# ON SOME CLASSES OF FIRST-ORDER DIFFERENTIAL SUBORDINATIONS

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**1. Introduction.** Let $f$ and $F$ be analytic in the unit disk $U$. The function $f$ is subordinate to $F$, written $f \prec F$ or $f(z) \prec F(z)$, if $F$ is univalent, $f(0) = F(0)$ and $f(U) \subset F(U)$.

In two previous articles [1 and 5] the authors investigated properties of the *Briot-Bouquet differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z). \tag{1}$$

This first-order differential subordination has many interesting applications in the theory of univalent functions [see 1, 5, and 8]. For special values of $\beta$, $\gamma$ and $h$ it has been investigated by many other authors.

The Briot-Bouquet differential subordination is but a special case of the general theory of differential subordinations introduced in [4, Section 4]. Restricting our attention to first-order differential subordinations, if $\psi: \mathbf{C}^2 \to \mathbf{C}$ is analytic in a domain $D$, if $h$ is univalent in $U$, and if $p$ is analytic in $U$ with $(p(z), zp'(z)) \in D$ when $z \in U$, then $p$ is said to satisfy the *first-order differential subordination*

$$\psi(p(z), zp'(z)) \prec h(z). \tag{2}$$

If we take $\psi(r,s) = r + s/(\beta r + \gamma)$ then the Briot-Bouquet differential subordination (1) can be written in the form (2).

Condition (2) is a generalization to complex function theory of the concept of first-order differential inequalities in real function theory. The similarity of the symbols $\prec$ and $<$ is appropriate since both represent an inclusion relation. In the real case there are many applications that require the finding of bounds on the function $p$ satisfying the differential inequality (2) with $\prec$ replaced by $<$. This is also the case for differential subordinations. We now repeat the following definitions of dominant and best dominant from [4, Definition 4], here restricted to the first-order case.

DEFINITION 1. The univalent function $q$ is said to be a *dominant* of the differential subordination (2) if $p \prec q$ for all $p$ satisfying (2). If $\tilde{q}$ is a dominant of (2) and $\tilde{q} \prec q$ for all dominants $q$ of (2), then $\tilde{q}$ is said to be the *best dominant* of (2). (Note that the best dominant is unique up to a rotation of $U$.)

In this article we determine dominants and best dominants of first-order differential subordinations. The special case of the Briot-Bouquet differential

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subordination given by (1) and its best dominant were analyzed in [5, Theorem 3]. Second-order differential subordinations, their dominants and best dominants were discussed in [4]. A good example of a best dominant for an $n$th-order differential subordination has been described in a recent paper by Goldstein, Hall, Sheil-Small, and Smith [2]. These authors show that $q(z)=z/2$ is the best dominant for the $n$th-order differential subordination

$$p(z)+zp'(z)+z^2p^{(2)}(z)+\cdots+z^np^{(n)}(z)\prec z.$$

We will present two major groupings of theorems for the differential subordination (2). The first grouping involves finding the dominants corresponding to a given class of functions $\psi$. The second grouping involves the reverse problem of finding the class of functions $\psi$ corresponding to a given dominant $q$. These results are given in Sections 3 and 4 respectively. Applications of these results in univalent function theory are presented in Section 5. Section 2 is concerned with two lemmas and a review of several definitions that are needed in the remainder of this article.

## 2. Preliminaries.

LEMMA 1. *Let $p$ be analytic in $U$ and let $q$ be analytic and univalent in $\bar{U}$ with $p(0)=q(0)$. If $p$ is not subordinate to $q$ then there exists points $z_0\in U$ and $\zeta_0\in\partial U$, and an $m\geq 1$ for which $p(|z|<|z_0|)\subset q(U)$,*

(i) $p(z_0)=q(\zeta_0)$, *and*

(ii) $z_0p'(z_0)=m\zeta_0q'(\zeta_0)$.

The proof of a more general form of this lemma may be found in [4, Lemma 1].

Our next lemma deals with the notion of a subordination chain. A function $L(z,t)$, $z\in U$, $t\geq 0$, is a *subordination chain* if $L(\cdot,t)$ is analytic and univalent in $U$ for all $t\geq 0$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z\in U$, and $L(z,s)\prec L(z,t)$ when $0\leq s\leq t$.

