SMOOTH S^1 ACTIONS ON HOMOTOPY CP^4'S

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Introduction. If the circle group S^1 acts smoothly on a $2n$-manifold P which is homotopy equivalent to the complex projective n-space CP^n, then the tangent and Hopf bundle fibers over fixed points give a set of S^1 representations. For $n=3$ Petrie constructed smooth S^1 actions which do not have the fixed point representations of linear actions ([6, II§4] and also [4, II§8]; a *linear* action on CP^n is one given in homogeneous coordinates by $t[z_0:\dots:z_n]=[t^{a_0}z_0:\dots:t^{a_n}z_n]$ for some $a_0,\dots,a_n \in Z$). For $n=4$, Theorem 4.1 of this paper has the following corollary.

THEOREM 1. *If P is a homotopy CP^4 which has a smooth S^1 action, then there is a linear S^1 action on CP^4 with the same fixed point tangent and Hopf bundle representations as those of the action on P.*

In particular, as pointed out by J. Shaneson, Petrie's exotic actions on CP^3 do not extend smoothly to CP^4.

Petrie conjectured that if P is a homotopy CP^n which admits a nontrivial S^1 action then $\hat{A}(P)=\hat{A}(CP^n)$ in $H^*(P;Q)$ ([6, p. 105]). This has been verified for $n=3$ by Dejter [2] and for various fixed-point set conditions by Wang [8], Tsukada and Washiyama [7], and Masuda [5]. Hattori [3, Prop. 4.15] has shown it for quasilinear actions (Definition 1B of this paper), which together with Theorem 1 yields the following.

THEOREM 2. *If P is a homotopy CP^4 which has a nontrivial smooth S^1 action then $\hat{A}(P)=\hat{A}(CP^4)$.*

In this paper the definition of Petrie's ψ polynomials and some of his results on them are quoted (§1) and the possibilities for these polynomials for homotopy CP^4's are restricted by using properties of stationary sets of subgroups of S^1 (§2 and §3), ultimately leaving quasilinearity as the only possibility (§4). Throughout, the real dimension of a manifold M is denoted by $\dim M$, the set of points of M fixed by $G\subset S^1$ by $F(M,G)$, and the order of the largest subgroup of S^1 which fixes all points of M by $|\mathrm{Stab}\, M|$; $N\subset M$ indicates that N is a smoothly and equivariantly embedded submanifold of M, with normal bundle $\nu(N,M)$.

1. Let P be a homotopy CP^n with an effective smooth S^1 action. Choose a lifting of the action to the Hopf bundle η over P so that η is an S^1 equivariant vector bundle (see [6, II, Prop. 1.1]). Let $P_0,\dots,P_k$ be the components of $F(P,S^1)$. For each P_i, $0\le i\le k$, there are elements $\eta\,|\,x=t^{a_i}$ and $\nu(P_i,P)\,|\,x=\sum_{j=1}^{n-m_i} t^{b_{ij}}$ of $R(S^1)=Z[t,t^{-1}]$, where $m_i=\frac{1}{2}\dim P_i$ and x is any point of P_i.

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DEFINITION 1A. (Notation as above.) For $x \in P_i$,

$$\psi(x,t)=\prod_{j\neq i}(1-t^{a_i-a_j})^{m_j}\prod_{k=1}^{n-m_i}(1-t^{b_{ik}})^{-1}.$$

Petrie defined these polynomials and established their major properties, some of which we quote as the following lemma (see [6, II§2]; also [2, p. 86]).

LEMMA 1.1. *Let* $x \in P_i \subset F(P, S^1)$.

(i) $\psi(x,t) \in R(S^1)$.

(ii) *For* $m \in Z$ *let* $n(m)$ *and* $d(m)$ *be the number of numerator exponents and denominator exponents, respectively, of* $\psi(x,t)$ *which are integral multiples of* m. *Then* $n(m) \geq d(m)$ *and if* m *is a prime power then* $n(m)=d(m)$.

(iii) $|\psi(x,1)|=1$ *and consequently* $\prod_{j\neq i}|a_i-a_j|=\prod_{k=1}^{n-m_i}|b_{ik}|$.

