omega\in f^*\wedge(dy_1,\ldots,dy_q)$ and $\omega_\theta\in g^*\wedge(dx_2,\ldots,dx_n,dy_1,\ldots,dy_q)$. Let $I=g^*\wedge(dx_2,\ldots,dy_q)$. $\omega$ and $\omega_\theta$ are in $I$.

$$\lambda(\nabla^\theta,\nabla+\nabla^{\text{fl}})(\varphi)=(2k)\int_{[0,1]}\varphi(\sigma\wedge\Omega_t^{2k-1})\,dt,$$

where $\sigma=\omega^\theta-\omega$, $\Omega_t=t^2\omega^2+t(d\sigma+[\sigma,\omega])+d\omega+\omega^2$. Since $\sigma,\omega$ are in $I$, $\Omega_t$ is in $I^2$ and so $\varphi(\sigma\wedge\Omega_t^{2k-1})\in I^{4k-1}=0$. □

**4. Proof of Theorem (1.6).** First, for a section of $E_0$ of the form

$$f=f_1\otimes s_1+f_2\otimes s_2,$$

we have $(D^+\otimes\rho)f=0$ if and only if $(D^+\otimes\rho)f_j=0$ $(j=1,2)$; similarly for $g_1\otimes t_1+g_2\otimes t_2$ and $D^-\otimes\rho$. For simplicity we will shorten $D^+\otimes\rho$ to $D^+$ and similarly for $D^-$. Recall we have also changed metrics so that $g(X,X)=1$.

Now recall $A$ from (1.4). Let $\lambda\neq 0$ be an eigenvalue of $A$ and let $A_\lambda\subset L_2(E_0)$ be the finite-dimensional subspace consisting of all eigenvectors with eigenvalue $\lambda$. Let $B_\lambda$ be the subspace of $A_\lambda$ spanned by

$$\{f_1\otimes s_1+f_2\otimes s_2+g_1\otimes t_1+g_2\otimes t_2\colon D^+f_j=D^-g_j=0,\ j=1,2\}.$$

Notice that $D^+f_j=0$ if and only if $-\sigma D^+f_j=0$ and similarly for $D^-$. Let $C_\lambda$ be a complementary subspace to $B_\lambda$.

LEMMA (4.1). *There is a map $E_\lambda\colon A_\lambda\to A_{-\lambda}$ mapping $C_\lambda$ injectively onto a complement for $B_{-\lambda}$. Hence* $\dim C_\lambda=\dim C_{-\lambda}$.

*Proof.* Consider $\sum f_j\otimes s_j+g_j\otimes t_j$ in $A_\lambda$. From the definition of $A$, the fact that $f$ is an eigenvector, and the relations

$$\sigma(\varphi\otimes s_j)=(-1)^{j+1}\deg(\varphi)\varphi\otimes t_{j+1}, \qquad \sigma(\varphi\otimes t_j)=(-1)^{j+1}\deg(\varphi)\varphi\otimes s_{j-1},$$
$$\deg(D^+\varphi)=\deg(D^-\varphi)=-\deg(\varphi),$$

it follows that

$$\begin{aligned} -iXf_1-\deg(g_2)D^-g_2&=\lambda f_1,\\ -iXf_2+\deg(g_1)D^-g_1&=\lambda f_2,\\ iXg_1-\deg(f_2)D^+f_2&=\lambda g_1,\\ iXg_2+\deg(f_1)D^+f_1&=\lambda g_2. \end{aligned}\tag{4.2}$$

Let $E_\lambda(f)=-D^-g_2\otimes s_1+D^-g_1\otimes s_2-D^+f_2\otimes t_1+D^+f_1\otimes t_2$.

A direct computation using (4.2) and the fact that $D^+, D^-$ commute with $X$ implies that $E_\lambda(f)$ is an eigenvector of $A$ with eigenvalue $-\lambda$. If $f\neq 0$ is in $C_\lambda$, then one of $D^+f_j, D^-g_j$ is nonzero and $D^+D^-\varphi$ or $D^-D^+\varphi=0$ implies $D^+\varphi$ or $D^-\varphi=0$ by adjointness. Hence for $f\neq 0$ in $C_\lambda$, $E_\lambda(f)\neq 0$ and $E_\lambda(f)$ is not in $B_{-\lambda}$. Thus $E_\lambda$ on $C_\lambda$ is injective into a complement for $B_{-\lambda}$. The same argument applied to $-\lambda$ yields the result. ☐

Thus $\eta_A(s;\rho)$ can be computed in terms of eigenfunctions of $A$ in $\ker(D^+\otimes\rho)$, $\ker(D^-\otimes\rho)$. Let $\{f_\lambda\}$, $\lambda\in P$ be a basis of eigenfunctions of $A$ in $\ker(D^+\otimes\rho)$ and $\{g_\lambda\}$, $\lambda\in Q$ for $A$ in $\ker(D^-\otimes\rho)$. From (4.2), $-iXf_\lambda=\lambda f_\lambda$ and $iXg_\lambda=\lambda g_\lambda$; thus

$$\begin{aligned}\eta_A(s;\rho) &= \Sigma_P \operatorname{sign}(-\lambda)|\lambda|^{-s}+\Sigma_Q \operatorname{sign}(\lambda)|\lambda|^{-s}\\ &= -2\eta^+(s;\rho)+2\eta^-(s;\rho).\end{aligned}$$

Hence Theorem (1.6) follows. ☐

**Appendix.** The following are theorems from [10, §2]. $O_*^G$ will denote the bordism group of oriented closed $G$ manifolds.

THEOREM 2 [10]. *Let $H$ be the identity component of a compact Abelian group $G$ and $M$ an oriented $G$ manifold with the action of $H$ having no fixed points on $M$. Then the bordism class of $M$ is null in $O_*^G\otimes Z[\frac{1}{2}]$.*

THEOREM 3 [10]. *Let $g$ be the number of connected components of the compact Abelian group $G$. Then $O_*^G\otimes Z[\frac{1}{2}g]$ is zero in odd dimensions.*

Theorem 3 is relevant to us. We take $G$ to be a torus $T$ to conclude that $O_*^T\otimes Z[\frac{1}{2}]$ is zero in odd dimensions. The proofs follow the lines of [11], using a detailed analysis of the relative bordism groups.

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