LEMMA 2 [7, p. 159]. *The function $L(z,t)=a_1(t)z+\cdots$, with $a_1(t)\neq 0$ for all $t\geq 0$, is a subordination chain if and only if*

$$\operatorname{Re}\left[z\,\frac{\partial L}{\partial z}\Big/\frac{\partial L}{\partial t}\right]>0,$$

*for $z\in U$ and $t\geq 0$.*

For consistency of notation we include the following definitions [7]. Suppose that $f$ is analytic in $U$. The function $f$ with $f'(0)\neq 0$ is *convex* (*univalent*) if and only if $\operatorname{Re}[1+zf''(z)/f'(z)]>0$, $z\in U$. The function $f$, with $f'(0)\neq 0$ and $f(0)=0$, is *starlike* (*univalent*) if and only if $\operatorname{Re}[zf'(z)/f(z)]>0$, $z\in U$. The function $f$ is *close-to-convex* (*univalent*) if and only if there is a starlike function $g$ such that $\operatorname{Re}[zf'(z)/g(z)]>0$, $z\in U$.

**3. Differential subordinations I.** This section deals with finding dominants and best dominants for several classes of $\psi$.

THEOREM 1. *Let h be analytic in U, let $\phi$ be analytic in a domain D containing h(U) and suppose*

(a) $\operatorname{Re}\phi(h(z))>0,\ z\in U$

*and either*

(b) *$h(z)$ is convex, or*

(b′) *$H(z)\equiv zh'(z)\phi(h(z))$ is starlike.*

*If p is analytic in U with $p(0)=h(0)$, $p(U)\subset D$, and*

$$p(z)+zp'(z)\phi(p(z))\prec h(z), \tag{3}$$

*then $p(z)\prec h(z)$.*

*Proof.* Without loss of generality we can assume that $p$ and $h$ satisfy the conditions of the theorem on the closed disc $\bar{U}$. If not, then we can replace $p(z)$ by $p_r(z)=p(rz)$, and $h(z)$ by $h_r(z)=h(rz)$, where $0<r<1$. These new functions satisfy the conditions of the theorem on $\bar{U}$. We would then prove $p_r(z)\prec h_r(z)$ for all $0<r<1$. By letting $r\to 1^-$, we obtain $p(z)\prec h(z)$.

*Case 1.* Suppose (a) and (b) are satisfied, but $p$ is not subordinate to $h$. According to Lemma 1, there are points $z_0\in U$ and $\zeta_0\in\partial U$, and an $m\geq 1$ such that

$$p(z_0)+z_0p'(z_0)\phi(p(z_0))=h(\zeta_0)+m\zeta_0h'(\zeta_0)\phi(h(\zeta_0)). \tag{4}$$

The complex number (or vector) $\zeta_0h'(\zeta_0)\phi(h(\zeta_0))$ satisfies

$$\arg[\zeta_0h'(\zeta_0)\phi(h(\zeta_0))]=\arg\zeta_0h'(\zeta_0)+\arg\phi(h(\zeta_0)). \tag{5}$$

From (a) we see that $|\arg\phi(h(\zeta_0))|<\pi/2$. Using this fact together with the fact that $\zeta_0h'(\zeta_0)$ is an outward normal to the boundary of the convex domain $h(U)$, (5) implies that (4) is a complex number outside of $h(U)$. This contradicts (3) and hence we conclude that $p\prec h$.

*Case 2.* If (a) and (b′) are satisfied then we obtain $\operatorname{Re}[zh'(z)/H(z)]=\operatorname{Re}[1/\phi(h(z))]>0$, which implies that $h$ is close-to convex (univalent), and hence that (3) is well defined. Since $h$ and $H$ are analytic in $U$ the function

$$L(z,t)\equiv h(z)+tzh'(z)\phi(h(z))=h(z)+tH(z) \tag{6}$$

is analytic in $U$ for all $t\geq 0$. From (a) we obtain

$$\frac{\partial L}{\partial z}(0,t)=h'(0)[1+t\phi(h(0))]\neq 0,\quad \text{for } t\geq 0.$$

This function is also continuously differentiable on $[0,\infty)$, for all $z\in U$. A simple calculation combined with (a) and (b′) yields

$$\operatorname{Re}\left[z\frac{\partial L}{\partial z}\bigg/\frac{\partial L}{\partial t}\right]=\operatorname{Re}\left[\frac{1}{\phi(h(z))}\right]+t\operatorname{Re}\left[\frac{zH'(z)}{H(z)}\right]>0,$$

for $z\in U$ and $t\geq 0$. Hence by Lemma 2, $L(z,t)$ is a subordination chain and we have $L(z,s)\prec L(z,t)$ for $0\leq s\leq t$. From (6) we obtain $h(z)=L(z,0)$ and hence

$$L(\zeta, t) \notin h(U), \tag{7}$$

for $|\zeta| = 1$ and $t \geq 0$.

