THE REDUCED MINIMUM MODULUS

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Introduction. Let T be a bounded linear operator acting in a Banach space. The reduced minimum modulus of T will be defined by the equation

$$\gamma(T)=\begin{cases}\inf\{\|Tx\|:\operatorname{dist}(x,\ker T)=1\} & \text{if } T\neq 0,\\ 0 & \text{if } T=0.\end{cases}$$

The definition of $\gamma(T)$ is taken from [9, Ch. IV, §5] for $T\neq 0$. (If $T=0$ we put $\gamma(T)=0$, whereas in [9], $\gamma(T)=\infty$). Thus $\gamma(T)>0$ if and only if T has closed non-zero range. If T is invertible then $\gamma(T)=\|T^{-1}\|^{-1}$ and this shows that the function $T\to\gamma(T)$ is not continuous but it could have good local continuity properties.

In general $\lim_{n\to\infty}\gamma(T^n)^{1/n}$ does not exist and it is not known what conditions on T are equivalent to the existence of $\lim_{n\to\infty}\gamma(T^n)^{1/n}$. We mention the following known cases when $\lim_{n\to\infty}\gamma(T^n)^{1/n}$ exists:

(1) if T is Fredholm, $\lim_{n\to\infty}\gamma(T^n)^{1/n}$ is the radius of the largest open disk centered at 0, included in the Fredholm domain of T, such that $\dim\ker(T-\lambda)=\text{const.}$ for $\lambda\neq 0$ in the disk ([8]);

(2) if T is surjective or bounded from below, $\lim_{n\to\infty}\gamma(T^n)^{1/n}$ is the radius of the largest open disk centered at 0 such that $T-\lambda$ is surjective or bounded from below for λ in the disk ([10], [11]).

In this paper we investigate the properties of the reduced minimum modulus of operators acting in Hilbert spaces. In Section 1 we develop some general properties of $\gamma(T)$ and a related matrix representation of T (Theorem 1.5). Section 2 will be devoted to the study of the continuity properties of the function $\lambda\to\gamma(T-\lambda)$, $\lambda\in C$. The discontinuities of this function form a countable set and $\lim_{\lambda\to\mu}\gamma(T-\lambda)$ always exists. The set

$$\sigma_\gamma(T)=\left\{\mu\in C:\lim_{\lambda\to\mu}\gamma(T-\lambda)=0\right\}$$

is closed, non-empty, and obeys the spectral mapping theorem (Theorem 2.7). As seen in Theorem 2.5 and Proposition 2.6, $\rho_\gamma(T)$, the complement of $\sigma_\gamma(T)$, is the minimal open set where $T-\lambda$ has an analytic generalized inverse.

Section 3 deals with the problem of the existence of $\lim_{n\to\infty}\gamma(T^n)^{1/n}$ and the role $\sigma_\gamma(T)$ plays in this problem. A positive new result and a direct generalization of the result of [8] is the existence of $\lim_{n\to\infty}\gamma(T^n)^{1/n}$ for semi-Fredholm operators (see Remark after Corollary 3.4).

The last part of the paper, Section 4, contains some results on the effect of a compact perturbation K on $\sigma_\gamma(T+K)$ (see Theorem 4.4).

1. Preliminaries. Throughout the paper we shall denote by H a fixed complex Hilbert space, $H\neq\{0\}$, and T will be a fixed bounded linear operator acting in H.

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The symbol $\mathcal{L}(H)$ will denote the algebra of all bounded linear operators acting in H. If X is a closed subspace of H, then P_X will denote the orthogonal projection of H onto X. If $G\subset H$, clm G is the closed linear manifold (or span) of G.

As in [3, §2], we shall call $\mu\in C$ a T-regular point if the function $\lambda\to P_{\ker(T-\lambda)}$, $\lambda\in C$ is norm-continuous at μ. If $\mu\in C$ is not T-regular we call it T-singular. Let us define (see [5])

$$\sigma_{\text{c.r.}}(T)=\{\lambda\in\sigma(T):(T-\lambda)H=((T-\lambda)H)^{-}\},$$
$$\sigma^{r}_{\text{c.r.}}(T)=\{\lambda\in\sigma_{\text{c.r.}}(T):\lambda \text{ is } T\text{-regular}\},$$
$$\sigma^{s}_{\text{c.r.}}(T)=\{\lambda\in\sigma_{\text{c.r.}}(T):\lambda \text{ is } T\text{-singular}\}.$$

Because T acts in a Hilbert space it is easy to see that in case $T\neq 0$ we have

$$\begin{aligned}\gamma(T)&=\inf\{\|Tx\|:x\in(\ker T)^{\perp},\ \|x\|=1\}\\&=\inf(\sigma((T^*T)^{1/2})\backslash\{0\})\\&=\inf(\sigma((TT^*)^{1/2})\backslash\{0\})=\gamma(T^*).\end{aligned}$$

1.1. PROPOSITION. *For every pair $A,B\in\mathcal{L}(H)$ we have*

(i) $\gamma(A)\|P_{(\ker A)^\perp}P_{\ker B}\|\le\|A-B\|$;

(ii) $\|P_{\ker A}-P_{\ker B}\|(\min\{\gamma(A),\gamma(B)\})^2\le 2(\|A\|+\|B\|)\|A-B\|$;

(iii) $|\gamma(A)-\gamma(B)|\le\|P_{\ker A}-P_{\ker B}\|\max\{\gamma(A),\gamma(B)\}+\|A-B\|$.

Proof. (i) The first relation is trivial if $\gamma(A)=0$. If $\gamma(A)>0$ we have as in [5, Lemma 2],

$$\gamma(A)\|P_{(\ker A)^\perp}P_{\ker B}\|\le\|AP_{\ker B}\|=\|(A-B)P_{\ker B}\|\le\|A-B\|.$$

(ii) To prove the second relation put $r=\frac{1}{2}(\min\{\gamma(A),\gamma(B)\})^2$. Since $r=0$ is a trivial case, assume $r>0$. Then (ii) follows from the relations:

$$\begin{aligned}\|P_{\ker A}-P_{\ker B}\|&=\frac{1}{2\pi}\left\|\int_{|\lambda|=r}[(\lambda-A^*A)^{-1}-(\lambda-B^*B)^{-1}]\,d\lambda\right\|\\&\le\frac{1}{2\pi}\int_{|\lambda|=r}\|(\lambda-A^*A)^{-1}\|\,\|A^*A-B^*B\|\,\|(\lambda-B^*B)^{-1}\|\,|d\lambda|\\&\le(1/r)\|A^*A-B^*B\|\le(1/r)(\|A\|+\|B\|)\|A-B\|.\end{aligned}$$

