COMPACT COMPOSITION OPERATORS ON $H^p(B_N)$

Barbara D. MacCluer

Introduction. Let B_N be the open unit ball in $\mathbf{C}^N$ and let $\Phi: B_N \to B_N$ be a holomorphic self-map of B_N. For f a holomorphic function on B_N, denote the composition $f \circ \Phi$ by $C_\Phi(f)$. This will again be a holomorphic function on B_N. We are concerned here with the question of when C_Φ, called the composition operator induced by Φ, will be a *bounded,* or respectively *compact,* operator on some Hardy space $H^p(B_N)$, for $0 < p < \infty$. Several authors ([1], [5]) have recently given examples to show that, in contrast to the case $N = 1$, C_Φ may indeed fail to be bounded on $H^p(B_N)$ when $N > 1$ and $p < \infty$. In Section 1 we give a necessary and sufficient condition, in terms of the measure $\sigma(\Phi^*)^{-1}$, for C_Φ to be bounded (respectively compact) on $H^p(B_N)$, and derive some consequences of this criterion.

In one variable, compact composition operators on the spaces $H^p(\mathbf{D})$ have been studied by J. Shapiro and P. Taylor in [9], where they examine the relationship between compactness of the operator C_Φ and certain geometric conditions on $\Phi(\mathbf{D})$. In particular, they show that any map Φ for which the range of Φ is contained in a region which touches the unit circle sufficiently "infrequently and sharply" will induce a compact composition operator. In Section 2 we study the question of whether there are geometric conditions on $\Phi(B_N)$ $(N > 1)$ which will guarantee that C_Φ be compact on $H^p(B_N)$. It is the existence of unbounded composition operators when $N > 1$ which makes this question much more difficult in several variables than in the case $N = 1$. Using the compactness criterion developed in Section 1, we show that any Φ with $\Phi(B_N)$ contained in a sufficiently small (depending on the dimension N) Koranyi approach region $D_\alpha(\zeta)$ will induce a compact composition operator on every $H^p(B_N)$, $p < \infty$. We give an example to show that this result is sharp in a strong sense; maps into larger Koranyi approach regions may even fail to induce bounded operators.

Finally we give an example of a map $\Phi: B_2 \to B_2$ for which C_Φ is compact on $H^p(B_2)$, but is not Hilbert-Schmidt on $H^2(B_2)$. To do this we use techniques developed in this paper to modify examples given in [9] for the case $N = 1$ of composition operators which are compact but not Hilbert-Schmidt on $H^2(\mathbf{D})$.

I would like to thank Professor Daniel Luecking for several helpful conversations regarding some of the material of Section 1, particularly Corollary 1.4 and Lemma 1.6.

1. A characterization of bounded (respectively compact) composition operators. The main goal of this section is a theorem which gives necessary and sufficient conditions for the operator C_Φ to be bounded (compact) on $H^p(B_N)$. We

Received July 24, 1984. Revision received September 25, 1984.
Research supported in part by National Science Foundation Grant DMS-8402721.
Michigan Math. J. 32 (1985).

begin with some notation. In all of what follows Φ will be a holomorphic map of B_N into B_N. Denote the rotation invariant probability measure on ∂B_N by σ. Recall that for almost every $[\sigma]$ point ζ of ∂B_N, $\Phi^*(\zeta) \equiv \lim_{r\to 1} \Phi(r\zeta)$ exists. Thus we may regard Φ as a map of $\overline{B_N}$ into $\overline{B_N}$, and we will usually continue to write Φ for this map, and reserve the notation Φ^* for the map from ∂B_N into $\overline{B_N}$ as defined above.

For a point $\eta \in \partial B_N$ and $t > 0$ let

$$S(\eta, t) = \{z \in \overline{B_N} : |1 - \langle z, \eta\rangle| < t\} \quad \text{and} \quad Q(\eta, t) = S(\eta, t) \cap \partial B_N.$$

Recall that $\sigma(Q(\eta, t)) \cong t^N$ [7, §5.1.4, p. 67]. Given $\Phi: B_N \to B_N$ we define a positive, finite Borel measure μ on $\overline{B_N}$ by $\mu(A) = \sigma((\Phi^*)^{-1}A)$. We can now state the main theorem of this section.

THEOREM 1.1. *Let $\Phi: B_N \to B_N$ be holomorphic and let μ be the measure on $\overline{B_N}$ defined by $\mu = \sigma(\Phi^*)^{-1}$. Then for $p < \infty$*

(i) *C_Φ is bounded on $H^p(B_N)$ if and only if there exists $C < \infty$ so that*

$$\mu(S(\eta, t)) \le Ct^N \quad (\eta \in \partial B_N,\ t > 0).$$

(ii) *C_Φ is compact on $H^p(B_N)$ if and only if*

$$\mu(S(\eta, t)) = o(t^N) \quad \text{as } t \to 0, \text{ uniformly in } \eta.$$

Before proceeding to the proof of Theorem 1.1, we give some corollaries, the first of which appears in [9], for the case $N = 1$, with a different proof.

COROLLARY 1.2. *If C_Φ is bounded (respectively compact) on $H^p(B_N)$ for some $p < \infty$, then C_Φ is bounded (respectively compact) on $H^p(B_N)$ for all $p < \infty$.*

Proof. This follows immediately, since conditions (i) and (ii) of Theorem 1.1 are independent of p. □

We give a second corollary, and some consequent examples of unbounded composition operators, after the following lemma.