Now assume that $p \nprec h$. As in Case 1 we have condition (4), which can be rewritten using (6) as

$$p(z_0) + z_0 p'(z_0)\phi(p(z_0)) = L(\zeta_0, m),$$

where $z_0 \in U$, $|\zeta_0| = 1$, and $m \geq 1$. Combined with (7) this contradicts (3). Hence $p \prec h$, completing the proof of the theorem. □

By carefully selecting the function $\phi$ we obtain the following corollaries.

COROLLARY 1.1. *Let $\beta$ and $\gamma$ be complex numbers with $\beta \neq 0$ and let $p$ and $h$ be analytic in $U$ with $h(0) = p(0)$. If $Q(z) \equiv \beta h(z) + \gamma$ satisfies*

(a) $\operatorname{Re} Q(z) > 0$, $z \in U$

*and either*

(b) *$Q$ is convex, or*

(b′) *$\log Q$ is convex*

*then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \tag{8}$$

*implies that $p(z) \prec h(z)$.*

*Proof.* If we set $\phi(w) = (\beta w + \gamma)^{-1}$ then (8) becomes (3) and since

$$\operatorname{Re} 1/\phi(h(z)) = \operatorname{Re} Q(z),$$

from (a) we obtain $\operatorname{Re}\phi(h) > 0$. Condition (b) implies that $h$ is convex, while condition (b′) implies that $H(z) \equiv \beta^{-1} z d[\log Q(z)]/dz = zh'(z)/(\beta h(z) + \gamma) = zh'(z)\phi(h(z))$ is starlike. All the conditions of Theorem 1 are thus satisfied, and so we have $p \prec h$. □

This corollary provides dominants for the Briot-Bouquet differential subordination (1). The authors in [1, Theorem 1] proved a weaker form of this corollary using only conditions (a) and (b). We present an example which requires the use of (a) and (b′), and which could not be previously handled. If we take $\gamma = 0$, $\beta = 1/\alpha > 0$, and $h(z) = e^{\lambda z}$ with $1 \leq |\lambda| \leq \pi/2$, then $Q(z) = e^{\lambda z}/\alpha$ satisfies (a) and (b′). The domain $Q(U)$ is lima-bean shaped and is clearly not convex. Thus (b) is not satisfied in these cases, but we still have

$$p(z) + \alpha \frac{zp'(z)}{p(z)} \prec e^{\lambda z} \Rightarrow p(z) \prec e^{\lambda z}.$$

By an analysis similar to that of Corollary 1.1, but using $\phi(w) = \beta w + \gamma$ and $\phi(w) = (\beta w + \gamma)^{-2}$ respectively in Theorem 1, we obtain the following corollaries.

COROLLARY 1.2. *Let $\beta$ and $\gamma$ be complex numbers with $\beta \neq 0$ and let $p$ and $h$ be analytic in $U$ with $h(0) = p(0)$. If $Q(z) = \beta h(z) + \gamma$ satisfies*

(i) $\operatorname{Re} Q(z) > 0$, $z \in U$ *and*
(ii) $Q$ *or* $Q^2$ *are convex,*
*then* $p(z)+zp'(z)[\beta p(z)+\gamma] \prec h(z)$ *implies* $p(z) \prec h(z)$.

COROLLARY 1.3. *Let* $\beta$ *and* $\gamma$ *be complex numbers with* $\beta \neq 0$ *and let* $p$ *and* $h$ *be analytic in* $U$ *with* $h(0)=p(0)$. *If* $Q(z)=\beta h(z)+\gamma$ *satisfies*
(i) $\operatorname{Re} Q^2(z) > 0$, $z \in U$ *and*
(ii) $Q$ *or* $1/Q$ *are convex,*
*then* $p(z)+zp'(z)[\beta p(z)+\gamma)^{-2} \prec h(z)$ *implies* $p(z) \prec h(z)$.

Our next theorem gives the best dominant for a different class of differential subordinations.