If P is the standard CP^n and the action is linear then each $\psi(x,t)$ is a unit in $R(S^1)$, as is clear from homogeneous coordinates. However, the ψ polynomials of Petrie's exotic actions on CP^3 are not units ([6, pp. 150–151]). This leads to the following definition (cf. [2, p. 87], [5, p. 131]).

DEFINITION 1B. An S^1 action on P is *quasilinear* if $\psi(x,t)$ is a unit in $R(S^1)$ for each $x \in F(P, S^1)$.

We will be using invariant submanifolds of P to relate the two sets of representations in the ψ polynomials. We quote a useful result of Petrie ([6, p. 135–137]).

LEMMA 1.2. *Let* $x, y \in F(P, S^1)$ *have Hopf bundle fiber representations* t^a, t^b *respectively. If* m *is a prime power then* x *and* y *are in the same component of* $F(P, Z_m)$ *if and only if* $a \equiv b (\text{mod } m)$.

The simplest nontrivial S^1 manifolds are 2-spheres. A nontrivial smooth S^1 action on S^2 clearly has exactly two fixed points and exactly one nonfixed orbit type, so that it is given by $t[z_0, z_1]=[z_0, t^m z_1]$, for a unique positive integer m, in some system of homogeneous coordinates on $S^2=CP^1$.

DEFINITION 1C. For $m \in Z$, $S^2(m)$ denotes an S^1 manifold which is equivariantly diffeomorphic to CP^1 with the S^1 action $t[z_0, z_1]=[z_0, t^m z_1]$.

This notation includes the trivial action, as $S^2(0)$, and the semifree action, as $S^2(1)$ or $S^2(-1)$. Clearly $S^2(m)$ and $S^2(n)$ are equivariantly diffeomorphic if and only if $|m|=|n|$. As usual an $S^2(m)$ in another manifold will be assumed smoothly and equivariantly embedded.

LEMMA 1.3. *Let* x, y *be two* S^1 *fixed points in an* $S^2(m) \subset P$ *which carries* k *times a generator of* $H_2(P; Z)$. *If the Hopf representations are* $\eta \mid x=t^a$ *and* $\eta \mid y=t^b$, *then* $a-b=km$ *and for* $m \neq 0$ *each of* $\psi(x,t), \psi(y,t)$ *contains* $(1-t^{km})/(1-t^m)$ *times a unit of* $R(S^1)$.

Proof. Chern class calculations imply that $\eta \mid S^2(m)$ has underlying vector bundle ξ^k, the k-fold tensor product of the Hopf bundle ξ over S^2. Now ξ and hence ξ^k can be made into S^1 equivariant bundles over $S^2(m)$ by lifting the homogeneous coordinate action, and then (switching x and y if necessary) $\xi^k \mid x = t^0 = 1$ and $\xi^k \mid y = t^{km}$. Since ξ^k and $\eta \mid S^2(m)$ have the same underlying bundle, by [6, I, Thm. 6.1] the first is the tensor product of the second by a unit in $R(S^1)$. The only unit which gives the right representation at x is t^{-a}, so $\xi^k = t^{-a} \otimes \eta \mid S^2(m)$, whence $\xi^k \mid y = t^{km} = t^{b-a}$. This proves the first statement, and the second follows straightforwardly using Definition 1A. □

LEMMA 1.4. *If x_1 and x_2 are points in different S^1 fixed point set components of a connected smooth S^1 manifold M with principal orbit type S^1/Z_m, and each of the tangent bundle representations $\tau M \mid x_i$, $i = 1, 2$, contains a t^m or t^{-m} component, then M contains an $S^2(m)$ which contains x_1 and x_2.*

Proof. The $S^2(m)$ can be constructed straightforwardly using, for example, [1, IV§3, VI§2].

2. Henceforth P is assumed to be a homotopy CP^4, so that there are at most five components $P_0, \ldots, P_k$ of $F(P, S^1)$ and $\sum_{i=0}^{k} (\frac{1}{2} \dim P_i + 1) = 5$ ([1, VII, Thm. 5.1]).

Choose and label five distinct points $x_0, \ldots, x_4$ so that for $0 \le i \le k$ exactly $\frac{1}{2} \dim P_i + 1$ of them are in P_i. (These are to be chosen once for all, but the labels—i.e., the subscripts—will be permuted as convenient.)