LEMMA 1.3. *Suppose λ is a positive measure on ∂B_N such that*

$$\lambda(Q(\eta, t)) \le Ct^N \quad (\eta \in \partial B_N,\ t > 0).$$

Then $d\lambda = g\,d\sigma$, where $g \in L^\infty(\partial B_N)$ with $\|g\|_\infty \le C'$, where C' is the product of C and a constant depending only on the dimension N.

Proof. The maximal function of the measure λ is by definition

$$M\lambda(\eta) = \sup_{t>0} \frac{\lambda(Q(\eta, t))}{\sigma(Q(\eta, t))}.$$

There exist constants A_1 and A_2, depending only on N, so that $A_1 t^N \le \sigma(Q(\eta, t)) \le A_2 t^N$ for all $t > 0$ and $\eta \in \partial B_N$ [7, §5.1.4, p. 67]. Thus $M\lambda(\eta) \le CA_1$ for every $\eta \in \partial B_N$. Theorems 5.2.7 and 5.3.1 of [7, p. 70] show that $\lambda \ll \sigma$, and $d\lambda = g\,d\sigma$, where $|g(\eta)| \le CA_1$ for almost every $[\sigma]\eta$ in ∂B_N. □

Lemma 1.3 and Theorem 1.1 have the following corollary.

COROLLARY 1.4. *If C_Φ is bounded on $H^p(B_N)$, then Φ^* cannot carry a set of positive σ-measure in ∂B_N into a set of σ-measure 0 in ∂B_N.*

Proof. Suppose $A \subseteq \partial B_N$ and $\Phi^*(A) \subseteq R \subseteq \partial B_N$ with $\sigma(A) > 0$ and $\sigma(R) = 0$. Let μ be the measure on $\overline{B_N}$ defined by $\mu = \sigma(\Phi^*)^{-1}$. Let $\mu_b = \mu \mid_{\partial B_N}$. Theorem 1.1(i) and Lemma 1.3 imply that $\mu_b \ll \sigma$. But $\mu_b(R) = \mu(R) = \sigma\Phi^{*-1}(R) \geq \sigma\Phi^{*-1}(\Phi^*(A)) \geq \sigma(A)$, since $\Phi^{*-1}(\Phi^*(A)) \supset A$. Since by hypothesis $\sigma(R) = 0$, this is a contradiction. □

APPLICATION. Corollary 1.4 shows immediately that any inner map $\Phi: B_N \to B_N$ with $\Phi^*(\partial B_N)$ contained in a set of σ-measure 0 induces an unbounded operator C_Φ. (We say Φ is an inner map if $|\Phi^*(\zeta)| = 1$ for almost every $\zeta \in \partial B_N$). Thus the maps Φ defined by $\Phi(z) = (A\phi_1(z), B\phi_2(z))$, where the ϕ_i are inner functions on B_2 and $(A, B) \in \partial B_2$, give unbounded operators, for the image of ∂B_2 is contained either in a torus or the boundary of a slice, which are sets of σ-measure 0. This example appears in [1].

The proof of Theorem 1.1 uses a variant of the following theorem, due to Hormander. We introduce the temporary notation $\mathcal{S}(\eta, t)$ for $\{z \in B_N : |1 - \langle z, \eta \rangle| < t\}$. Thus $\mathcal{S}(\eta, t) = S(\eta, t) \cap B_N$.

THEOREM 1.5 ([4], [2], [6]). *If λ is a positive measure on B_N, and if there exists a constant C so that*

$$(*) \qquad \lambda\mathcal{S}(\eta, t) \leq Ct^N \quad (t > 0,\ \eta \in \partial B_N),$$

then there exists a constant C' so that

$$(**) \qquad \int_{B_N} |f|^p \, d\lambda \leq C' \int_{\partial B_N} |f^*|^p \, d\sigma$$

for all $f \in H^p(B_N)$, $p < \infty$.

Conversely, if $(**)$ holds for some p, then there exists a constant C so that $(*)$ holds.

What we need is a slight variation of this result, where λ is a positive measure on $\overline{B_N}$, the sets $\mathcal{S}(\eta, t)$ in condition $(*)$ are replaced by the sets $S(\eta, t)$, and in $(**)$ the left-hand integral is over $\overline{B_N}$. In this setting the direction $(**) \Rightarrow (*)$ follows as before, using standard estimates on the test functions $f_\alpha(z) = (1 - \langle z, \alpha \rangle)^{-4N/p}$ with $\alpha = (1-t)\eta$.

For the other direction, suppose λ is a positive measure satisfying $\lambda S(\eta, t) \leq Ct^N$. Write $\lambda = \lambda_i + \lambda_b$, where $\lambda_i = \lambda \mid_{B_N}$ and $\lambda_b = \lambda \mid_{\partial B_N}$. By Lemma 1.3, $d\lambda_b = g\, d\sigma$ for some $g \in L^\infty(\partial B_N)$. Thus, using Theorem 1.5,

$$\begin{aligned}
\int_{\overline{B_N}} |f|^p \, d\lambda &= \int_{B_N} |f|^p \, d\lambda_i + \int_{\partial B_N} |f|^p g \, d\sigma \\
&\leq C' \int_{\partial B_N} |f|^p \, d\sigma + \|g\|_\infty \int_{\partial B_N} |f|^p \, d\sigma \\
&= C'' \int_{\partial B_N} |f|^p \, d\sigma.
\end{aligned}$$

A careful check of the constants shows that C'' may be taken to be the product of C and a constant depending only on the dimension N, and p. Thus, if C is small, C'' can be chosen small.