(iii) For every $x\in(\ker B)^\perp$, $\|x\|=1$, we have

$$\begin{aligned}\|(A-B)x\|&\ge\|Ax\|-\|Bx\|\ge\gamma(A)\|P_{(\ker A)^\perp}x\|-\|Bx\|\\&\ge\gamma(A)-\|Bx\|-\gamma(A)\|P_{\ker A}x\|\\&=\gamma(A)-\|Bx\|-\gamma(A)\|(P_{\ker A}-P_{\ker B})x\|\\&\ge\gamma(A)-\|Bx\|-\gamma(A)\|P_{\ker A}-P_{\ker B}\|,\end{aligned}$$

and this easily implies that

$$\|A-B\|\ge\gamma(A)-\gamma(B)-\|P_{\ker A}-P_{\ker B}\|\max\{\gamma(A),\gamma(B)\}.$$

But interchanging A and B, we obtain

$$\pm(\gamma(A)-\gamma(B)) \leq \|P_{\ker A}-P_{\ker B}\| \max\{\gamma(A), \gamma(B)\}+\|A-B\|. \qquad \square$$

1.2. COROLLARY. *The set*

$$\Gamma_\epsilon(H)=\{A \in \mathcal{L}(H): \gamma(A) \geq \epsilon\} \quad (\epsilon \geq 0)$$

is norm-closed and the functions

$$A \to P_{\ker A}, \quad A \to \gamma(A), \qquad A \in \Gamma_\epsilon(A), \quad \epsilon>0$$

are continuous.

Proof. Since $\Gamma_0(H)=\mathcal{L}(H)$, we shall assume $\epsilon>0$. Let $B \in \Gamma_\epsilon(H)^-$, $\{A_n\}_{n=1}^\infty \subset \Gamma_\epsilon(H)$ be such that $\lim_{n \to \infty}\|A_n-B\|=0$. By Proposition 1.1(i) and (ii), $\lim_{n \to \infty} P_{\ker A_n}$ exists and majorizes $P_{\ker B}$ and by Proposition 1.1(iii), $\lim_{n \to \infty} \gamma(A_n)=\gamma(B)$. This implies $B \in \Gamma_\epsilon(H)$ (see Proposition 1.1). $\square$

REMARK. A Banach space version of the proof that $\Gamma_\epsilon(H)$ is norm-closed is given in [4, Lemma 1.9].

1.3. LEMMA. *If T has a matrix representation of the form $T=\begin{pmatrix} A & * \\ 0 & B \end{pmatrix}$, where A has dense range and $B \neq 0$, then we have $\gamma(T) \leq \gamma(B)$.*

Proof. Since A^* is injective we deduce that B^* is the restriction of T^* to an invariant subspace including $\ker T^*$. It follows that

$$\gamma(T)=\gamma(T^*) \leq \gamma(B^*)=\gamma(B). \qquad \square$$

1.4. LEMMA. *Let $\mu \in C$ be such that $T-\mu$ has closed range. Then the following conditions are equivalent:*

(i) *μ is T-regular,*

(ii) $\ker(T-\mu) \subset \operatorname{clm}_{\lambda \neq \mu} \ker(T-\lambda)$,

(iii) *T has a matrix representation of the form $T=\begin{pmatrix} A & * \\ 0 & B \end{pmatrix}$, where $A-\mu$ is surjective and $B-\mu$ is bounded from below.*

Proof. The equivalence (i)$\Leftrightarrow$(ii) is proved in [5, Lemma 1]. If we put

$$H_1=\operatorname*{clm}_{\lambda \neq \mu} \ker(T-\lambda), \qquad H_2=H \ominus H_1$$

and if $T=\begin{pmatrix} A & * \\ 0 & B \end{pmatrix}$ is the matrix representation of T determined by the decomposition $H=H_1+H_2$, then obviously $A-\mu$ has dense range and, by Lemma 1.3, $B-\mu$ will have closed range. If μ is T-regular then $\ker(T-\mu)=\ker(A-\mu)$; consequently $A-\mu$ will be a closed range surjection. Since by [3, Proposition 1.3], we derive that $B-\mu$ is bounded from below, and the implication (i)$\Rightarrow$(iii) follows. To prove (iii)$\Rightarrow$(i), observe that we have $\ker(T-\lambda)=\ker(A-\lambda)$ for λ in a neighborhood of μ and the function $\lambda \to P_{\ker(A-\lambda)}$ is continuous at μ by [3, Lemma 1.5]. $\square$

1.5. THEOREM. *Let us put*

$$H_r'(T) = \operatorname*{clm}_{\lambda \in \sigma_{\text{c.r.}}^r(T)} \ker(T-\lambda), \qquad H_l'(T) = \operatorname*{clm}_{\lambda \in \sigma_{\text{c.r.}}^r(T)} \ker(T-\lambda)^*,$$

$$H_0'(T) = (H_r'(T) + H_l'(T))^\perp.$$

Then $H = H_r'(T) + H_0'(T) + H_l'(T)$ *is an orthogonal decomposition and determines the matrix representation*

$$T = \begin{pmatrix} T_r' & * & * \\ 0 & T_0' & * \\ 0 & 0 & T_l' \end{pmatrix}$$

such that

$$\sigma(T) \setminus \sigma_{\text{c.r.}}^r(T) = \sigma_r(T_r') \cup \sigma(T_0') \cup \sigma_l(T_l').$$

Proof. The inclusion "$\subset$" follows immediately from Lemma 1.4, thus we shall prove the inclusion "$\supset$" only. Using Lemma 1.3 and Lemma 1.4 we easily derive the inclusion

$$\sigma_{\text{c.r.}}^r(T) \subset \rho_r(T_r') \cap \rho(T_0') \cap \rho_l(T_l'),$$

and this implies

$$C \setminus \sigma_{\text{c.r.}}^r(T) \supset \sigma_r(T_r') \cup \sigma(T_0') \cup \sigma_l(T_l').$$

Since obviously the right-hand side of the above inclusion is a subset of $\sigma(T)$, we conclude that

$$\sigma(T) \setminus \sigma_{\text{c.r.}}^r(T) \supset \sigma_r(T_r') \cup \sigma(T_0) \cup \sigma_l(T_l'). \qquad \square$$

REMARK. If we put

$$H_r(T) = \operatorname*{clm}_{\lambda \in \rho_{s\text{-}F}^r(T)} \ker(T-\lambda), \qquad H_l(T) = \operatorname*{clm}_{\lambda \in \rho_{s\text{-}F}^r(T)} \ker(T-\lambda)^*,$$

$$H_0(T) = (H_r(T) + H_l(T))^\perp,$$

then the properties of the matrix representation

$$T = \begin{pmatrix} T_r & * & * \\ 0 & T_0 & * \\ 0 & 0 & T_l \end{pmatrix}$$

are described in [3, §2] (here $\rho_{s\text{-}F}^r(T)$ denotes the set of T-regular points in the semi-Fredholm domain of T). The inclusion $\rho(T) \cup \sigma_{\text{c.r.}}^r(T) \supset \rho_{s\text{-}F}^r(T)$ can be easily deduced.

2. The function γ_T. To simplify the statements we shall denote by γ_T the function defined by the equation $\gamma_T(\lambda) = \gamma(T-\lambda)$, $\lambda \in C$.