Henceforth, for $0 \le i \le 5$, a_i is the exponent of the Hopf fiber representation $\eta \mid x_i$, and $\psi(x_i, t)$ will be abbreviated $\psi_i(t)$.

The following definition is motivated by the fact that the standard CP^2 contains three nicely embedded (in homogeneous coordinates) 2-spheres which determine the fixed point representations of a linear action.

DEFINITION 2. A component L of $F(P, G)$, G a subgroup of S^1, will be called *standard* if $\dim L = 4$, the rational Euler characteristic $\chi(L) = 3$, and either $G = S^1$ or else L contains three distinct $S^2(m)$'s (for up to three values of m), each of which carries a generator of $H_2(P; Z)$ and contains exactly two of the $\{x_i\}$, and any two of which intersect in exactly one point.

LEMMA 2.1. *If x_i is in a standard submanifold of P and $\psi_i(t)$ is not a unit, then there are positive integers a, b and r, with $a > 1$, $b > 1$ and $(a, b) = 1$, such that $\psi_i(t)$ is equal to $(1 - t^{abr})(1 - t^r)/[(1 - t^{ar})(1 - t^{br})]$ times a unit in $R(S^1)$.*

Proof. This follows straightforwardly from Definition 1A, Definition 2, Lemma 1.3, and Lemma 1.1. □

The next two lemmas show that certain submanifolds of P are standard.

LEMMA 2.2. *If* $\dim L = 4$, $\chi(L) = 3$, *$F(L, S^1)$ contains nonisolated points, and L is a component of $F(P, Z_n)$ for $Z_n \subset S^1$, then L is standard.*

Proof. By [1, VII, Thm. 5.1] and the hypotheses, $F(L, S^1) \subset F(P, S^1)$ is either a homotopy CP^2—in which case it is all of L, which is thereby standard—or an S^2 and a single point x. In this case L has only one non-fixed orbit type, so by Lemma 1.4 it contains two $S^2(r)$'s, $r = |\mathrm{Stab}\, L|$, each of which contains x and one of the two labeled points in the S^1 fixed $S^2 = S^2(0)$. These carry generators of $H_2(P; Z)$ by [1, VII, Thm. 5.1] for the $S^2(0)$ and by this and intersection class considerations for the $S^2(r)$'s. □

LEMMA 2.3. *If* $\dim L = 4$ *and* L *is a component of* $F(P, Z_p)$ *for* $Z_p \subset S^1$ *and* p *a prime power, then* L *is standard.*

Proof. [1, VII, Thm. 3.1] implies that $\chi(L) = 3$, so that the conclusion follows from Lemma 2.2 if $F = F(L, S^1)$ contains nonisolated points. Otherwise F consists of three isolated points, and the required three $S^2(m)$'s are obtained in the obvious way from the components of $\tau(L) \mid x_i$ for $x_i \in F$—that is, if $\tau(L) \mid x_i = t^a + t^b$ then x_i is in an $S^2(a)$ and an $S^2(b)$ in L. Two such 2-spheres with nonprincipal orbit types clearly intersect in (at least) one point of F, and cannot intersect in two: if L contained $S^2(ar)$, $S^2(br)$ with $a > 1$, $b > 1$, $r = |\mathrm{Stab}\, L|$ (so that $(a, b) = 1$) and $S^2(ar) \cap S^2(br) = \{x_i, x_j\} \subset F$, then intersection class considerations would imply that one, say $S^2(ar)$, carried a generator of a Z summand of $H_2(L; Z)$, and then (because of its Chern class) $\nu = \nu(S^2(ar), L)$ would be the Hopf bundle or its dual, so that by Lemma 1.3 the exponents of $\nu \mid x_i, \nu \mid x_j$—clearly some combination of $\pm br$—would differ by ar, contradicting $(a, b) = 1$. It follows that principal orbit (i.e., Lemma 1.4) spheres, if any, can be arranged consistently with Definition 2 also. To show, finally, that these 2-spheres carry generators of $H_2(P; Z)$, let one carry $k[P] \cap c^3$ where $[P] \in H_4(P; Z)$ is the fundamental class and c generates $H^*(P; Z)$. Then L carries $\pm k^2[P] \cap c^2$, and $(p, k) = 1$ by [1, VII, Thm. 3.1]. Suppose $|k| > 1$. Let q be the greatest power dividing $k|\mathrm{Stab}\, L|$ of a prime dividing k. Since $k|\mathrm{Stab}\, L|$, and hence q, divides $a_i - a_j$ for each $x_i, x_j \in F$ (Lemma 1.3 and the foregoing), by Lemma 1.2 F is contained in a component K of $F(P, Z_q)$. Then $\dim K \geq 4$. Since Z_q can fix at most one of the $S^2(m)$'s in L ($q \nmid |\mathrm{Stab}\, L|$) and so moves at least two, $K \cap L$ contains at least one isolated point. Hence $\dim K = 4$ and K also contains three 2-spheres, each of which carries $\pm h[P] \cap c^3$ for some h prime to q. Thus $K \cap L = F$, which implies using intersection classes that $3 \geq |[K] \cdot [L]| = h^2 k^2$. Hence $|k| = 1$ after all, and the proof is complete. □