A positive measure λ on $\overline{B_N}$ satisfying (*) $\lambda S(\eta, t) \leq Ct^N$ will be called a σ_N-Carleson measure; the smallest C which satisfies (*) will be called the Carleson constant of λ and denoted $K(\lambda)$.

We can now give the proof of Theorem 1.1. We divide the proof into two parts and begin with the boundedness characterization.

Proof of Theorem 1.1(i). Suppose C_Φ is bounded on $H^p(B_N)$. Then there exists $C_1 < \infty$ so that for every $f \in A(B_N) = H(B_N) \cap C(\overline{B_N})$

$$\|f \circ \Phi\|_p^p = \int_{\partial B_N} |f \circ \Phi^*|^p \, d\sigma \leq C_1 \int_{\partial B_N} |f|^p \, d\sigma,$$

where we have used the fact that $(f \circ \Phi)^* = f \circ \Phi^*$, which follows from the continuity of f on $\bar{B}_N$. But

$$\int_{\partial B_N} |f \circ \Phi^*|^p \, d\sigma = \int_{\overline{B_N}} |f|^p \, d\mu \quad \text{where } \mu = \sigma(\Phi^*)^{-1}.$$

Thus

$$\int_{\overline{B_N}} |f|^p \, d\mu \leq C_1 \int_{\partial B_N} |f|^p \, d\sigma \quad (f \in A(B_N)).$$

As in the proof of Theorem 1.5, the test functions $(1 - \langle z, \alpha \rangle)^{-4N/p}$ (in $A(B_N)$), where $\alpha = (1-t)\eta$, show that there is a constant C so that $\mu S(\eta, t) \leq Ct^N$, for all $t > 0$ and $\eta \in \partial B_N$.

Conversely, suppose that for some constant C we have $\mu S(\eta, t) \leq Ct^N$ for all t, η. Then our variant of Theorem 1.5 shows that

$$\int_{\overline{B_N}} |f|^p \, d\mu \leq C' \int_{\partial B_N} |f^*|^p \, d\sigma \quad (f \in H^p(B_N)).$$

In particular, for $f \in A(B_N)$,

$$C' \int_{\partial B_N} |f|^p \, d\sigma \geq \int_{\overline{B_N}} |f|^p \, d\mu = \int_{\partial B_N} |f \circ \Phi^*|^p \, d\sigma = \int_{\partial B_N} |(f \circ \Phi)^*|^p \, d\sigma,$$

and thus $\|f \circ \Phi\|_{H^p(B_N)} \leq C'' \|f\|_{H^p(B_N)}$ if f is in $A(B_N)$. Since $A(B_N)$ is dense in $H^p(B_N)$ we are done. □

Before turning to the proof of the compactness characterization we give one lemma. In the case $N = 1$ this lemma is due to J. Ryff [8], where the key ingredient of the proof was an application of Lindelof's theorem on asymptotic values of functions in $H^\infty(\mathbf{D})$. Ryff's argument does not extend to the case $N > 1$, where Lindelof's theorem fails. The following alternate argument using Corollary 1.4 was shown to me by Daniel Luecking.

LEMMA 1.6. *Suppose C_Φ is bounded on $H^p(B_N)$ and let f be in $H^p(B_N)$. Then for almost every* $[\sigma]$ ζ *in* ∂B_N, $(f\circ\Phi)^*(\zeta)=f^*\circ\Phi^*(\zeta)$. (*Here the notation* f^* *denotes the function defined on* $\overline{B_N}$ *by* $f^*(z)=\lim_{r\to 1} f(rz)$.)

Proof. For $r<1$ let $f_r\in A(B_N)$ be defined by $f_r(z)=f(rz)$. Now $f_r\to f$ in $H^p(B_N)$ and, since C_Φ is bounded on $H^p(B_N)$ by hypothesis, $f_r\circ\Phi\to f\circ\Phi$ in $H^p(B_N)$. Thus as $r\uparrow 1$

$$(*)\qquad \int_{\partial B_N}|f_r\circ\Phi^*|^p\,d\sigma=\int_{\partial B_N}|(f_r\circ\Phi)^*|^p\,d\sigma\to\int_{\partial B_N}|(f\circ\Phi)^*|^p\,d\sigma.$$

Since, by Corollary 1.4, Φ^* cannot carry a set of positive measure in ∂B_N into a set of measure 0 in ∂B_N, and since the radial limit functions of both Φ and f exist on a set of full measure in ∂B_N, we have $\lim_{r\to 1} f_r\circ\Phi^*=f^*\circ\Phi^*$ at almost every point of ∂B_N. This combined with $(*)$ shows that $f^*\circ\Phi^*=(f\circ\Phi)^*$ almost everywhere $[\sigma]$. □

We now complete the proof of Theorem 1.1 by showing that C_Φ is compact on $H^p(B_N)$ if and only if $\mu(S(\eta,t))=o(t^N)$ uniformly in η. The proof relies upon the following easily obtained characterization of compact composition operators [5]: C_Φ is compact on $H^p(B_N)$ if and only if for every sequence $\{f_n\}$ which is bounded in $H^p(B_N)$ and for which $f_n\to 0$ uniformly on compacta, we have $\|f_n\circ\Phi\|_p\to 0$.