2.1. PROPOSITION. *For every* $\mu \in C$, $\lim_{\lambda \to \mu} \gamma_T(\lambda)$ *exists and the following implications hold true:*

(i) μ *is T-regular* $\Rightarrow \gamma_T$ *is continuous at* μ;

(ii) $\lim_{\lambda \to \mu} \gamma_T(\lambda) > 0 \Rightarrow \mu$ *is T-regular.*

Proof. The implication (i) is a consequence of Proposition 1.1(iii). If

$$\overline{\lim_{\lambda \to \mu}} \gamma_T(\lambda) > 0$$

then μ is T-regular by Corollary 1.2, and the existence of $\lim_{\lambda \to \mu} \gamma_T(\lambda)$ follows by (i). This proves the implication (ii) as well as the existence of $\lim_{\lambda \to \mu} \gamma_T(\lambda)$.

2.2. THEOREM. *The set $\sigma^s_{\text{c.r.}}(T)$ is at most countable. The set*

$$\{\mu \in \sigma^s_{\text{c.r.}}(T) : \gamma_T(\mu) > 0\}$$

is the set of discontinuity points of γ_T.

Proof. By Proposition 2.1 we easily derive

$$\left\{\mu \in C : \lim_{\lambda \to \mu} \gamma_T(\lambda) \neq \gamma_T(\mu)\right\} = \{\mu \in \sigma^s_{\text{c.r.}}(T) : \gamma_T(\mu) > 0\}.$$

Suppose first that T is not a scalar multiple of the identity operator. If we put $\sigma_n = \{\mu \in \sigma^s_{\text{c.r.}}(T) : \gamma_T(\mu) \geq 1/n\}$ we have $\sigma^s_{\text{c.r.}}(T) = \bigcup_{n=1} \sigma_n$. If σ_n has an accumulation point μ_n, then $\mu_n \in \sigma^r_{\text{c.r.}}(T)$ by Proposition 2.1(ii), contradicting the fact that $\sigma^r_{\text{c.r.}}(T)$ is open (see Proposition 1.6). This shows that σ_n is finite and $\sigma^s_{\text{c.r.}}(T)$ is at most countable. If $T = \mu I$ then $\sigma^s_{\text{c.r.}}(T) = \{\mu\}$. □

2.3. COROLLARY. *If T is not a scalar multiple of I then $\sigma^s_{\text{c.r.}}(T)$ coincides with the set of discontinuity points of γ_T.*

Proof. Since we have $\sigma^s_{\text{c.r.}}(T) = \{\mu \in \sigma^s_{\text{c.r.}}(T) : \gamma_T(\mu) > 0\}$, we apply Theorem 2.2.

REMARK. Suppose that $T - \lambda$ has closed range for every $\lambda \in C$. Since $\partial\sigma(T) \subset \sigma^s_{\text{c.r.}}(T)$ we derive that $\partial\sigma(T)$ is at most countable. It follows that $\sigma(T)$ is at most countable, and we recapture Theorem 1 of [5].

In the sequel we shall need the following notation:

$$\sigma_\gamma(T) = \left\{\mu \in \mathbf{C} : \lim_{\lambda \to \mu} \gamma_T(\lambda) = 0\right\}, \qquad \rho_\gamma(T) = \mathbf{C} \setminus \sigma_\gamma(T).$$

2.4. PROPOSITION. *The set $\sigma_\gamma(T)$ is closed and we have*

(i) $\partial\sigma(T) \subset \sigma_\gamma(T) \subset \sigma(T)$,

(ii) $\sigma_\gamma(T) = \sigma^s_{\text{c.r.}}(T) \cup \{\mu \in C : \gamma_T(\mu) = 0\}$,

(iii) $\rho_\gamma(T) = \sigma^r_{\text{c.r.}}(T) \cup \rho(T)$.

Proof. For every $\mu \in \partial\sigma(T)$ we have

$$\lim_{\lambda \to \mu} \gamma_T(\lambda) = \lim_{\substack{\lambda \in \rho(T) \\ \lambda \to \mu}} \gamma_T(\lambda) = \lim_{\substack{\lambda \in \rho(T) \\ \lambda \to \mu}} \|(T-\lambda)^{-1}\|^{-1} = 0.$$

Thus $\mu \in \sigma_\gamma(T)$. Since obviously $\sigma_\gamma(T) \subset \sigma(T)$, (i) follows. The relation (iii) is an easy consequence of Proposition 2.1; thus $\sigma_\gamma(T) = \mathbf{C} \setminus \rho_\gamma(T) = \sigma(T) \setminus \sigma^r_{\text{c.r.}}(T)$, and this shows that $\sigma_\gamma(T)$ is closed. Further, observe that we obviously have

$$\sigma^s_{\text{c.r.}}(T) \cup \{\mu \in C : \gamma_T(\mu) = 0\} = \sigma(T) \setminus \sigma^r_{\text{c.r.}}(T),$$

which concludes the proof. □

REMARK. Suppose that T is not a scalar multiple of I. Then

$$\mathbf{C}=\rho(T)\cup\sigma^r_{\text{c.r.}}(T)\cup\sigma^s_{\text{c.r.}}(T)\cup\{\mu\in\mathbf{C}:\gamma_T(\mu)=0\}$$

is a partition with the following properties:

(1) $\rho(T)$ is a set of continuity points for both functions $\lambda\to(T-\lambda)^{-1}$ and γ_T.

(2) $\sigma^r_{\text{c.r.}}(T)$ is a set of continuity points for both functions γ_T and $\lambda\to P_{\ker(T-\lambda)}$.

(3) $\sigma^s_{\text{c.r.}}(T)$ is a set of discontinuity points for both functions γ_T and $\lambda\to P_{\ker(T-\lambda)}$.

(4) $\{\mu\in C:\gamma_T(\mu)=0\}$ is a set of continuity points for γ_T.

2.5. THEOREM. *There exists an analytic function* $F\colon\rho_\gamma(T)\to\mathcal{L}(H)$ *such that*

$$(T-\lambda)F(\lambda)(T-\lambda)=T-\lambda,\quad F(\lambda)(T-\lambda)F(\lambda)=F(\lambda),\quad \lambda\in\rho_\gamma(T).$$

Proof. Consider the matrix representation

$$T=\begin{pmatrix} T_r' & * & * \\ 0 & T_0' & * \\ 0 & 0 & T_l' \end{pmatrix}$$

given by Theorem 1.5. Since we have

$$\rho_\gamma(T)=\sigma(T)\setminus\sigma^r_{\text{c.r.}}(T)=\sigma_r(T_r')\cup\sigma(T_0')\cup\sigma_l(T_l'),$$

we can apply the results of [6, §2] to produce an analytic function

$$F\colon\rho_\gamma(T)\to\mathcal{L}(H)$$

such that $(T-\lambda)F(\lambda)$ is a projection onto the range of $T-\lambda$ and $I-F(\lambda)(T-\lambda)$ is a projection onto $\ker(T-\lambda)$. The function F will fulfill the conditions required by our theorem. □