The following lemma prepares for Lemma 2.5, which, with Lemma 3.5, limits the possibilities for nonquasilinear actions to what are in effect extensions of Petrie's actions on CP^3.

LEMMA 2.4. *If the action on* P *is not quasilinear and no* $F(P, Z_p)$ *contains a six-dimensional component for any prime power* p, *then* $\psi_i(t)$ *is a unit for at least four values of* i.

Proof. We may assume that $\psi_1(t)$ is not a unit. Then it follows from Lemma 1.1 that the denominator of $\psi_1(t)$ contains terms $1 - t^m$ with $|m| > 1$, and from the second hypothesis and Lemma 2.3 that each such t^m determines an $S^2(m)$ in P containing x_1. If all such 2-spheres carried generators of $H_2(P; Z)$ then $\psi_1(t)$ would be a unit by Lemma 1.3, so for some $a > 1$, x_1 is in an $S^2(a)$ carrying $k > 1$ times a generator of $H_2(P; Z)$. Let x_2 be in this $S^2(a)$ also, so that $|a_1 - a_2| = ak$ (Lemma 1.3), and then Lemma 1.2 implies that x_1 and x_2 are in an $S^2(b)$ with

$b \mid k$ and $b>1$. Hence for $h=k/b$ each of $\psi_1(t), \psi_2(t)$ contains a unit times $(1-t^{abh})/[(1-t^a)(1-t^b)]$, and Lemma 1.1 implies that each of these numerators contains $1-t$ also. Assign subscripts 3, 4 so that $|a_1-a_4|=|a_2-a_3|=1$. If $m=|a_1-a_3|$ and $n=|a_3-a_4|$ then clearly $m=abh\pm 1$, $n\equiv \pm 1 (\bmod m)$, and neither a nor b is congruent to 1 mod m. Now if $\psi_3(t)$ were a unit, there would be $S=S^2(m)\subset P$ containing x_1 and x_3, and since $S\subset F(P, Z_m)$ the S^1 representations $\nu(S,P)\mid x_1=t^a+t^b+t^c$, $\nu(S,P)\mid x_3=t+t^n+t^d$ (up to signs of exponents, some $c, d\in Z$) would pass to the same element of $R(Z_m)=R(S^1)/(1-t^m)$—that is, $\{a,b,c\}$ and $\{\pm 1, n, d\}=\{\pm 1, \pm 1, d\}$ would be the same subset of Z_m. As this is not the case, $\psi_3(t)$ and, by a similar argument, $\psi_4(t)$ are not units. □

LEMMA 2.5. *Under the hypotheses of Lemma* 2.4, *for an appropriate subscripting* $\psi_0(t)$ *is a unit and* $\psi_i(t)$, $1\le i\le 4$, *is a unit times*

$$(1-t^{ab})(1-t)/[(1-t^a)(1-t^b)]$$

for coprimes $a>1$, $b>1$.