Proof of Theorem 1.1(ii). Suppose first that $\mu S(\eta,t)\neq o(t^N)$, uniformly in η. Then there exists $\eta_n\in\partial B_N$, $t_n\to 0$ and $\beta>0$ such that $\mu S(\eta_n,t_n)\geq\beta t_n^N$. Let $f_n(z)=(1-\langle z,\alpha_n\rangle)^{-4N/p}$, where $\alpha_n=(1-t_n)\eta_n$. A computation [7, §1.4.10, p. 17] shows that $\|f_n\|_p^p\cong t_n^{-3N}$. Let $g_n=f_n/\|f_n\|_p$. Note that $g_n\to 0$, uniformly on compacta, since if $|z|\leq r<1$, $|g_n(z)|^p\leq C(t_n/(1-r)^{4N})$. Thus $\{g_n\}$ is a bounded sequence in $H^p(B_N)$ with $g_n\to 0$ almost uniformly. Another calculation shows that

$$\begin{aligned}\|g_n\circ\Phi\|_p&\geq C_2\|f_n\|_p^{-p}\mu S(\eta_n,t_n)t_n^{-4N}\\&\geq C_3t_n^{3N}(\beta t_n^N)t_n^{-4N}\\&=C\beta.\end{aligned}$$

Thus $\|g_n\circ\Phi\|_p\not\to 0$ and this implies that C_Φ is not compact.

Finally suppose that $\mu S(\eta,t)=o(t^N)$ as $t\to 0$, uniformly in η. It is convenient at this point to replace the sets $S(\eta,t)$ by the sets $D(\eta,t)=\{z\in\overline{B_N}:|z|>1-t$ and $z/|z|\in Q(\eta,t)\}$. Since $D(\eta,t)\subseteq S(\eta,2t)$, the hypothesis that $\mu S(\eta,t)=o(t^N)$ implies that $\mu D(\eta,t)=o(t^N)$, uniformly in η.

Given $\epsilon>0$, choose t_0 sufficiently small so that $\mu D(\eta,t)\leq\epsilon t^N$ for all η and for all $t\leq t_0$. Let μ' be the measure supported on $R_N\equiv\overline{B_N}\setminus(1-t_0)\overline{B_N}$, defined by $\mu'(A)=\mu(A\cap R_N)$. We claim that μ' is a σ_N-Carleson measure with $K(\mu')\leq C\epsilon$, where C is an absolute constant depending only on the dimension N. We need to verify that $\mu'(S(\eta,t))\leq C\epsilon t^N$, for all η and t. This is immediate for $t\leq t_0$ since in this case $\mu'(S(\eta,t))=\mu S(\eta,t)\leq\mu D(\eta,t)\leq\epsilon t^N$.

In the case $t > t_0$, we have $\mu' S(\eta, t) = \mu(S(\eta, t) \cap R_N) \leq \mu(D(\eta, t) \cap R_N)$. Cover $\bar{Q}(\eta, t) = \{\zeta \in \partial B_N : |1 - \langle \zeta, \eta \rangle| \leq t\}$ by a finite collection of balls $Q(\alpha_j, t_0/3)$ with centers α_j in $\bar{Q}(\eta, t)$. Note that there is an absolute constant s (depending on N) so that $Q(\eta, st) \supset Q(\alpha_j, t_0/3)$ [9, p. 29]. Now, as in Lemma 5.23 of [7, p. 68], we may extract a disjoint collection of the balls $Q(\alpha_j, t_0/3)$ so that $\bar{Q}(\eta, t) \subset \bigcup_\Gamma Q(\alpha_j, t_0)$. Since each $Q(\alpha_j, t_0/3)$ is contained in $Q(\eta, st)$, and Γ is a disjoint collection, we must have

$$\sigma\left(\bigcup_\Gamma Q(\alpha_j, t_0/3)\right) = \sum_\Gamma \sigma Q(\alpha_j, t_0/3) \leq \sigma Q(\eta, st) \leq C_1 t^N.$$

But $\sigma Q(\alpha_j, t_0/3) \geq C_2 t_0^N$, so we can have at most $C_3(t/t_0)^N$ balls $Q(\alpha_j, t_0/3)$ in the collection Γ. (Each constant C_i depends only on the dimension N). Since $\bigcup_\Gamma Q(\alpha_j, t_0)$ covers $Q(\eta, t)$, we have $D(\eta, t) \cap R_N$ covered by $\bigcup_\Gamma D(\alpha_j, t_0)$. Thus

$$\begin{aligned} \mu' S(\eta, t) &\leq \mu(D(\eta, t) \cap R_N) \\ &\leq \sum_\Gamma \mu D(\alpha_j, t_0) \leq C_3 (t/t_0)^N \epsilon t_0^N \\ &= C_3 \epsilon t^N, \end{aligned}$$

which verifies our claim.