2.6. PROPOSITION. *Let G be an open subset of C and let $F\colon G\to\mathcal{L}(H)$ be an analytic function such that $(T-\lambda)F(\lambda)(T-\lambda)=T-\lambda$, $\lambda\in G$. Then $G\subset\rho_\gamma(T)$ and we have*

$$(T-\lambda)^{n+1}\frac{d^nF(\lambda)}{d\lambda^n}(T-\lambda)^{n+1}=n!\,(T-\lambda)^{n+1},\quad \lambda\in G,\ n\geq 0,$$

$$\gamma((T-\lambda)^{n+1})\geq n!\left\|\frac{d^nF(\lambda)}{d\lambda^n}\right\|^{-1},\quad \lambda\in G,\ n\geq 0.$$

Proof. Suppose we have

$$(T-\lambda)^{n+1}\frac{d^nF(\lambda)}{d\lambda^n}(T-\lambda)^{n+1}=n!\,(T-\lambda)^{n+1},\quad \lambda\in G,$$

for some fixed $n\geq 0$. Then differentiating and multiplying both sides by $T-\lambda$ we derive

$$(T-\lambda)^{n+2}\frac{d^{n+1}F(\lambda)}{d\lambda^{n+1}}(T-\lambda)^{n+2}=(n+1)!\,(T-\lambda)^{n+2},$$

and the first relation will follow by induction. Now using the first relation we can check easily that we have

$$P_{(\ker(T-\lambda)^{n+1})^\perp} = P_{(\ker(T-\lambda)^{n+1})^\perp} F_n(\lambda)(T-\lambda)^{n+1},$$

where $F_n(\lambda) = (1/n!)(d^n F(\lambda)/d\lambda^n)$. Since for every $x \in (\ker(T-\lambda)^{n+1})^\perp$, $\|x\| = 1$, we have

$$1 \leq \|P_{(\ker(T-\lambda)^{n+1})^\perp} F_n(\lambda)\| \, \|(T-\lambda)^{n+1} x\|,$$

we infer

$$\gamma((T-\lambda)^{n+1}) \geq \|P_{(\ker(T-\lambda)^{n+1})^\perp} F_n(\lambda)\|^{-1} \geq \|F_n(\lambda)\|^{-1}.$$

In particular, $\gamma(T-\lambda) \geq \|F_0(\lambda)\|^{-1}$ and the inclusion $G \subset \rho_\gamma(T)$ becomes obvious. □

REMARK. The second relation of the above proposition is proved in [8]. We included a proof for completeness. Theorem 1.5 and Proposition 1.6 together show that $\rho_\gamma(T)$ is the set of points where $T-\lambda$ has a local (or equivalently a global) analytic generalized inverse. As is natural to expect, $\sigma_\gamma(T)$ obeys the spectral mapping theorem:

2.7. THEOREM. *Let f be a complex analytic function defined in a neighborhood of $\sigma(T)$. Then we have*

$$\sigma_\gamma(f(T)) = f(\sigma_\gamma(T)).$$

Proof. Consider the matrix representation

$$T = \begin{bmatrix} T_r' & * & * \\ 0 & T_0' & * \\ 0 & 0 & T_l' \end{bmatrix}$$

given by Theorem 1.5. Since we have

$$\sigma_\gamma(T) = \sigma_r(T_r') \cup \sigma(T_0') \cup \sigma_l(T_l'), \qquad f(T) = \begin{bmatrix} f(T_r') & * & * \\ 0 & f(T_0') & * \\ 0 & 0 & f(T_l') \end{bmatrix},$$

and since one-side spectra obey the spectral mapping theorem, applying Lemma 1.4(iii) we easily derive the inclusion $\sigma_\gamma(f(T)) \subset f(\sigma_\gamma(T))$. To prove the opposite inclusion put $S = f(T)$ and consider the matrix representation

$$S = \begin{bmatrix} S_r' & * & * \\ 0 & S_0' & * \\ 0 & 0 & S_l' \end{bmatrix}$$

given by Theorem 1.5. Then T has the matrix representation

$$T=\begin{bmatrix} A & * & * \\ 0 & B & * \\ 0 & 0 & C \end{bmatrix},$$

where $S_r'=f(A)$, $S_0'=f(B)$, $S_l'=f(C)$. Because by Lemma 1.4(iii) and Theorem 1.5 we have

$$\begin{aligned}\sigma_\gamma(f(T))&=\sigma_r(f(A))\cup\sigma(f(B))\cup\sigma_l(f(C))\\&=f(\sigma_r(A))\cup f(\sigma(B))\cup f(\sigma_l(C))\\&=f(\sigma_r(A)\cup\sigma(B)\cup\sigma_l(C))\supset f(\sigma_\gamma(T)),\end{aligned}$$

the proof is concluded. □

3. The existence of $\lim_{n\to\infty}\gamma(T^n)^{1/n}$.

3.1. LEMMA. *If T is similar to A then we have*

$$\overline{\lim_{n\to\infty}}\,\gamma(T^n)^{1/n}=\overline{\lim_{n\to\infty}}\,\gamma(A^n)^{1/n},$$

$$\underline{\lim_{n\to\infty}}\,\gamma(T^n)^{1/n}=\underline{\lim_{n\to\infty}}\,\gamma(A^n)^{1/n}.$$

Proof. If $T=S^{-1}AS$, where S is invertible, then we can easily check that we have

$$\begin{aligned}\gamma(T^n)&=\gamma(S^{-1}A^nS)\geq\gamma(S^{-1})\gamma(A^nS)\\&=\gamma(S^{-1})\gamma(S^*A^{*n})\geq\gamma(S^{-1})\gamma(S^*)\gamma(A^{*n})\\&=\|S\|^{-1}\|S^{-1}\|^{-1}\gamma(A^n).\end{aligned}$$

Analogously we derive

$$\gamma(A^n)\geq\|S\|^{-1}\|S^{-1}\|^{-1}\gamma(T^n),$$

hence the relations in the statement follow. □

3.2. THEOREM. *Suppose $0\in\rho_\gamma(T)$ and let r denote the radius of the largest open disk centered at 0 and included in $\rho_\gamma(T)$. Then $\lim_{n\to\infty}\gamma(T^n)^{1/n}$ exists and we have* $\lim_{n\to\infty}\gamma(T^n)^{1/n}=r$.

Proof. Let $H_r'(T)$ be as in Theorem 1.5 and let $T=\begin{pmatrix} A & * \\ 0 & B\end{pmatrix}$ be the matrix representation of T determined by the decomposition $H=H_r'(T)+H_r'(T)^\perp$. If we denote by r_1 (respectively r_2) the radii of the largest open disks centered at 0 and included in $\rho_r(A)$ (resp. $\rho_l(B)$), then Theorem 1.5 implies $r=\min\{r_1,r_2\}$. There are three cases we must consider:

(1) $H_r'(T)^\perp=\{0\}$, $r_2=\infty$, $r=r_1$, $T=A$. Since T is surjective the existence of $\lim_{n\to\infty}\gamma(T^n)^{1/n}$ and the relation $\lim_{n\to\infty}\gamma(T^n)^{1/n}=r$ follow by [10, Theorem 1] (see also [11]).