Proof. By [1, VII, Thm. 3.1] and the hypotheses, $F(P, Z_2)=S\cup L$ where S is a 2-sphere and $\dim L=4$ and hence (Lemma 2.3) L is standard. Let $x_0, x_1, x_3\in L$, so that $x_2, x_4\in S$ and a_2-a_4 is even. By Lemma 2.4 we may assume that $\psi_1(t)$ and $\psi_3(t)$ are not units. Lemma 2.1 gives $a, b, r, \lambda\in Z$ with $a>1$, $b>1$, $(a,b)=1$, $r\ge 1$, such that $\psi_1(t)=t^\lambda(1-t^{abr})(1-t^r)/[(1-t^{ar})(1-t^{br})]$. Then $r=1$ by the hypotheses and Lemma 2.3, and $L\subset F(P, Z_2)$ implies that a and b are odd. Hence the numerator exponents ab, 1 of $\psi_1(t)$ may be taken to be $|a_1-a_2|$, $|a_1-a_4|$ respectively, and then $\psi_2(t)$ is a multiple of $\psi_1(t)$ by Lemma 1.2 and Lemma 1.3. Similarly there are odd coprimes $c>1$, $d>1$ and $\mu\in Z$ such that $\psi_3(t)=t^\mu(1-t^{cd})(1-t)/[(1-t^c)(1-t^d)]$. Now cd is equal to one of $|a_3-a_4|$, $|a_3-a_2|$, and if it were the latter then $\psi_2(t)$ would be a multiple of $\psi_1(t)\psi_3(t)$, implying $|a_2-a_4|=1$. As a_2-a_4 is even, $cd=|a_3-a_4|$ instead, and $\psi_4(t)$ is a multiple of $\psi_3(t)$. Similar considerations vis-a-vis $\psi_2(t), \psi_4(t)$ show that $\psi_0(t)$ is a unit, and this and computations similar to those of [2, pp. 90–92] show that $\{a,b\}=\{c,d\}$.

3. Throughout this section we assume that there is a prime p and a p-subgroup $G\subset S^1$ such that $F(P,G)=M\cup\{x_0\}$. Then $\dim M=6$ and M is a mod p cohomology CP^3 ([1, VII, Thm. 3.1]).

Note that $\{x_1, x_2, x_3, x_4\}\subset M$. Define b_i for $1\le i\le 4$ by $\nu(M,P)\mid x_i=t^{b_i}$. The next four lemmas cancel $(1-t^{a_i-a_0})/(1-t^{b_i})$ in $\psi_i(t)$.

LEMMA 3.1. *For* $1\le i\le 4$, $a_i-a_0\mid b_i$.

Proof. Any prime power which divides a_i-a_0 must divide b_i also by Lemma 1.2 and the fact that M has codimension 2 in P.

LEMMA 3.2. *If* x_0 *is in no six-dimensional component of* $F(P, Z_q)$ *for any prime power* q, *then* $|b_i|=|a_i-a_0|$ *for* $1\le i\le 4$ *and moreover* $\psi_0(t)$ *is a unit.*

Proof. Either $|a_i-a_0|>1$ or $|a_i-a_0|=1$. For the first case, let m be a prime power which divides a_i-a_0. Then x_0 and x_i are in one component L of $F(P, Z_m)$ by Lemma 1.2, and $\dim L\le 4$ by hypothesis. Thus L contains (possibly by

Lemma 2.3) an $S^2(b_i)$ containing x_0 and x_i, and it follows from Lemma 1.3 and Lemma 3.1 that $|b_i|=|a_i-a_0|$. For the other case $|a_i-a_0|=1$, suppose that b_i is a multiple of a prime n. By Lemma 1.1, $n \mid a_i-a_k$ for some $k \neq 0, i$. Then $x_k \in M$ and also, by Lemma 1.2, x_i and x_k are in a component N of $F(P, Z_n)$. Since $N \not\subset M$, $n \mid b_k$, and this, $x_0 \notin N$ and the first part of this proof imply that $|a_k-a_0|=1$. But then $2=|a_k-a_0|+|a_i-a_0| \geq |a_k-a_i| \geq np$ (the last inequality by Lemma 1.2; p is the prime determining M). Consequently no such n can divide b_i, and again $|b_i|=|a_i-a_0|$. Finally, by the first part of this proof $\psi_0(t)$ cancels to a unit times a quotient of $(1-t)$'s and so is a unit by Lemma 1.1. □

LEMMA 3.3. *If $F(P, S^1)$ is isolated then* $|b_i|=|a_i-a_0|$ *for* $1 \leq i \leq 4$.