To finish the proof, suppose $\{f_n\}$ is a sequence in $H^p(B_N)$, where $f_n \to 0$ almost uniformly, and $\|f_n\|_p^p \leq M$. Since by Theorem 1.1(i) C_Φ is bounded, we have

$$\begin{aligned} \|f_n \circ \Phi\|_p^p &= \int_{\partial B_N} |(f_n \circ \Phi)^*|^p \, d\sigma \\ &= \int_{\partial B_N} |f_n^* \circ \Phi^*|^p \, d\sigma \qquad \text{(Lemma 1.6)} \\ &= \int_{\overline{B_N}} |f_n^*|^p \, d\mu = \int_{\overline{B_N}} |f_n^*|^p \, d\mu' + \int_{(1-t_0)\overline{B_N}} |f_n|^p \, d\mu \\ &\leq C(N,p) K(\mu') \int_{\partial B_N} |f_n^*|^p \, d\sigma + \int_{(1-t_0)\overline{B_N}} |f_n|^p \, d\mu \\ &\leq C(N,p) M \epsilon + \int_{(1-t_0)\overline{B_N}} |f_n|^p \, d\mu. \end{aligned}$$

The first term can be made as small as desired by choosing ϵ small (this determines $t_0 > 0$). Then the second term is made small for n sufficiently large using the hypothesis that $f_n \to 0$ uniformly on $(1 - t_0)\overline{B_N}$. Thus $\|f_n \circ \phi\|_p \to 0$, and we are done. □

We remark that a similar "little o" Carleson condition appears in connection with the characterization of compact Toeplitz operators on Bergman spaces. See [2] and the references therein.

2. Maps into Koranyi approach regions. Our main goal in this section is a result in the spirit of the one-dimensional work due to J. Shapiro and P. Taylor [9], where certain geometric conditions on $\Phi(\mathbf{D})$ are shown to be sufficient to guarantee that C_Φ be compact on $H^p(\mathbf{D})$. For example, they show that if $\Phi(\mathbf{D})$ is contained in a polygon inscribed in the unit circle, then C_Φ must be compact [9, p. 482].

Our main result here involves Koranyi approach regions $D_\alpha(\zeta)$ in B_N, defined as follows. For $\alpha > 1$ and $\zeta \in \partial B_N$ let

$$D_\alpha(\zeta) = \{z : |1 - \langle z, \zeta \rangle| < \tfrac{1}{2}\alpha(1 - |z|^2)\}.$$

Note that the intersection of $D_\alpha(\zeta)$ with the complex line through 0 and ζ is a standard non-tangential approach region in a disc, while $D_\alpha(\zeta)$ allows "parabolic" approach to ∂B_N in the orthogonal directions. Basic facts about the Koranyi approach regions can be found in [7, §5.4].

We will show that if $\Phi(B_N)$ is contained in $D_\alpha(\zeta)$, for $\alpha < \alpha_0 = \alpha_0(N)$, then C_Φ will be compact on $H^p(B_N)$, $p < \infty$. The limiting value α_0 decreases from ∞ to 1 as N increases from 1 to ∞. An example will be given to show that this result is sharp, and moreover it is possible for $\Phi(B_N)$ to be contained in $D_\gamma(\zeta)$ for some $\gamma > \alpha_0$ and yet C_Φ fail to be even bounded on $H^p(B_N)$.

For simplicity we will work with approach regions based at the point $e_1 = (1, 0')$. This involves no loss of generality since for every unitary map U, $UD_\alpha(\zeta) = D_\alpha(U\zeta)$. Note that $D_\alpha(e_1) = \{z : |1 - z_1| < \frac{1}{2}\alpha(1 - |z|^2)\}$.

We begin with the following lemma, which exploits the compatibility of the regions $D_\alpha(e_1)$ and the non-isotropic balls $S(e_1, t)$.

LEMMA 2.1. *Suppose* $\Phi : B_N \to B_N$ *is holomorphic and suppose further that* $\Phi(B_N) \subseteq D_\alpha(e_1)$. *Then*

(i) C_Φ *is bounded on* $H^p(B_N)$ *if there is a constant* M *such that*

$$\sigma(\Phi^{*-1} S(e_1, t)) \le Mt^N \quad (t > 0).$$

(ii) C_Φ *is compact on* $H^p(B_N)$ *if*

$$\sigma(\Phi^{*-1} S(e_1, t)) = o(t^N) \quad \text{as} \quad (t \to 0).$$

Proof. By Theorem 1.1, to verify (i) we need to show that there exists a constant M' with $\sigma(\Phi^{*-1} S(\eta, t)) \le M' t^N$ for all $\eta \in \partial B_N$ and all $t > 0$.

For $z, w \in \overline{B_N}$ let

$$d(z, w) = |1 - \langle z, w \rangle|.$$

Suppose η is in ∂B_N and $t < (4\alpha)^{-1} d(e_1, \eta)$. We claim that $S(\eta, t) \cap D_\alpha(e_1) = \emptyset$. Lemma 5.4.3 of [7, p. 74] shows that if z is in $D_\alpha(e_1)$ and η is a point of ∂B_N then $d(e_1, \eta) < 4\alpha d(z, \eta)$. So $d(z, \eta) > (4\alpha)^{-1} d(e_1, \eta) > t$ and thus $z \notin S(\eta, t)$. Hence for $t < (4\alpha)^{-1} d(e_1, \eta)$, $\sigma\Phi^{*-1} S(\eta, t) = 0$.

On the other hand if $t \ge (4\alpha)^{-1} d(e_1, \eta)$ then $Q(\eta, t) \cap Q(e_1, 4\alpha t) \ne \emptyset$ and there exists an absolute constant s (depending on N) so that $Q(e_1, 4\alpha st) \supset Q(\eta, t)$. Thus $S(e_1, 8\alpha st) \supset S(\eta, t)$, and writing μ for $\sigma\Phi^{*-1}$ we have

$$\mu S(\eta, t) \leq \mu S(e_1, 8\alpha st)$$
$$\leq CM(8\alpha st)^N = M't^N.$$

The above calculations, together with Theorem 1.1, also give (ii), since for a fixed α the constant M' can be taken to be the product of M and an absolute constant depending only on N. □

We can now state our main result. Both statements in this theorem are sharp, as will be shown following the proof.