(2) $H_r'(T)=\{0\}$. Passing to the adjoint we reduce to the case (1).

(3) $H_r'(T)\neq 0$, $H_r'(T)^\perp\neq\{0\}$. It is plain that $A\neq 0$ and $B\neq 0$ (via $0\in\rho_\gamma(T)$), thus by Lemma 1.3 we have $\gamma(T^n)\leq\min\{\gamma(A^n),\gamma(B^n)\}$.

Using (1) and (2) we derive

$$\varlimsup_{n\to\infty} \gamma(T^n)^{1/n} \le \min\{r_1, r_2\} = r.$$

Let F be the function defined in Theorem 2.5. If we put

$$F_n = \frac{1}{n!}\frac{d^nF(\mu)}{d\mu^n}\bigg|_{\mu=0}$$

then we have

$$F(\lambda) = \sum_{n=0} \lambda^n F_n, \qquad |\lambda| < r, \quad \varlimsup_{n\to\infty} \|F_n\|^{1/n} = r^{-1}.$$

By Proposition 2.6 we see that

$$\varliminf_{n\to\infty} \gamma(T^n)^{1/n} \ge \varliminf_{n\to\infty} \|F_n\|^{-1/n} = \left(\varlimsup_{n\to\infty} \|F_n\|^{1/n}\right)^{-1} = r. \qquad \square$$

REMARK. The analyticity argument used in the proof of (iii) appears in both [8] and [10]. A combinatorial argument replaces it in [11].

3.3. PROPOSITION. *Suppose* $0 \in \sigma_\gamma(T)$ *and let* σ *denote the connected component of* $\sigma_\gamma(T)$ *containing* 0. *Suppose also that we have*

$$\varlimsup_{n\to\infty} \gamma(T^n)^{1/n} = r > \sup_{\zeta\in\sigma} |\zeta|.$$

Then $\lim_{n\to\infty} \gamma(T^n)^{1/n}$ *exists,* T *is similar to* $Q \oplus T'$ *where* Q *is nilpotent,* $0 \in \rho_\gamma(T')$, *and we have*

$$\lim_{n\to\infty} \gamma(T^n)^{1/n} = \lim_{n\to\infty} \gamma(T'^n)^{1/n}.$$

Proof. Using Lemma 3.1 and Theorem 3.2 we only need to show the existence of Q and T' as above. Consider the matrix representation

$$T = \begin{pmatrix} T_r' & * & * \\ 0 & T_0' & * \\ 0 & 0 & T_l' \end{pmatrix}$$

given by Theorem 1.5. Let σ_1, σ_2 be clopen subsets of $\sigma_\gamma(T)$ such that

$$\sigma \subset \sigma_1, \qquad \sigma_\gamma(T) = \sigma_1 \cup \sigma_2, \qquad \sigma_1 \cap \sigma_2 = \varnothing.$$

If we put $\sigma_1' = \sigma_1 \cap \sigma(T_0')$ and $\sigma_2' = \sigma_2 \cap \sigma(T_0')$, then the inclusion $\sigma(T_0') \subset \sigma_\gamma(T)$ implies $\sigma(T_0') = \sigma_1' \cup \sigma_2'$. Using now [7, Ch. VII, Theorem 20] we may suppose that T has a matrix representation of the form

$$T = \begin{pmatrix} A & * & * \\ 0 & B & * \\ 0 & 0 & C \end{pmatrix},$$

where A has dense range, $\sigma(B) = \sigma_1'$, and C is injective. Suppose that C does not act on a $\{0\}$-space, and put $S = \begin{pmatrix} B & * \\ 0 & C \end{pmatrix}$. Using Lemma 1.3 we derive

$\lim_{n\to\infty}\gamma(S^n)^{1/n} > r$. Now we want to show the inclusion $\sigma_1' \subset \rho_l(C)$. To this aim assume the contrary and pick $\mu \in \sigma_1' \cap \rho_l(C)$. Since every vector of the form $\binom{0}{x}$ is orthogonal to $\ker S^n$ we easily derive that

$$\begin{aligned}\overline{\lim_{n\to\infty}}\,\gamma(S^n)^{1/n} &\le \overline{\lim_{n\to\infty}}\left(\inf_{\|x\|=1}\left\|\begin{pmatrix} B & * \\ 0 & C\end{pmatrix}^n \begin{pmatrix} 0 \\ x\end{pmatrix}\right\|\right)^{1/n} \\ &\le \lim_{n\to\infty}\left\|\begin{pmatrix} B & * \\ 0 & \mu\end{pmatrix}^n\right\|^{1/n} \le \left|\begin{pmatrix} B & * \\ 0 & \mu\end{pmatrix}\right|_{\mathrm{sp}} \\ &\le \sup_{\zeta\in\sigma_1}|\zeta|\end{aligned}$$

and this is a contradiction. If C acts on a $\{0\}$-space we trivially have $\sigma_1' \subset \rho_l(C)$, thus in any case we have $\sigma_1' \subset \rho_l(C)$ and analogously $\sigma_1' \subset \rho_r(A)$.

Applying the results of [6, §2] we can find T_1, T_2 such that T is similar with $T_1 \oplus T_2$, $\sigma(T_1) \subset \sigma_1'$, $0 \in \rho_\gamma(T_2)$. But because we assumed $r > \sup_{\zeta\in\sigma}|\zeta|$ we can choose σ_1 such that $r > \sup_{\zeta\in\sigma_1}|\zeta|$. Thus if $T_1^n \neq 0$ $(\forall n)$, applying Lemma 3.1 we derive the contradiction

$$r = \overline{\lim_{n\to\infty}}\,\gamma((T_1\oplus T_2)^n)^{1/n} \le |T_1|_{\mathrm{sp}} \le \sup_{\zeta\in\sigma_1}|\zeta|.$$

This shows that T_1 is nilpotent and we can take $Q = T_1$, $T' = T_2$. □

3.4. COROLLARY. *Suppose* $0 \in \sigma_\gamma(T)$ *and the connected component of* $\sigma_\gamma(T)$ *containing* 0 *is a singleton. Then* $\lim_{n\to\infty}\gamma(T^n)^{1/n}$ *exists and, if strictly positive, coincides with the radius of the largest open disk centered at* 0 *and included in* $\rho_\gamma(T) \cup \{0\}$.

