ON THE APPROXIMATION OF SINGULARITY SETS BY ANALYTIC VARIETIES II

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Introduction. Let D be the open unit disc. For a compact subset X of the closed unit polydisc in $\mathbf{C}^2$ we denote $X \cap (D \times \mathbf{C})$ by X^0. We say that X is a *singularity set* if there exists a function which is holomorphic on $D \times \mathbf{C} \setminus X^0$ and singular at each point of X^0. One class of examples arises from polynomial hulls: one can show, using the "continuity theorem", that if $K \subseteq \partial D \times \mathbf{C} \subseteq \mathbf{C}^2$ is compact then the polynomially convex hull $\hat{K}$ of K is a singularity set; see Słodkowski [4].

A basic fact [6] about singularity sets X is that the maximum principle is satisfied for polynomials in the coordinate functions z and w; i.e., if P is a polynomial in z and w, and N is a compact subset of X^0 with relative boundary ∂N, then $|P(z_0, w_0)| \leq \max_{\partial N}|P|$, for each $(z_0, w_0) \in N$. The presence of the maximum principle suggests that it may be possible, in some sense, to approximate X^0 by analytic subvarieties of $\mathbf{C}^2$. This is a variant of the general problem of approximating polynomial convex hulls by varieties.

For λ in $\bar{D}$ let $X_\lambda = \{w \in \mathbf{C}: (\lambda, w) \in X\}$. We assume that $X_\lambda \neq \varnothing$ for all λ. Under the assumption that each X_λ is contained in a single disc of radius r, we showed in [1] that X^0 can be approximated by an analytic variety, in fact, by the graph of a holomorphic function on D. In the case of one-point fibres, it is a classical result of Hartogs that X is an analytic variety. The analyticity of X in the case of finite or countable fibres was studied by Bishop, Basener and others; see [7] for a survey and further references. It is natural to examine the case of totally disconnected fibres. It then may be possible to cover the fibres X_λ by a finite number of disks. This is the kind of assumption made in our main result which we now state; the assumption is for a set Ω containing all λ sufficiently close to the unit circle and also the point $\lambda = 0$. The latter condition should be viewed only as a normalization. The notation $D(p, r)$ will denote $\{w \in \mathbf{C}: |w-p| < r\}$.

THEOREM. *Let X be a singularity set in $\mathbf{C}^2$ with $|w| < \frac{1}{2}$ on X. Suppose that there exists r with $0 < r < \frac{1}{2}$ and that there exist a domain $\Omega \subseteq D$ and an integer $n > 0$ with the following properties:*

(i) *For all $\lambda \in \bar{\Omega}$ there exist $p_1(\lambda), p_2(\lambda), \ldots, p_n(\lambda) \in \mathbf{C}$ such that*

$$X_\lambda \subseteq \bigcup_{j=1}^{n} D(p_j(\lambda), r),$$

where $|p_j(\lambda)| < \frac{1}{2} - 4r$ (the p_j are *not* assumed to be continuous functions of λ) *and $X_\lambda \cap D(p_j(\lambda), r) \neq \varnothing$, $1 \leq j \leq n$.*

(ii) *$|p_j(\lambda) - p_k(\lambda)| > 8r$ if $j \neq k$.*

(iii) *The set $\partial\Omega$ is smooth; $\partial\Omega = \Gamma_0 \cup \partial D$ where distance $(\Gamma_0, \partial D) \equiv \rho > 0$; $0 \in \Omega$; $\omega_0(\Gamma_0)$ denotes the harmonic measure of Γ_0 with respect to $0 \in \Omega$.*

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Then there exist polynomials $f_j(\lambda)$, $1 \le j \le n$, *such that for*

$$P(\lambda, w) \equiv w^n + f_1(\lambda) w^{n-1} + \cdots + f_n(\lambda),$$

the algebraic variety $\{P=0\}$ *approximates* X *in the following sense. If* $(\lambda_0, w_0) \in X$, *then*

$$|P(\lambda_0, w_0)| \le \eta \equiv 5(2n+1)r + (4/\rho)\omega_0(\Gamma_0) \tag{1}$$

and

(2) *there exists* (λ_0, w_0^*) *in* $\{P=0\}$ *over* λ_0 *such that* $|w_0^* - w_0| \le \eta^{1/n}$.