THEOREM 2.2. *Let $\Phi: B_N \to B_N$ be holomorphic and set $\alpha_0 = (\cos(\pi/2N))^{-1}$.*
(1) *If $\Phi(B_N) \subseteq D_{\alpha_0}(e_1)$, then C_Φ is bounded on $H^p(B_N)$.*
(2) *If $\Phi(B_N) \subseteq D_\gamma(e_1)$ for some $1 < \gamma < \alpha_0$, then C_Φ is compact on $H^p(B_N)$.*

We remark that in (2) the operators C_Φ are moreover Hilbert-Schmidt on $H^2(B_N)$, as will be shown later. Before beginning the proof of Theorem 2.2 we give one lemma. Let σ_1 denote normalized Lebesgue measure on $\partial\mathbf{D}$.

LEMMA 2.3. *Suppose $\psi: \mathbf{D} \to \mathbf{D}$ is holomorphic such that $\psi(\mathbf{D})$ is contained in the non-tangential approach region $D_\alpha = \{\lambda \in \mathbf{D}: |1-\lambda| < \frac{1}{2}\alpha(1-|\lambda|^2)\}$. Then there exists a constant C depending on $\psi(0)$ and α, such that*

$$\sigma_1(\psi^{*-1}S(1, t)) \leq Ct^b,$$

where $b = \pi/(2\cos^{-1}(\alpha^{-1}))$. (Note that in the special case $\alpha = (\cos(\pi/2N))^{-1}$ we have $b = N$).

Proof. The region D_α is contained in a polygon $P \subseteq \bar{\mathbf{D}}$ with one vertex at 1 and with vertex angle $= 2\cos^{-1}(\alpha^{-1})$ at this point. Let ρ be the biholomorphic map of $\mathbf{D}$ onto the interior of P with $\rho(1) = 1$ and $\rho(0) = \psi(0)$. As in Corollary 3.2 of [9] a standard local mapping argument shows that there is a neighborhood $\mathbf{N}$ of 1 and a non-vanishing holomorphic function h on that neighborhood so that for all z in $\mathbf{N}$

$$1 - \rho(z) = (1-z)^{1/b} h(z)$$

where $b = \pi/(2\cos^{-1}(\alpha^{-1}))$. Thus there is a $t_0 > 0$ such that

$$\rho^{-1}S(1, t) \subseteq S(1, C_1 t^b) \quad (t < t_0),$$

where t_0 and C_1 depend on the map ρ; that is, on the geometry determined by α and on $\psi(0)$. Since $\psi(\mathbf{D}) \subset D_\alpha \subset P$ we may write $\psi = \rho \circ (\rho^{-1} \circ \psi) = \rho \circ \tau$. Thus for $t < t_0$

$$\psi^{-1}(S(1, t)) \subset \tau^{-1}S(1, C_1 t^b).$$

Now C_τ is necessarily bounded on $H^p(\mathbf{D})$, by Littlewood's subordination principle [3, Theorem 1.7], with $\|C_\tau\| \leq 1$ since $\tau(0) = 0$. Thus there is an absolute constant C_2 so that

$$\sigma_1(\psi^{-1}S(1, t)) \leq \sigma_1(\tau^{-1}S(1, C_1 t^b))$$
$$\leq C_2 C_1 t^b$$

for all $t < t_0$. Trivially for $t \geq t_0$ we have

$$\sigma_1(\psi^{-1}S(1,t)) \leq 1 \leq t_0^{-b} t^b.$$

Thus there is a constant C such that, for all t, $\sigma_1(\psi^{*-1}S(1,t)) \leq Ct^b$. □

We now give the proof of Theorem 2.2.

Proof of Theorem 2.2. By Lemma 2.1 it suffices to show that there is a constant C so that

$$\sigma(\Phi^{*-1}S(e_1,t)) \leq Ct^N \quad (t>0).$$

Let $A = \Phi^{*-1}S(e_1,t)$ and let ζ be a point of ∂B_N. We consider first $A \cap [\zeta]$, the intersection of A with the boundary of the slice through 0 and ζ (that is, points of the form $e^{i\theta}\zeta$ in A).

Let Φ^ζ be the map of $\mathbf{D}$ into $\mathbf{D}$ given by $\Phi^\zeta(\lambda) = \Phi_1(\lambda\zeta)$. Here Φ_1 denotes the first coordinate function of Φ. By hypothesis $\Phi(B_N) \subseteq D_{\alpha_0}(e_1)$, so

$$\begin{aligned}|1-\Phi_1(\lambda\zeta)| &< \tfrac{1}{2}\alpha_0(1-|\Phi(\lambda\zeta)|^2)\\ &< \tfrac{1}{2}\alpha_0(1-|\Phi_1(\lambda\zeta)|^2).\end{aligned}$$