Proof. If $A=\{b_i : |b_i|>|a_i-a_0|\}$ is not empty, we may assume that $b_1 \in A$ and also that

$$|b_1|=\max\{|b_i| : b_i \in A\}. \tag{3.3}$$

For $b=|b_1|$ and n a prime power which divides b_1 and not a_1-a_0 let B, N be the components containing x_1 of $F(P, Z_b), F(P, Z_n)$ respectively. B contains more than one S^1 fixed point ([1, IV, Cor. 2.3]) and $x_0 \notin B \subset N$ (Lemma 1.2), so we may assume $x_2 \in B$. Then $n \mid b_2$ and $n \nmid a_2-a_0$, so $b_2 \in A$. For $i=1,2$, let $k_i=|b_i/(a_i-a_0)|>1$. Now $|b_2| \leq |b_1|$ by (3.3) and $b=|b_1|$ divides b_2 since $B \subset F(P, Z_b)$, so $|b_2|=|b_1|$ and by Lemma 1.4 there is an $S^2(b) \subset B$ containing x_1 and x_2. Now a_1-a_2 is nonzero by Lemma 1.2 and the hypothesis that $F(P, S^1)$ is isolated, and $(b, p)=1$ (p the prime determining M). These, Lemma 1.3, and Lemma 1.2 imply the second inequality in

$$b<pb \leq |a_1-a_2| \leq |a_1-a_0|+|a_0-a_2|=|b_1|/k_1+|b_2|/k_2=b/k_1+b/k_2 \leq b.$$

This contradiction implies that S is empty, as was to be shown. □

If neither of the hypotheses of the previous two lemmas apply, we have the following conclusion.

LEMMA 3.4. *If each point of $F(P, S^1)$ is contained in a six-dimensional component of $F(P, Z_q)$ for some prime power q, and $F(P, S^1)$ contains nonisolated points, then the action on P is quasilinear.*

Proof. By hypothesis there are powers p, q of different primes such that $F(P, Z_p)=M \cup \{x_0\}$ and $F(P, Z_q)=N \cup \{x_1\}$ where $x_0 \in N$, $x_1 \in M$ and $\dim M = \dim N = 6$. Then the rational Euler characteristic $\chi(M \cap N)=\chi(F(M \cap N, S^1))=3$ and by general position $\dim(M \cap N)=4$. $F(P, S^1)$ contains a component S with $\dim S \geq 2$ by hypothesis, and clearly $S \subseteq M \cap N$, so by [1, VII, Thm. 5.1] $F(M \cap N, S^1)$ is either a homotopy CP^2 or an isolated point and an S^2. Hence (use [1, IV, Cor. 2.3] if necessary) one component L of $M \cap N$ contains $F(M \cap N, S^1)$ and, by Lemma 2.2, is standard. Thus for $2 \leq i \leq 4$, $x_i \in L \subset M \cap N$, $p \mid a_i-a_1$ and $q \mid a_i-a_0$, the conclusion of Lemma 2.1 is impossible, and $\psi_i(t)$ is a unit. Next, if some prime power not dividing $|\mathrm{Stab}\, N|$ divides three members of $A=\{a_0-a_j : 1 \leq j \leq 4\}$, then using Lemma 1.2 it follows as above that $\psi_0(t)$ is a unit. Otherwise, each prime which divides one or two members of A determines by Lemma 1.2 one or (by Lemma 2.3) two $S^2(m)$'s each of which contains x_0 and one point of M, and by an intersection class argument and Lemma 1.3 the relevant terms in $\psi_0(t)$ cancel. This leaves $\psi_0(t)$ a unit times, possibly, a quotient of $(1-t)$'s or $(1-t^q)$'s, so that by Lemma 1.1 it is in any case a unit. Similarly $\psi_1(t)$ is a unit. □

We may now in effect restrict the nonquasilinear part of the action to M and use Dejter's calculations for CP^3. (We again use the notation of the first two paragraphs of this section.)