One of our basic tools will be the construction of certain functions of the fibres which are subharmonic. In the first section we obtain a result of this type which is a variant of a lemma of Senitchkin [2]. We then apply it in the third section to prove our main theorem.

1. Subharmonicity theorem. We shall formulate our result in the context of maximum modulus algebras ([6], [7]). Let X be a locally compact Hausdorff space, A an algebra of continuous complex valued functions on X, Ω an open set in $\mathbf{C}$, and $f \in A$ with $f(X) \subseteq \Omega$. We say that (A, X, Ω, f) is a *maximum modulus algebra* provided the following hold:

(i) A separates the points of X and contains the constants.

(ii) For each compact $K \subseteq \Omega$, $f^{-1}(K) = \{x \in X : f(x) \in K\}$ is compact; i.e., $f: X \to \Omega$ is a proper mapping.

(iii) For each closed disc $\Delta \subseteq \Omega$ with center λ_0 and for each $x^0 \in f^{-1}(\lambda_0)$,

$$|g(x^0)| \le \max_{f^{-1}(\partial\Delta)} |g|, \quad g \in A.$$

It was proved in [6] that if $X \subseteq \mathbf{C}^2$ is a singularity set then (P, X^0, D, z_1) is a maximum modulus algebra, where P is the restriction of the polynomials in z_1 and z_2 to X^0.

Let (A_k, X_k, Ω, p_k), $1 \le k \le n$ be n maximum modulus algebras over the same plane set Ω. Let $\otimes^n A$ be the algebra of all functions g on ΠX_j of the form

$$g(x) = \sum_{j=1}^{N} g_{j1}(x_1) g_{j2}(x_2) \ldots g_{jn}(x_n)$$

where $x = (x_1, x_2, \ldots, x_n)$, $x_j \in X_j$ and $g_{ji} \in A_i$. Let

$$X^{(n)} = \{x = (x_1, \ldots, x_n) \in \Pi X_j : p_1(x_1) = p_2(x_2) = \cdots = p_n(x_n)\}$$

and let $\pi: X^{(n)} \to \Omega$ be $\pi(x) = p_1(x_1)$ $(= p_2(x_2) = \cdots = p_n(x_n))$. Let $A^{(n)}$ be the restriction of $\otimes^n A$ to $X^{(n)}$. Then $\pi \in A^{(n)}$.

PROPOSITION. $(A^{(n)}, X^{(n)}, \Omega, \pi)$ *is a maximum modulus algebra.*

REMARK 1. Senitchkin [2] proved this for the case of uniform algebras when all of the n algebras and projections are the same. His proof also applies in our case but we prefer to give a new proof which depends mainly on the maximum principle for subvarieties of the polydisc.

REMARK 2. By a result of ([5], [6]), if $g \in A$ and (A, X, Ω, π) is a maximum modulus algebra then $\log Z_g$ is subharmonic on Ω, where $Z_g(\lambda) = \sup_{\pi(x)=\lambda}|g(x)|$. Applying this to the approximations $\sum_{k=0}^{N}(g^k/k!)$ of e^g for $g \in A^{(n)}$ and letting $N \to \infty$, we conclude that $\lambda \to \max_{x \in \pi^{-1}(\lambda)} \operatorname{Re} g(x)$ is subharmonic on Ω for $g \in A^{(n)}$. It is in this form that we shall use the result.

Proof. (i) and (ii) are clear. We shall verify (iii) and, without loss of generality, may assume that Δ is $\bar{D}$ the closed unit disc with center 0.