Proof. If $\overline{\lim}_{n\to\infty}\gamma(T^n)^{1/n} > 0$ we apply Proposition 3.3 and Theorem 3.2 to derive the existence and the significance of $\lim_{n\to\infty}\gamma(T^n)^{1/n}$. If $\overline{\lim}_{n\to\infty}\gamma(T^n)^{1/n} = 0$, then $\lim_{n\to\infty}\gamma(T^n)^{1/n}$ trivially exists. □

REMARK. If T is semi-Fredholm then $\lim_{n\to\infty}\gamma(T^n)^{1/n}$ exists and coincides with the radius of the largest open disk centered at 0 and included in $\rho_{s\text{-}F}^r(T) \cup \{0\}$. In this case T is similar to $Q \oplus T'$ as in Proposition 3.3 (see [3, Theorem 3.3]).

3.5. THEOREM. *The set* $\{\mu \in \sigma_\gamma(T) : \overline{\lim}_{n\to\infty}\gamma((T-\lambda)^n)^{1/n} > 0\}$ *is at most countable.*

Proof. If we put

$$\sigma = \{\mu \in \sigma_\gamma(T) : \overline{\lim_{n\to\infty}}\,\gamma((T-\mu)^n)^{1/n} > 0\},$$

$$\sigma_{n,m} = \{\mu \in \sigma_\gamma(T) : \gamma((T-\mu)^n) > 1/m\}, \quad n, m \ge 1,$$

we have $\sigma \subset \bigcup_{n,m}\sigma_{n,m}$. By Corollary 1.2 we know that $\sigma_{n,m}$ is closed and the function $\mu \to P_{\ker(T-\mu)^n}$, $\mu \in \sigma_{n,m}$, is continuous. Suppose that $\sigma_{n,m}$ has an accumulation point $\mu_{n,m}$ and put $H_{n,m} = \mathrm{clm}_{\lambda\neq\mu_{n,m}}\ker(T-\lambda)$. It is plain that we have

$$(T-\mu_{n,m})^n H_{n,m} = H_{n,m}, \quad \ker(T-\lambda) \subset H_{n,m}, \quad \lambda \neq \mu_{n,m},$$

and this easily implies that $\mu_{n,m} \in \rho_\gamma(T)$. This contradiction shows that $\sigma_{n,m}$ is finite and concludes the proof. □

As seen before, $\lim_{n\to\infty} \gamma(T^n)^{1/n}$ exists and has a precise spectral meaning if either $0 \in \rho_\gamma(T)$ or 0 is an isolated point of $\sigma_\gamma(T)$. In general the sequence $\{\gamma(T^n)^{1/n}\}_{n=1}^\infty$ can be spread all over the closed interval $[0, \|T\|]$, as seen in the following example.

EXAMPLE. Let T_n, $n \geq 1$, be the $(n+1)\times(n+1)$ matrix operator defined in $\underbrace{H\oplus\cdots\oplus H}_{(n+1)\text{ times}}$ by the equations

$$T_1 = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \qquad T_n = \begin{bmatrix} 0 & A_n & 0 & 0 & \cdots & 0 \\ 0 & 0 & I & 0 & \cdots & 0 \\ 0 & 0 & 0 & I & 0 \,.. & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & & & 0 & B_n \\ 0 & \cdots & & & 0 & 0 \end{bmatrix}, \quad n \geq 2,$$

where $\|A\| \leq 1$ and A_n, B_n are self-adjoint projections. If we put $S = \bigoplus_{n=1}^\infty T_n$, then it is easy to see that we have

$$\gamma(S) = \gamma(A), \quad \gamma(S^n) = \gamma(A_n B_n), \quad n \geq 2.$$

The numbers $\{\gamma(S^n)\}$ can be prescribed in $[0,1]$ if we choose A, A_n, B_n properly, thus the set of limit points of the sequence $\{\gamma(S^n)^{1/n}\}$ can be made compact. Proposition 3.3 shows that if $0 \in \sigma(T)$ and $\overline{\lim}_{n\to\infty} \gamma(T^n)^{1/n} > 0$ then the connected component of $\sigma(T)$ containing 0 is not a singleton. Is 0 an interior point of $\sigma(T)$ in this case? Equivalently, we can formulate the following:

QUESTION 1. If $0 \in \partial\sigma(T)$, must $\overline{\lim}_{n\to\infty} \gamma(T^n)^{1/n} = 0$ (or $\underline{\lim}_{n\to\infty} \gamma(T^n)^{1/n} = 0$)?

We do not know if the function $\overline{\lim}_{n\to\infty} \gamma((T-\lambda)^n)^{1/n}$ behaves as Proposition 2.1 suggests.

QUESTION 2. Does

$$\lim_{\lambda\to\mu} \overline{\lim_{n\to\infty}} \gamma((T-\lambda)^n)^{1/n} \quad \left(\text{or} \lim_{\lambda\to\mu} \underline{\lim_{n\to\infty}} \gamma((T-\lambda)^n)^{1/n}\right)$$

exist for every $\mu \in \mathbf{C}$?

4. Compact perturbation. Throughout this section we shall assume that H is infinite-dimensional.

Let $\tilde{T}$ denote the image of T in the Calkin algebra $\mathcal{L}(H)/\mathcal{K}(H)$. Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{L}(H)/\mathcal{K}(H)$ containing $\tilde{T}$ and $\tilde{I}$ (where I denotes the identity operator in H) and let $\phi: \mathcal{A} \to \mathcal{L}(H')$ be a faithful *-representation, where H' is a complex Hilbert space. It is easy to see that $\gamma(\phi(\tilde{T}))$ and $\sigma_\gamma(\phi(\tilde{T}))$ do not depend on $\mathcal{A}$ or ϕ and that we have

$$\gamma(\phi(\tilde{T}))=\inf(\sigma((T^*T)^{1/2})\setminus\{0\}) \quad \text{if } \tilde{T}\neq 0.$$

This allows us to define $\sigma_\gamma(\tilde{T})$, $\rho_\gamma(\tilde{T})$ by

$$\sigma_\gamma(\tilde{T})=\sigma_\gamma(\phi(\tilde{T})), \qquad \rho_\gamma(\tilde{T})=\mathbf{C}\setminus\sigma_\gamma(\tilde{T}).$$

Since obviously $\gamma(\tilde{T}-\lambda)\geq\gamma(T-\lambda)$, $\lambda\in C$, by applying Proposition 2.1 we derive the inclusion $\rho_\gamma(T)\subset\rho_\gamma(\tilde{T})$. We shall use further the following notation.

$$\rho_{s\text{-}F}(T)=\{\lambda\in\mathbf{C}: T-\lambda \text{ is semi-Fredholm}\},$$

$$\rho_{s\text{-}F}^r(T)=\{\lambda\in\rho_{s\text{-}F}(T): \lambda \text{ is } T\text{-regular}\} \quad ([3,\ \S2]),$$

$$\rho_{s\text{-}F}^s(T)=\{\lambda\in\rho_{s\text{-}F}(T): \lambda \text{ is } T\text{-singular}\} \quad ([3,\ \S2]),$$

$$\sigma_p^0(T)=\{\lambda\in\sigma(T)\cap\rho_{s\text{-}F}(T): \lambda \text{ is isolated in } \sigma(T)\} \quad ([3,\ \S2]),$$

$$\sigma_l(T)=\text{the left spectrum of } T,\ \rho_l(T)=\mathbf{C}\setminus\sigma_l(T),$$

$$\sigma_r(T)=\text{the right spectrum of } T,\ \rho_r(T)=\mathbf{C}\setminus\sigma_r(T),$$

$$\mathcal{S}_m(H)=\{S\in\mathcal{L}(H): \rho_{s\text{-}F}(S)=\rho_l(S)\cup\rho_r(S)\} \quad ([3,\ \S4]).$$

Any operator $S\in\mathcal{S}_m(H)$ will be called smooth.