Thus, in the notation of Lemma 2.3, $\Phi^\zeta(\mathbf{D}) \subseteq D_{\alpha_0} \subseteq \mathbf{D}$. Since $\Phi^\zeta(0) = \Phi_1(0)$ for all ζ, Lemma 2.3 shows that

$$\begin{aligned}\sigma_1(A \cap [\zeta]) &= \sigma_1((\Phi^\zeta)^{-1}S(1,t))\\ &\leq Ct^N,\end{aligned}$$

where C is a constant depending on $\Phi(0)$ and on the geometry fixed by α_0, but not on ζ. Then by slice integration [7, §1.3.7, p. 15]

$$\sigma(A) = \int_{\partial B_N} \chi_A(\zeta)\, d\sigma(\zeta) = \int_S d\sigma(\zeta) \int_{-\pi}^{\pi} \chi_A(e^{i\theta}\zeta)\frac{d\theta}{2\pi},$$

where each of the inner integrals is at most Ct^N. This gives $\sigma(A) \leq Ct^N$ and completes the proof of the first statement in Theorem 2.2.

If $\Phi(B_N) \subseteq D_\gamma(e_1)$ where $\gamma < \alpha_0$, then by Lemma 2.3 again

$$\sigma_1((\Phi^\zeta)^{-1}S(1,t)) \leq Ct^b,$$

where $b = \pi/(2\cos^{-1}(\gamma^{-1}))$ satisfies $b > N$. Again slice integration shows that $\sigma(A) \leq Ct^b = o(t^N)$. Thus by Lemma 2.1, C_Φ is compact on $H^p(B_N)$. □

EXAMPLE. The following example shows that Theorem 2.2 is sharp. Let ψ be an inner function on B_N with $\psi(0) = 0$. Recall that ψ^* is a measure-preserving map of ∂B_N into $\partial\mathbf{D}$ [7, §19.1.5, p. 405]. Construct a map Φ of B_N into B_N by

$$\Phi(z) = (1-(1-\psi(z))^b, 0, \ldots, 0).$$

Consider first the case $1 > b > 1/N$.

$$\begin{aligned}\sigma(\Phi^{*-1}S(e_1,t)) &= \sigma\{\zeta : |(1-\psi(\zeta))^b| < t\}\\ &= \sigma\{\zeta : \psi(\zeta) \in S(1, t^{1/b})\}\\ &\cong t^{1/b}\end{aligned}$$

for t sufficiently small. Since $t^{1/b} \neq O(t^N)$ if $b > 1/N$, C_Φ is not bounded. If $b = 1/N$ then C_Φ is bounded but not compact since $t^N \neq o(t^N)$. Since $\Phi(B_N)$ is (essentially) contained in $D_\alpha(e_1)$ where $\alpha = (\cos(\pi b/2))^{-1}$, this shows that Theorem 2.2. is sharp.

REMARKS ON THEOREM 2.2. Lemma 2.3 may also be used to show that if Φ is such that $\Phi(B_N) \subseteq D_\gamma(e_1)$ where $\gamma < (\cos(\pi/2N))^{-1}$, then C_Φ will be Hilbert-Schmidt on $H^2(B_N)$. It is easy to see [5] that C_Φ will be Hilbert-Schmidt precisely when

$$\int_{\partial B_N} (1 - |\Phi(\zeta)|)^{-N} d\sigma(\zeta) < \infty.$$

If $\Phi(B_N) \subset D_\gamma(e_1)$ then $|1 - \Phi_1(z)| < \frac{1}{2}\gamma(1 - |\Phi(z)|^2) < \gamma(1 - |\Phi(z)|)$. Thus C_Φ will be Hilbert-Schmidt if

$$(*) \qquad \infty > \int_{\partial B_N} |1 - \Phi_1(\zeta)|^{-N} d\sigma(\zeta) = \int_{\partial B_N} d\sigma(\zeta) \int_{-\pi}^{\pi} |1 - \Phi^\zeta(e^{i\theta})|^{-N} \frac{d\theta}{2\pi}.$$

As in the proof of Theorem 2.2, the map Φ^ζ takes $\mathbf{D}$ into the nontangential approach region D_γ. If $N(2\cos^{-1}(\gamma^{-1}))/\pi < 1$ the techniques of Lemma 2.3 show that each of the inner integrals above is bounded by a finite constant independent of ζ. Thus C_Φ will be Hilbert-Schmidt on $H^2(B_N)$ if $\gamma < (\cos(\pi/2N))^{-1}$.

We finish by applying the methods of this Section to construct an example of a map $\Phi: B_2 \to B_2$ for which C_Φ is compact on $H^p(B_2)$, but *not* Hilbert-Schmidt on $H^2(B_2)$. This example relies heavily on the work done by Shapiro and Taylor to construct analogous examples when $N = 1$. We will use the relevant results from [9, §4] as needed, and refer the reader to their paper for further details.

Let $f(z) = z(-\log z)$ on $\{\operatorname{Re} z \geq 0,\ |z| < 1\}$. By [9, p. 485] there exists $0 < \epsilon < 1$ and a one-to-one conformal map g of the disc $\mathbf{D}$ onto $H(\epsilon) = \{|z| < \epsilon,\ \operatorname{Re} z > 0\}$, with $g(1) = 0$ so that $\tau(z) = 1 - f(g(z))$ maps $\mathbf{D}$ univalently onto a Jordan domain in $\mathbf{D}$ whose boundary touches $\partial\mathbf{D}$ only at 1 and for which C_τ is compact on $H^p(\mathbf{D})$. Moreover there is a constant M so that $1 - |1 - f(iy)| \leq My$ for all $y \in [0, \epsilon]$.