Let $x^0 = (x_1^0, \ldots, x_n^0)$ where $p_k(x_k^0) = 0$. Since (iii) holds for A_k (i.e., $|g_k(x_k^0)| \leq \sup_{p_k^{-1}(\partial D)}|g_k|$ for $g_k \in A_k$) there is a representing measure μ_k on $p_k^{-1}(\partial D) \subseteq X_k$ for evaluation at x_k^0; i.e., $g_k(x_k^0) = \int g_k \, d\mu_k$ for $g_k \in A_k$. Then $p_{k*}(\mu_k)$ (the projection of μ_k to ∂D given by $\int f dp_{k*}(\mu) = \int f \circ p_k \, d\mu$) represents the origin for the disc algebra, and so by uniqueness $p_{k*}(\mu)$ is normalized Lebesgue measure $dm = d\theta/2\pi$ on ∂D. Let $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$ on ΠX_k. Clearly $F(x^0) = \int F \, d\mu$ for $F \in \otimes^n A$. Also $\Pi_*(\mu)$ has its support in the torus T^n (where $T = \partial D$) and it is straightforward to check that $\Pi_*(\mu) = dm \times \cdots \times dm = d\sigma$, Haar measure on the torus, where $\Pi = (p_1, p_2, \ldots, p_n)$.

Now for $F \in L^\infty(d\mu)$ define $F_* \in L^\infty(d\sigma, T^n)$ by $\Pi_*(F d\mu) = F_* \, d\sigma$. We claim that if $F \in \otimes^n A$ then $F_* \in H^\infty(T^n)$. To see this it suffices to show that $I = \int e^{is_1\theta_1} e^{is_2\theta_2} \ldots e^{is_n\theta_n} F_* \, d\sigma$ vanishes if $F \in \otimes^n A$ and the s_j are integers with at least one of them being positive. Without loss of generality we may assume that $F(x) = g_1(x_1) \ldots g_n(x_n)$ for $g_k \in A_k$ and that $s_1 > 0$. Then

$$I = \int p_1^{s_1} \ldots p_n^{s_n} F \, d\mu$$

$$= \int p_2^{s_2} \ldots p_n^{s_n} \left(\int p_1^{s_1} g_1 g_2 \ldots g_n \, d\mu_1(x_1) \right) d\mu_2(x_2) \ldots d\mu_n(x_n)$$

the inner integral is $p_1(x_1^0)^{s_1} \cdot g_1(x_1^0) g_2 \ldots g_n = 0$ since $p_1^{s_1} g_1 \in A_1$ and $p_1(x_1^0) = 0$. Hence $I = 0$.

Now we shall also view F_* as a bounded holomorphic function on the polydisc D^n. We claim

$$(*) \qquad \overline{\lim_{\substack{\zeta \to \zeta_0 \\ \zeta \in D^n}}} |F_*(\zeta)| \leq \max_{x \in \Pi^{-1}(\zeta_0)} |F(x)|$$

for each $\zeta_0 \in T^n$, $F \in \otimes^n A$. Let $\varphi(\zeta) = \sup\{|F(x)| : x \in \Pi^{-1}(\zeta)\}$ for $\zeta \in T^n$. It is easy to see that φ is upper semicontinuous on T^n and that $|F_*| \leq \varphi$ σ-almost everywhere on T^n. We shall apply the following lemma which will be proved later. The Poisson integral of $u \in L^1(\sigma, T^n)$ will be denoted $P[u]$.

LEMMA. *If u is upper semicontinuous on T^n then*

$$\overline{\lim_{\substack{\zeta \to \zeta_0 \in T^n \\ \zeta \in D^n}}} P[u](\zeta) \leq u(\zeta_0) \quad \textit{for } \zeta_0 \in T^n.$$

We have $|F_*(\zeta)| = |P[F_*](\zeta)| \leq P[|F_*|](\zeta) \leq P[\varphi](\zeta)$ for $\zeta \in D^n$ and so from the lemma

$$\overline{\lim_{\zeta\to\zeta_0}}\,|F_*(\zeta)|\le\overline{\lim_{\zeta\to\zeta_0}}\,P[\varphi](\zeta)\le\varphi(\zeta_0).$$

This is (*).