Let $K\in\mathcal{K}(H)$ be given. Using Lemma 1.4 we easily derive

$$\rho_{s\text{-}F}^r(T+K)\subset\rho_\gamma(T+K), \qquad \rho_{s\text{-}F}(T)\setminus\rho_\gamma(T+K)=\rho_{s\text{-}F}^s(T),$$

and in general $\rho_\gamma(T+K)\neq\rho_\gamma(T)$. In the sequel we shall show that we can choose K such that $\rho_\gamma(T+K)=\rho_{s\text{-}F}(T)$ and the function γ_{T+K} is continuous (see Theorem 4.4 below).

4.1. LEMMA. *Suppose that we have*

$$\dim\ker T=\infty, \quad \dim(\ker T^{m+1}\ominus\ker T^m)<\infty, \quad \textit{for some } m\geq 1,$$

and let $\epsilon>0$ *be given. Then there exists* $K\in\mathcal{K}(H)$ *such that*

$$\|K\|<\epsilon, \quad \gamma((T+K)^n)=0, \quad \textit{for all } n\geq 1,$$

$$\sigma_l(T+K)=\sigma_l(T), \qquad \sigma_r(T+K)=\sigma_r(T).$$

Proof. Without loss of generality we may suppose that we have

$$\dim(\ker T^{k+1}\ominus\ker T^k)=\infty, \quad 0\leq k<m.$$

If H is not separable we can find a separable subspace $H'\subset H$ such that H' reduces T and $T'=T\,|\,H'$ verifies the hypothesis of our lemma. This allows us to assume that H is separable.

The decomposition $H=\sum_{j=1}^{m+1}H_j$, where

$$H_j=\ker T^j\ominus\ker T^{j-1}, \quad 1\leq j\leq m, \quad H_{m+1}=(\ker T^m)^\perp,$$

determines the matrix representation

$$T=\begin{bmatrix} 0 & T_{1,2} & \cdots\cdots\cdots & & \\ 0 & 0 & T_{2,3} & \cdots\cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots\cdots & & 0 & T_{m,m+1} \\ 0 & \cdots\cdots & & 0 & T_{m+1,m+1} \end{bmatrix}$$

with $\dim\ker T_{m+1,m+1}<\infty$. Because H is assumed separable we can choose $K\in\mathcal{K}(H)$ of the form

$$K=\begin{bmatrix} K_1 & 0 & 0 & & 0 \\ 0 & K_2 & 0 & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & K_m & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, \quad \sigma(K)=\{0\},$$

with $\|K_j\|<\epsilon$, $\ker K_j=\ker K_j^*=\{0\}$, $1\leq j\leq m$. Suppose that $\gamma(T+K)>0$ and put

$$H_0=\left(\sum_{j=1}^{m} H_j\right)+\ker T_{m+1,m+1}, \qquad T_0=(T+K)\,|\,H_0 \quad (T_0\in\mathcal{L}(H_0,H)).$$

Since $\ker(T+K)\subset H_0$ and $\dim\ker(T+K)<\infty$, we conclude that T_0 is semi-Fredholm, an obvious contradiction. This implies $\gamma(T+K)=0$ and, analogously, we have $\gamma((T+K)^n)=0$, $n>1$. To prove the relations

$$\sigma_l(T+K)=\sigma_l(T), \qquad \sigma_r(T+K)=\sigma_r(T)$$

is an easy exercise. □

4.2. LEMMA. *Suppose* $\dim\ker T^m=\infty$ *and* $\dim\ker T^{*m}=\infty$ *for some fixed* $m\geq 1$, *and put* $S=T\oplus T$. *Let* $\epsilon>0$ *be given. Then there exists* $K\in\mathcal{K}(H\oplus H)$ *such that*

$$\|K\|<\epsilon, \qquad \gamma((S+K)^m)=0,$$
$$\sigma_l(S+K)=\sigma_l(T), \qquad \sigma_r(S+K)=\sigma_r(T).$$

Proof. Using Lemma 4.1 we can easily reduce to the case

$$\dim(\ker T^j\ominus\ker T^{j-1})=\infty, \qquad \dim(\ker T^{*j}\ominus\dim\ker T^{*(j-1)})=\infty, \quad 1\leq j\leq m.$$

Choose $K_{1,2}\in\mathcal{K}(H)$ such that

$$\|K_{1,2}\|<\epsilon, \qquad (\ker K_{1,2})^\perp\subset(T^{m-1}\ker T^m)^-,$$
$$\operatorname{rank} K_{1,2}=\infty, \qquad K_{1,2}H\subset(T^mH)^\perp\cap(T^{m-1}H)^-.$$

If we put

$$K=\begin{pmatrix} 0 & K_{1,2} \\ 0 & 0 \end{pmatrix},$$

then we have

$$S^m = \begin{pmatrix} T^m & \sum_{j=1}^m T^{m-j}K_{1,2}T^{j-1} \\ 0 & T^m \end{pmatrix}.$$

If $x \oplus y \in \ker S^m$ we have

$$P_{(T^mH)^\perp}\left(\sum_{j=1}^m T^{m-j}K_{1,2}T^{j-1}y\right) = K_{1,2}T^{m-1}y = 0, \quad T^m y = 0,$$

and this implies that

$$\ker S^m \subset H \oplus \ker(K_{1,2}T^{m-1} \mid \ker T^m).$$

Now, putting

$$X = \{0\} \oplus (\ker T^m \ominus \ker(K_{1,2}T^{m-1} \mid \ker T^m)),$$

we have $X \subset (\ker S^m)^\perp$, $\operatorname{rank}(S^m \mid X) = \infty$ (in view of $\operatorname{rank} K_{1,2} = \infty$), and $S^m \mid X$ is compact. This shows that $S^m(H \oplus H)$ is not closed; consequently, $\gamma(S^m) = 0$. Since the spectral relations in the statement are obvious, the proof is concluded. □

4.3. LEMMA. *Suppose* $T \in \mathcal{S}_m(H)$, $\rho_\gamma(T) = \rho_{s\text{-}F}(T)$, *and let* $\epsilon > 0$ *be given. Then there exists* $T' \in \mathcal{S}_m(H)$ *such that*

$$T' - T \in \mathcal{K}(H), \qquad \|T' - T\| < \epsilon, \qquad \gamma((T' - \lambda)^n) = 0,$$

for all $\lambda \notin \rho_{s\text{-}F}(T)$, $n \geq 1$.