The map we wish to consider is $\Phi(z) = (1 - \phi(\rho(z)), 0)$, where $\rho: B_2 \to \mathbf{D}$ is an inner function in the ball with $\rho(0) = 0$ and ϕ is defined on $\mathbf{D}$ by $\phi(z) = F(g(z))$ with g as above and $F(z) = (z(-\log z))^{1/2}$. (Both the logarithm and square root denote the principal branch.)

We show first that C_Φ is compact on $H^p(B_2)$. Since $\Phi(B_2)$ is contained in a nontangential approach region based at 1 in the complex line through 0 and e_1, Lemma 2.1 shows that we need only verify that $\sigma(\Phi^{*-1}S(e_1, t)) = o(t^2)$. Tracing back through the definition of Φ we see that if ζ is in ∂B_2 the point $e^{i\theta}\zeta$ is in $\Phi^{*-1}S(e_1, t)$ if and only if

$$\begin{aligned} |F \circ g(\rho(e^{i\theta}\zeta))| < t &\Leftrightarrow |f \circ g(\rho_\zeta(e^{i\theta}))| < t^2 \\ &\Leftrightarrow 1 - f \circ g(\rho_\zeta(e^{i\theta})) \in S(1, t^2) \\ &\Leftrightarrow \tau(\rho_\zeta(e^{i\theta})) \in S(1, t^2). \end{aligned}$$

But C_τ is compact on $H^p(\mathbf{D})$, and therefore Theorem 1.1 shows that

$$\sigma_1\{e^{i\theta}: \tau(\rho_\zeta(e^{i\theta})) \in S(1, t^2)\} = o(t^2),$$

independent of ζ, since $\rho_\zeta(0)=0$. By slice integration, $\sigma\{\zeta : \Phi^*(\zeta) \in S(e_1, t)\} = o(t^2)$, as desired.

To show that C_Φ is not Hilbert-Schmidt on $H^2(B_2)$ we show that

$$\int_{\partial B_2} (1-|\Phi(\zeta)|)^{-2} d\sigma(\zeta) = \infty.$$

We claim that there is a constant m so that for iy in the interval $I_k = [i(m(k+1))^{-2}, i(mk)^{-2})$ on the imaginary axis, where k is a sufficiently large integer, we have $1-|1-F(iy)| \leq k^{-1}$. Note that $|I_k| \cong k^{-3}$.

Assume for the moment that the claim is verified. If $h=g^{-1}$, then as in [9] h extends conformally to a neighborhood of 0 and there is a $\delta > 0$ so that both h' and its reciprocal are bounded on $[-i\delta, i\delta]$. Thus if k is sufficiently large, say $k \geq K_0$, so that $I_k \subseteq [-i\delta, i\delta]$, then

$$\sigma_1\{g^{-1}(I_k)\} \cong k^{-3}.$$

Since ρ is measure preserving as a map from ∂B_2 to $\partial \mathbf{D}$

$$\sigma\{\rho^{-1}(g^{-1}I_k)\} \cong k^{-3}.$$

Thus

$$\begin{aligned} \int_{\partial B_2} (1-|\Phi(\zeta)|)^{-2} d\sigma(\zeta) &\geq \sum_{k \geq K_0} \int_{\rho^{-1}g^{-1}(I_k)} (1-|\Phi(\zeta)|)^{-2} d\sigma(\zeta) \\ &\geq C \sum_{k \geq K_0} (k^{-3})(k^2) = \infty, \end{aligned}$$

since on $\rho^{-1}g^{-1}(I_k)$ we have $1-|\Phi(\zeta)| = 1-|1-F(iy)| \leq k^{-1}$ by our claim.

To verify the claim, a calculation shows that there are absolute constants M_1 and M_2 such that

$$\begin{aligned} 1-|1-F(iy)| &\leq M_1((1-|1-f(iy)|)^{1/2} + |f(iy)|) \\ &\leq M_2 y^{1/2} \end{aligned}$$

for y sufficiently small. The second inequality follows from Lemma 4.1(c) of [9] and the fact that $|f(iy)| \cong y(-\log y) = o(y^{1/2})$. This verifies the claim and we are done. □

REFERENCES

1. J. Cima, C. Stanton, and W. Wogen, *On boundedness of composition operators,* Proc. Amer. Math. Soc., to appear.
2. J. Cima and W. Wogen, *A Carleson measure theorem for the Bergman space on the ball,* J. Operator Theory 7 (1982), 157–165.
3. P. Duren, *Theory of H^p spaces,* Academic Press, New York, 1970.

4. L. Hormander, *L^p estimates for (pluri-) subharmonic functions,* Math. Scand. 20 (1967), 65–78.
5. B. MacCluer, *Spectra of compact composition operators on $H^p(B_N)$,* Analysis 4 (1984), 87–103.
6. S. Power, *A simple proof of Hormander's Carleson theorem for the ball,* preprint.
7. W. Rudin, *Function theory in the unit ball of $\mathbf{C}^N$,* Springer, New York, 1980.
8. J. Ryff, *Subordinate H^p functions,* Duke Math. J. 33 (1966), 347–354.
9. J. Shapiro and P. Taylor, *Compact, nuclear, and Hilbert-Schmidt composition operators on H^2,* Indiana Univ. Math. J. 23 (1973), 471–496.

Department of Mathematics
University of Virginia
Charlottesville, Virginia 22903