Thus for $g\in\otimes^n A$ we have

$$|g(x^0)|=\left|\int g\,d\mu\right|=\left|\int_{T^n}g_*\,d\sigma\right|=|g_*(0)|$$

$$\le\overline{\lim_{\substack{|\lambda|\to1\\(\lambda,\lambda,\dots,\lambda)\in D^n}}}|g_*(\lambda,\lambda,\dots,\lambda)|$$ (by the maximum principle on the variety $z_1=z_2=\cdots=z_n$)

$$\le\sup_{\substack{\Pi(x)=\zeta=(\zeta_1,\zeta_2,\dots,\zeta_n)\in T^n\\ \zeta_1=\zeta_2=\cdots=\zeta_n}}|g(x)|$$ (by (*))

$$=\sup_{\pi^{-1}(\partial D)}|g|.$$

This gives the proposition. It remains to verify the lemma. Let $\epsilon>0$. Fix $\zeta_0\in T^n$. Since u is usc there exists a continuous function ψ on T^n such that

(a) $u<\psi$ on T^n and

(b) $\psi(\zeta_0)<u(\zeta_0)+\epsilon$.

Then $P[u]\le P[\psi]$ and $\lim_{\zeta\to\zeta_0}P[\psi](\zeta)=\psi(\zeta_0)$ (ψ is a continuous function) give

$$\overline{\lim_{\substack{\zeta\to\zeta_0\\ \zeta\in D^n}}}\,P[u](\zeta_0)\le\psi(\zeta_0)<u(\zeta_0)+\epsilon.$$

Since ϵ is arbitrary the lemma follows. □

2. We precede the proof of the theorem with a lemma. In this section a "disk" will mean the intersection of $\bar\Omega$ with a disk.

LEMMA. *Under the hypotheses of the Theorem, there exists a finite covering $\{D_k\}$ of $\bar\Omega$ by disks and for all k there exist n continuous functions $\tilde p_1^k,\tilde p_2^k,\dots,\tilde p_n^k$ on D_k satisfying:*

(i) *For all k and i such that $D_k\cap D_i\neq\emptyset$ and for all $z\in D_k\cap D_i$, the unordered n-tuples $\{\tilde p_j^k(z)\}_{1\le j\le n}$ and $\{\tilde p_j^i(z)\}_{1\le j\le n}$ coincide. We thus obtain an unordered n-tuple of complex numbers for each $\lambda\in\bar\Omega$ which we call $\{w_j(\lambda)\}_{1\le j\le n}$; i.e., $\{w_j(\lambda)\}=\{\tilde p_j^k(z)\}$ for $\lambda\in D_k$.*

(ii) *$X_\lambda\subseteq\bigcup_1^n D(w_j(\lambda),5r)$, for all $\lambda\in\bar\Omega$. If Δ is a sufficiently small closed disk in $\bar\Omega$ then there exists a decomposition $X\cap\lambda^{-1}(\Delta)=\bigcup_{k=1}^n X_k$, where $X_k\subseteq\{(\lambda,w):\lambda\in\Delta,|w_k(\lambda)-w|<5r\}$ and the X_k are compact, pairwise disjoint sets with $(X_k)_\lambda$ non-empty for all $\lambda\in\Delta$. Moreover the X_k are singularity sets in $\Delta^0\times\mathbf{C}$.*

(iii) *$|w_j(\lambda)|<\frac12$ for $\lambda\in\bar\Omega$.*

Proof.

Assertion 1. There exists a $\delta>0$ such that for all $\lambda_1,\lambda_2\in\bar\Omega$, if $|\lambda_1-\lambda_2|<\delta$ then

$$X_{\lambda_2} \subseteq \bigcup_{j=1}^{n} D(p_j(\lambda_1), 3r)$$

and X_{λ_2} has a non-empty intersection with each set $D(p_j(\lambda_1), 3r)$, $1 \le j \le n$.

This follows from the compactness of $\bar{\Omega}$ and the following assertion: For all $\lambda_0 \in \bar{\Omega}$, there exists $\delta(\lambda_0)$ such that if $|\lambda - \lambda_0| < \delta(\lambda_0)$ and $\lambda \in \bar{\Omega}$, then $X_\lambda \subseteq \bigcup_{j=1}^{n} D(p_j(\lambda_0), r)$ and X_λ meets each set $D(p_j(\lambda_0), r)$, $1 \le j \le n$.