LEMMA 3.5. *If the action on P is not quasilinear then* $\psi_0(t)$ *is a unit and* $\psi_i(t)$, $1 \leq i \leq 4$, *is a unit times* $(1-t^{abr})(1-t^r)/[(1-t^{ar})(1-t^{br})]$ *for integers* a, b, r *with* $a>1$, $b>1$, $(a,b)=1$ *and r a nonzero multiple of p.*

Proof. By hypothesis some $\psi_k(t)$ is not a unit and by Lemma 3.2 we may assume $k \neq 0$. Then for $1 \leq i \leq 4$ the terms in $\psi_i(t)$ with exponents $a_i - a_0$ and b_i cancel to a unit by Lemma 3.2, Lemma 3.3 and Lemma 3.4, and calculations identical to those at [2, pp. 90–92] show that these polynomials are as claimed, with $r = |\mathrm{Stab}\, M|$. As for $\psi_0(t)$, if a prime power q divides $a_0 - a_i$ then by Lemma 1.2 x_0 and x_i are in a component L of $F(P, Z_q)$. By the above the exponents of the tangent bundle S^1 representation $\tau P \mid x_i$, which appear in the denominator of $\psi_i(t)$, are, up to signs, ar, br, $(ab \pm 1)r$ and $a_i - a_0$. Since $(q,r)=1$ and q can divide at most one of a, b, $ab \pm 1$, $\dim L \leq 4$ and (possibly by Lemma 2.3) L contains an $S^2(a_i - a_0)$ containing x_0 and $x_1 \in M$ and (by intersection class considerations) carrying a generator of $H_2(P;Z)$. Hence $\psi_0(t)$ cancels to a quotient of $(1-t)$'s by Lemma 1.3, and is a unit by Lemma 1.1. □

4. The previous two sections imply that if the action on P is not quasilinear then there is an $S^2(m)$ which contains fixed points x_0, x_1 such that $\psi_0(t)$ is a unit and $\psi_1(t)$ is not. The $\nu(S^2(m), P)$ representations at these points turn out to be incompatible.

THEOREM 4.1. *If P is a homotopy* CP^4 *with a smooth* S^1 *action then the action is quasilinear.*

Proof. Assume to the contrary that P has a smooth S^1 action which is not quasilinear. By [1, VII, Thm. 5.1], Lemma 2.5 and Lemma 3.5, there are $\{x_0, \ldots, x_4\} \subset F(P, S^1)$ such that $\psi_0(t)$ is a unit; $\psi_1(t) = \pm t^\lambda (1-t^{abr})(1-t^r)/[(1-t^{ar})(1-t^{br})]$ for integers a, b, r, λ with $a>1$, $b>1$, $(a,b)=1$, $r \neq 0$; and $|a_1 - a_0| \geq |a_2 - a_0|$, $|a_1 - a_2| = abr$, $|a_1 - a_3| = r$, $|a_1 - a_4| = (ab \pm 1)r$. Now $m = |a_0 - a_1|$ is prime to r and at least two of a, b, $ab \pm 1$, so $(m,a) = (m,b) = 1$ by Lemma 1.2 and Lemma 2.3, and, since $2m \geq |a_0 - a_1| + |a_0 - a_2| \geq |a_1 - a_2| = abr$, $m > abr/2$. Moreover, since $1 - t^m$ cancels in both $\psi_0(t)$ and $\psi_1(t)$, Lemma 1.4 yields an $S^2(m)$ which contains x_0 and x_1 and, by Lemma 1.3, carries a generator of $H_2(P;Z)$. The $\nu = \nu(S^2(m), P)$ representations at x_0 and x_1 (determined from $\tau P \mid x_i$ as reflected in the denominator of $\psi_i(t)$, $i = 0, 1$) are up to signs of exponents

$$\nu \mid x_0 = t^{a_0-a_2}+t^{a_0-a_3}+t^{a_0-a_4}, \qquad \nu \mid x_1 = t^{ar}+t^{br}+t^{abr+er}$$

for $|e|=1$. From above, the exponents of $\nu \mid x_0$ are equal to $m \pm abr$, $m \pm r$, $m \pm (ab+e)r$ respectively. Since ν is a Z_m equivariant bundle over the trivial Z_m space $S^2(m)$, $\nu \mid x_0$ and $\nu \mid x_1$ pass to the same element of $R(Z_m) = R(S^1)/(1-t^m)$—that is, for some choice of signs

$$\{\pm abr, \pm r, \pm(ab+e)r\} = \{\pm ar, \pm br, \pm(ab+e)r\} \subset Z_m.$$

Straightforward computations show that this is incompatible with $(a,m) = (b,m) = 1$ and $m > abr/2$.

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