Proof. Let us put

$$\sigma = \{(\lambda, n) \in \sigma_\gamma(T) \times \mathbf{N} : \gamma((T - \lambda)^n) > 0\}.$$

We know by the proof of Theorem 3.5 that σ is at most countable and to avoid trivial situations we assume $\sigma \neq \emptyset$. We also assume that H is separable, otherwise we proceed as in the proof of Lemma 4.1.

Let $\mathfrak{A}$ denote the C^*-algebra generated by T, I, and $\mathcal{K}(H)$, and let $\phi : \tilde{\mathfrak{A}} \to \mathcal{L}(H)$ be a representation of infinite multiplicity. For every $(\lambda, n) \in \sigma$ fix $\epsilon_{(\lambda,n)} > 0$ such that $\sum_{(\lambda,n)\in\sigma} \epsilon_{(\lambda,n)} = \eta < \epsilon$ and put $T_{(\lambda,n)} = \phi(\tilde{T}) \oplus \phi(\tilde{T})$. Since ϕ has infinite multiplicity we easily derive that $\phi(\tilde{T}) \in \mathcal{S}_m(H)$ and

$$\dim\ker(\phi(\tilde{T}) - \lambda)^n = \dim\ker(\phi(\tilde{T}) - \lambda)^{*n} = \infty, \quad \text{for} \quad (\lambda, n) \in \sigma.$$

Thus, applying Lemma 4.2, we can find $K_{(\lambda,n)} \in \mathcal{K}(H \oplus H)$ such that

$$\|K_{(\lambda,n)}\| < \epsilon_{(\lambda,n)}, \qquad T_{(\lambda,n)} + K_{(\lambda,n)} \in \mathcal{S}_m(H \oplus H), \qquad \gamma((T_{(\lambda,n)} + K_{(\lambda,n)} - \lambda)^n) = 0.$$

If we put

$$A = \bigoplus_{(\lambda,n)\in\sigma} (T_{(\lambda,n)} + K_{(\lambda,n)}), \quad B = T \oplus A$$

we obviously have

$$\gamma((B - \lambda)^n) = 0, \quad \forall \lambda \notin \rho_{s\text{-}F}(T), \ n \geq 1, \quad \rho_{s\text{-}F}(B) = \rho_{s\text{-}F}(T).$$

Let $\mu \in \rho_l(T)$ be given and assume $\mu \in \sigma_l(B)$. Then we derive $\mu \in \sigma_l(A) \cap \rho_{s\text{-}F}(A)$ and we can find $(\lambda, n) \in \sigma$ such that $\mu \in \sigma_p(T_{(\lambda,n)} + K_{(\lambda,n)})$, which is a contradiction because we have

$$\sigma_l(T_{(\lambda,n)}+K_{(\lambda,n)})=\sigma_l(\phi(\tilde{T}))\subset\sigma_l(T).$$

This implies that $\rho_l(T)\subset\rho_l(B)$ and, analogously, $\rho_r(T)\subset\rho_r(B)$, whence we derive

$$\rho_{s\text{-}F}(B)=\rho_{s\text{-}F}(T)\subset\rho_l(B)\cup\rho_r(B)\subset\rho_{s\text{-}F}(B),$$

or equivalently, B is smooth.

Now put $B'=T\oplus\bigoplus_{(\lambda,n)\in\sigma}T_{(\lambda,n)}$ and apply [12, Theorem 1.3] to produce a unitary operator U such that

$$U^*B'U-T\in\mathcal{K}(H),\qquad \|U^*B'U-T\|+\eta<\epsilon.$$

Taking $T'=U^*BU$ we have

$$T'-T=U^*(B-B')U+U^XB'U-T\in\mathcal{K}(H),\qquad \|T'-T\|<\epsilon.$$

The other properties of T' required by the statement of our lemma follow from the fact that T' is unitarily equivalent with B.

4.4. THEOREM. *There exists $K\in\mathcal{K}(H)$ such that*

$$T+K\in\mathcal{S}_m(H)\quad \textit{and}\quad \gamma((T+K-\lambda)^n)=0,\quad \textit{for all } \lambda\notin\rho_{s\text{-}F}(T)\ \textit{and}\ n\geq 1.$$

If moreover $\sigma_p^0(T)=\varnothing$ we may suppose that $\|K\|$ is arbitrarily small.

Proof. Using [3, Theorem 4.5], we reduce the proof to the case $T\in\mathcal{S}_m(H)$. Proceeding as before, we shall also assume that H is separable.

Let $\mathcal{A},\phi$ be as in the proof of Lemma 4.3 and let $\{\lambda_n\}_{n=1}^{\infty}$ be dense in $\sigma(T)\setminus\rho_{s\text{-}F}(T)$. If we choose $K_n\in\mathcal{K}(H\oplus H)$ such that $\sum_{n=1}^{\infty}\|K_n\|<\epsilon$, and if we put

$$T_n=\phi(\tilde{T})\oplus\phi(\tilde{T}),\qquad A=\bigoplus_{n=1}^{\infty}(T_n+K_n),\qquad B=T\oplus A,$$

we may suppose (via Lemma 4.2) that

$$\gamma(T_n+K_n-\lambda_n)=0,\qquad \sigma_l(T_n+K_n)=\sigma_l(\phi(\tilde{T})),\qquad \sigma_r(T_n+K_n)=\sigma_r(\phi(\tilde{T})).$$

Arguing as in the proof of Lemma 4.3 we derive that B is smooth and $U^*BU-T\in\mathcal{K}(H)$, $\|U^*BU-T\|<\epsilon$ for some unitary operator U; thus we may assume $T=U^*BU$. Since we also have $\gamma(B-\lambda_n)\leq\gamma(T_n+K_n-\lambda_n)=0$ and $\{\lambda_n\}_{n=1}^{\infty}$ is dense in $\sigma(T)\setminus\rho_{s\text{-}F}(T)$, we infer

$$\rho_\gamma(T)=\rho_\gamma(B)=\rho_{s\text{-}F}(T),$$

and this allows us to apply Lemma 4.3. □

REMARK. Theorem 4.4 shows that there exists a compact perturbation K such that γ_{T+K} is continuous, $\rho_\gamma(T+K)=\rho_{s\text{-}F}(T)$, and $\sigma_\gamma(T+K)$ is the zero set of γ_{T+K}. As we have observed at the beginning of this section, we have $\rho_\gamma(T+K)\subset\rho_\gamma(\tilde{T})$. We do not know the answer to the following two questions:

QUESTION 3. Can we choose $K\in\mathcal{K}(H)$ such that $\rho_\gamma(T+K)=\rho_\gamma(\tilde{T})$?

QUESTION 4 (weakened version). Is $\bigcup_{K\in\mathcal{K}(H)}\rho_\gamma(T+K)=\rho_\gamma(\tilde{T})$?

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