To see this note that X_{λ_0} is disjoint from $\sigma \equiv \bigcup_{j=1}^{n} \partial D(p_j(\lambda_0), r)$, and so there is a $\delta(\lambda_0)$ such that X_λ is disjoint from σ for $|\lambda - \lambda_0| \le \delta(\lambda_0)$. The compactness of X yields that if $\delta(\lambda_0)$ is sufficiently small, then $X_\lambda \subseteq \bigcup D(p_j(\lambda_0), r)$ for $|\lambda - \lambda_0| \le \delta(\lambda_0)$. It follows that the sets

$$X_k = \{(\lambda, w) \in X : |\lambda - \lambda_0| \le \delta(\lambda_0),\ |w - p_k(\lambda_0)| < r\}$$

are disjoint, compact, non-empty subsets of X and that their union is $X \cap \lambda^{-1}(\Delta)$ where $\Delta = \{\lambda : |\lambda - \lambda_0| \le \delta(\lambda_0)\} \cap \bar{\Omega}$.

We next verify that X_k is a singularity set over Δ. In fact, the definition of X as a singularity set gives a function f which is analytic in $\Delta^0 \times \mathbf{C} \setminus X$ and singular at X. A standard Laurent decomposition applied in the second variable shows that f is a sum of n functions $f_1, f_2, \ldots, f_n$, where f_k is analytic in $\Delta^0 \times \mathbf{C} \setminus X_k$, $1 \le k \le n$. Then f_k must be singular at X_k and so X_k is a singularity set over Δ.

Now suppose $(X_k)_\lambda$ were empty for some $\lambda \in \Delta$, then the fibers of X_k would be empty for an open subset about λ. From this one easily concludes by a Cauchy integral argument that any function analytic on $\Delta^0 \times \mathbf{C} \setminus X_k$ extends to be analytic on all of $\Delta^0 \times \mathbf{C}$. This contradicts the fact that X_k is a non-empty singularity set. Hence $(X_k)_\lambda$ is non-empty for all $\lambda \in \Delta$. This completes the proof of Assertion 1; δ below will be the quantity given by this assertion.

Assertion 2. If $\lambda_1, \lambda_2 \in \bar{\Omega}$ and $|\lambda_1 - \lambda_2| < \delta$, then for all i $(1 \le i \le n)$ there exists a unique j $(1 \le j \le n)$ such that $|p_i(\lambda_1) - p_j(\lambda_2)| < 4r$.

The uniqueness follows from hypothesis (ii) of the Theorem. For the existence observe that $X_{\lambda_2} \cap D(p_i(\lambda_1), 3r)$ is non-empty by Assertion 1 and therefore $D(p_i(\lambda_1), 3r)$ meets some $D(p_j(\lambda_2), r)$ by hypothesis (i) of the Theorem. The following is a direct consequence of Assertion 2.

Assertion 3. In every subset W of $\bar{\Omega}$ of diameter $< \delta$ the $\{p_i(\lambda)\}$ can be given as single-valued (discontinuous) functions satisfying $|p_i(\lambda_1) - p_i(\lambda_2)| < 4r$ if $\lambda_1, \lambda_2 \in W$, $1 \le i \le n$. The functions $\{p_i(\lambda)\}$ are unique up to order.

Now cover $\bar{\Omega}$ by a finite set of disks $\{D_k\}_{1 \le k \le N}$ such that $D_k \subseteq \bar{D}_k \subseteq W_k \subseteq \bar{\Omega}$, where $\{W_k\}_{1 \le k \le N}$ are open disks of diameter $< \delta$. Let $\{\varphi_t\}_{1 \le t \le T}$ be a smooth partition of unity of $\bar{\Omega}$ such that diameter(support φ_t) $< \delta$ for $1 \le t \le T$ and such that if support(φ_t) meets D_k then support(φ_t) $\subseteq W_k$ for $1 \le t \le T$, $1 \le k \le N$. Fix $u_t \in$ support(φ_t), $1 \le t \le T$.

Let $\{p_j^k\}_{1 \le j \le n}$ be single-valued branches of p_j in W_k, $1 \le k \le N$, which are given by Assertion 3. We define $\tilde{p}_j^k(z)$ for $z \in D_k$ by

$$\tilde{p}_j^k(z) = \sum_{t=1}^{T} p_j^k(u_t) \varphi_t(z).$$

Note that for $z \in D_k$, if $\varphi_t(z) \neq 0$, then $\operatorname{spt}\varphi_t$ meets D_k and hence $\operatorname{spt}(\varphi_t) \subseteq W_k$; hence $u_k \in W_k = \operatorname{domain} p_j^k$. Thus the $\tilde{p}_j^k$ are well-defined continuous functions; for $z \in D_k$,

$$|\tilde{p}_j^k(z) - p_j^k(z)| = |\sum (p_j^k(u_t) - p_j^k(z))\varphi_t(z)|.$$

For $z \in D_k$, $\varphi_t(z) = 0$ unless $z \in \operatorname{spt}(\varphi_t)$ and then $|u_t - z| < \delta$ and so

$$|p_j^k(u_t) - p_j^k(z)| < 4r$$

by Assertion 3. Hence we get

$$(*) \qquad |\tilde{p}_j^k(z) - p_j^k(z)| < 4r$$

for $z \in D_k$.

Assertion 4. The unordered n-tuples $\{\tilde{p}_j^k\}$ and $\{\tilde{p}_j^i\}$ coincide on $D_k \cap D_i$.

In fact, by Assertion 3, we can relabel the p_j^i such that $p_j^i(z) = p_j^k(z)$ for all $z \in W_k \cap W_i$. With this labeling it follows that $\tilde{p}_j^i(z) = \tilde{p}_j^k(z)$ for $z \in D_k \cap D_i$.

We have now verified (i) of the Lemma. For (ii) we observe that (*) implies $D(p_j(z), r) \subseteq D(w_j(z), 5r)$ (after a possible reordering of the $p_j(z)$). This gives the first part of (ii). The second part follows from the proof of Assertion 1 where the X_k are defined. Finally (iii) also follows from (*) and (i) of the Theorem.

3. Proof of the theorem. Fix j $(1 \leq j \leq n)$ and φ in the disc algebra with $\varphi(0) = 0$. By the lemma we have n locally defined functions $w_1(\lambda), \dots, w_n(\lambda)$ on $\bar{\Omega}$, and moreover we can write $X_\lambda = \bigcup_{k=1}^n X_k(\lambda)$ locally over Ω with $X_k(\lambda) \subseteq \{(\lambda, w) \in X : |w - w_k(\lambda) < 5r\}$. For a fixed disc Δ in Ω, $Y_k = \bigcup_{\lambda \in \Delta} X_k(\lambda)$ are well defined and $(\mathcal{P}_k, Y_k, \Delta, z_1)$ are maximum modulus algebras where $\mathcal{P}_k$ is the restriction of the polynomials to Y_k, $1 \leq k \leq n$. Thus we can apply the result of Section 1 over small disks in Ω to conclude that the function

$$\psi_1(\lambda) = \max_{\substack{w_s \in X_s(\lambda) \\ 1 \leq s \leq n}} \operatorname{Re}\left[\varphi(\lambda)(-1)^j \sum_{1 \leq i_1 < \cdots < i_j \leq n} w_{i_1} w_{i_2} \dots w_{i_j}\right]$$

is subharmonic on all of Ω. If ψ_2 is defined in the same way but with the negative of the function in square brackets it is also subharmonic in Ω. We define

$$\begin{aligned} G(\lambda) &= \operatorname{Re}\left[(-1)^j \varphi(\lambda) \sum_{1 \leq i_1 < \cdots < i_j \leq n} w_{i_1}(\lambda) w_{i_2}(\lambda) \dots w_{i_j}(\lambda)\right] \\ &= \operatorname{Re}(\varphi(\lambda) a_j(\lambda)) \end{aligned}$$

for all $\lambda \in \Omega$, where the continuous functions a_j are defined on $\bar{\Omega}$ by

$$X^n + a_1(\lambda) X^{n-1} + \cdots + a_n(\lambda) = \prod_{j=1}^n (X - w_j(\lambda)).$$

LEMMA.

$$|G(\lambda) - \psi_1(\lambda)| < 5\binom{n}{j-1}\left(\frac{1}{2}\right)^{j-1} nr|\varphi(\lambda)| \quad \textit{for } \lambda \in \Omega.$$

The same estimate holds with ψ_1 replaced by $-\psi_2$.

Proof. Fix $\lambda\in\Omega$. Let $F(w_1,w_2,\dots,w_n)=\sum_{1\le i_1<i_2<\cdots<i_j\le n} w_{i_1}w_{i_2}\dots w_{i_j}$. Then there exists $w_s^*\in X_s(\lambda)$ for $1\le s\le n$ such that $\psi_1(\lambda)=\mathrm{Re}[(-1)^j\varphi(\lambda)F(w_1^*,\dots,w_n^*)]$, and so $|G(\lambda)-\psi_1(\lambda)|\le|\varphi(\lambda)|\,|F(w_1^*,\dots,w_n^*)-F(w_1(\lambda),\dots,w_n(\lambda))|$. From

$$\partial F/\partial w_1=\sum_{2\le i_2<i_3<\cdots<i_j\le n} w_{i_2}w_{i_3}\dots w_{i_j}$$

we get

$$|\partial F/\partial w_1|<\binom{n}{j-1}\left(\frac{1}{2}\right)^{j-1}$$

if all $|w_k|<\frac{1}{2}$. Now $|w_k^*|<\frac{1}{2}$ and $|w_k(\lambda)|<\frac{1}{2}$, and so we get

$$|\nabla F|<\sqrt{n}\binom{n}{j-1}\left(\frac{1}{2}\right)^{j-1}$$

on the line segment from w^* to $w(\lambda)$. Also $\|w^*-w(\lambda)\|\le 5\sqrt{n}r$ as $|w_k^*-w_k(\lambda)|\le 5r$ for $1\le k\le n$. The estimate for ψ_1 now follows from the mean value theorem. The same argument applies to $-\psi_2$. □

Let h be the harmonic extension of $\mathrm{Re}(\varphi a_j)$ from $\partial\Omega$ to Ω. Let ω_0 be harmonic measure on $\partial\Omega$ for the origin.

LEMMA.

$$|h(0)|<5n\binom{n}{j-1}\left(\frac{1}{2}\right)^{j-1} r\int|\varphi|\,d\omega_0.$$

Proof. From the previous lemma we have $\psi_1-q|\varphi|<G<-\psi_2+q|\varphi|$ on Ω, where

$$q=5n\binom{n}{j-1}\left(\frac{1}{2}\right)^{j-1} r.$$

Hence

$$\int\psi_1\,d\omega_0-q\int|\varphi|\,d\omega_0<\int G\,d\omega_0<-\int\psi_2\,d\omega_0+q\int|\varphi|\,d\omega_0.$$

Since the ψ_i are subharmonic, we have $\int\psi_i\,d\omega_0\ge\psi_i(0)=0$. This and $\int G\,d\omega_0=h(0)$ yield the lemma. □

Let g be the harmonic extension of $\mathrm{Re}(\varphi a_j)$ from ∂D to D. We have, recalling $\partial\Omega=\partial D\cup\Gamma_0$.

$$\begin{aligned}\mathrm{Re}\int_{\partial D}\varphi a_j\,dm&=\int_{\partial D} g\,dm=g(0)\\&=\int_{\partial\Omega} g\,d\omega_0=\int_{\partial D}\mathrm{Re}(\varphi a_j)\,d\omega_0+\int_{\Gamma_0} g\,d\omega_0\\&=\int_{\partial\Omega}\mathrm{Re}(\varphi a_j)\,d\omega_0+\int_{\Gamma_0}(g-\mathrm{Re}(\varphi a_j))\,d\omega_0.\end{aligned}$$

Since $\int_{\partial\Omega}\mathrm{Re}\,\varphi a_j\,d\omega_0=h(0)$ we get

$$(*) \qquad \left|\operatorname{Re}\int_{\partial\Omega} \varphi a_j \, dm\right| \le |h(0)| + \omega_0(\Gamma_0)(\|g\|_{\Gamma_0} + \|\varphi\|_{\Gamma_0}\|a_j\|_D).$$

By the previous lemma $|h(0)| < q \int |\varphi| \, d\omega_0$. Letting u be the harmonic extension of $|\varphi|$ from ∂D to D, and using the fact that $|\varphi| \le u$ (since $|\varphi|$ is subharmonic), we get $\int |\varphi| \, d\omega_0 \le \int u \, d\omega_0 = u(0) = \int u \, dm = \int |\varphi| \, dm$; hence $|h(0)| \le q \int |\varphi| \, dm$. By estimating the Poisson integral we get $\|g\|_{\Gamma_0} \le (2/\rho)\|a_j\|_{\partial D} \int |\varphi| \, dm$ and $\|\varphi\|_{\Gamma_0} \le (2/\rho) \int |\varphi| \, dm$. Finally, using

$$\|a_j\|_{\partial D} \le \binom{n}{j}\left(\frac{1}{2}\right)^j$$

(since $|w_j(\lambda)| \le \frac{1}{2}$) in (*) gives

$$(**) \qquad \left|\operatorname{Re}\int \varphi a_j \, dm\right| < \left[q + \omega_0(\Gamma_0)\cdot\frac{4}{\rho}\binom{n}{j}\left(\frac{1}{2}\right)^j\right]\int |\varphi| \, dm.$$

Applying (**) for $e^{i\theta}\varphi$ with arbitrary real θ yields

$$\left|\int \varphi a_j \, dm\right| < \eta_j \cdot \int |\varphi| \, dm \text{ where } \eta_j = 5n\binom{n}{j-1}\left(\frac{1}{2}\right)^{j-1} r + \frac{4}{\rho}\omega_0(\Gamma_0)\binom{n}{j}\left(\frac{1}{2}\right)^j.$$

A standard duality argument now gives the following.

LEMMA. *This distance from a_j to the disc algebra in $C(\partial D)$ is $< \eta_j$. In particular there exists a polynomial f_j such that $|a_j - f_j|_{\partial D} < \eta_j$.*

We can now verify (1) of the main theorem. Let $(\lambda_0, w_0) \in X$ and let μ be a representing measure for this point which lives on $X_T = X \cap \{(\lambda, w) : |\lambda| = 1\}$ relative to the algebra of polynomials. Then, setting $a_0 \equiv 1 \equiv f_0$ and recalling $P = \sum f_j w^{n-j}$:

$$\begin{aligned} P(\lambda_0, w_0) &= \int P \, d\mu \\ &= \int_{X_T}\left(\sum_0^n f_j(\lambda) w^{n-j} - \sum_0^n a_j(\lambda) w^{n-j}\right) d\mu + \int_{X_T}\left(\sum_0^n a_j(\lambda) w^j\right) d\mu \\ &= \int_{X_T} \sum_{j=1}^n (f_j(\lambda) - a_j(\lambda)) w^{n-j} \, d\mu + \int_{X_T} \prod_{j=1}^n (w - w_j(\lambda)) \, d\mu. \end{aligned}$$

By the choice of f_j we have

$$\|(f_j(\lambda) - a_j(\lambda)) w^{n-j}\|_{X_T} \le \eta_j \cdot \left(\frac{1}{2}\right)^{n-j}$$

and

$$\begin{aligned} \sum_1^n \eta_j \left(\frac{1}{2}\right)^{n-j} &\le 10nr \sum_1^n \binom{n}{j-1}\left(\frac{1}{2}\right)^{j-1}\left(\frac{1}{2}\right)^{n-j+1} \\ &\quad + \frac{4}{\rho}\omega_0(\Gamma_0)\sum_1^n \binom{n}{j}\left(\frac{1}{2}\right)^j\left(\frac{1}{2}\right)^{n-j}. \end{aligned}$$

Each of these last two sums is <1. Also in $\Pi|w-w_j(\lambda)|$ for $(\lambda, w)\in X$ all factors are ≤ 1 and at least one factor is $\leq 5r$. Thus we obtain

$$|P(\lambda_0, w_0)| \leq 5(2n+1)r+(4/\rho)\,\omega_0(\Gamma_0).$$

This is (1); (2) follows directly from (1). □

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