THEOREM 2. *Let* $h$ *be convex in* $U$ *and* $\theta$ *and* $\phi$ *be analytic in a domain* $D$. *Let* $p$ *be analytic in* $U$, *with* $p(0)=h(0)=\theta(p(0))$ *and* $p(U) \subset D$. *If the differential equation*

$$\theta(q(z))+zq'(z)\phi(q(z))=h(z) \tag{9}$$

*has a univalent solution in* $U$ *that satisfies* $q(0)=h(0)$ *and*

$$\theta(q(z)) \prec h(z), \tag{10}$$

*then the relation*

$$\theta(p(z))+zp'(z)\phi(p(z)) \prec h(z) \tag{11}$$

*implies* $p(z) \prec q(z)$. *The function* $q$ *is the best dominant of* (11).

*Proof.* As in Theorem 1, we can assume that the functions $p$, $q$ and $h$ satisfy the conditions of this theorem in the closed disc $\bar{U}$.

If $p$ is not subordinate to $q$, then by Lemma 1 there are points $z_0 \in U$ and $\zeta_0 \in \partial U$, and an $m \geq 1$, such that

$$\theta(p(z_0))+z_0 p'(z_0)\phi(p(z_0))=\theta(q(\zeta_0))+m\zeta_0 q'(\zeta_0)\phi(q(\zeta_0)).$$

We can simplify this latter expression by using (9) and obtain

$$\theta(p(z_0))+z_0 p'(z_0)\phi(p(z_0))=h(\zeta_0)+(m-1)[h(\zeta_0)-\theta(q(\zeta_0))].$$

From (10) and the fact that $h(U)$ is convex we conclude that the above right-hand term (and hence the left-hand term) is not in $h(U)$. This contradicts (11) and hence we must have $p \prec q$.

From (9) we see that $q$ also satisfies (11). This implies that the dominant $q$ is also the best dominant of (11). □

For the special case $\theta(w)=w$ we can combine Theorems 1 and 2, and obtain the following result.

COROLLARY 2.1. *Let* $h$ *be convex in* $U$ *and let* $\phi$ *be analytic in a domain* $D$ *containing* $h(U)$. *Let* $p$ *be analytic in* $U$ *with* $p(0)=h(0)$ *and* $p(U) \subset D$, *and suppose that the differential equation*

$$(12)\qquad q(z)+zq'(z)\phi(q(z))=h(z)$$

*has a univalent solution that satisfies* $q(0)=h(0)$. *If* $\operatorname{Re}\phi(h(z))>0$, *for* $z\in U$, *then*

$$(13)\qquad p(z)+zp'(z)\phi(p(z))\prec h(z)$$

*implies* $p\prec q\prec h$, *where* $q$ *is the best dominant of* (13).

*Proof.* From (12) we see that $q$ satisfies (3) and since (a) and (b) of Theorem 1 are satisfied we have $q\prec h$. If we take $\theta(w)=w$, then we see that all the conditions of Theorem 2 are satisfied, and we obtain the conclusion of the corollary.

The special case $\phi(w)=(\beta w+\gamma)^{-1}$, corresponds to the Briot-Bouquet differential subordination (1), which was analyzed in [5].

**4. Differential subordinations II.** In this section we will again consider differential subordinations of the form (11). However, in this section we first select the dominant and then find the appropriate $h$ corresponding to this dominant.

THEOREM 3. *Let* $q$ *be univalent in* $U$ *and let* $\theta$ *and* $\phi$ *be analytic in a domain* $D$ *containing* $q(U)$, *with* $\phi(w)\neq 0$ *when* $w\in q(U)$. *Set* $Q(z)=zq'(z)\phi(q(z))$, $h(z)=\theta(q(z))+Q(z)$ *and suppose that*

(i) $Q$ *is starlike (univalent) in* $U$, *and*

$$\text{(ii)}\qquad \operatorname{Re}\frac{zh'(z)}{Q(z)}=\operatorname{Re}\left[\frac{\theta'(q(z))}{\phi(q(z))}+\frac{zQ'(z)}{Q(z)}\right]>0,\quad z\in U.$$

*If* $p$ *is analytic in* $U$, *with* $p(0)=q(0)$, $p(U)\in D$ *and*

$$(14)\qquad \theta(p(z))+zp'(z)\phi(p(z))\prec\theta(q(z))+zq'(z)\phi(q(z))=h(z),$$

*then* $p\prec q$, *and* $q$ *is the best dominant of* (14).

*Proof.* As was done in Theorem 1, we can assume that the functions $p$, $q$ and $h$ satisfy the conditions of this theorem in the closed disc $\bar{U}$.

Since $Q$ is starlike, from (ii) we deduce that the function $h$ is univalent (close-to-convex) and hence (14) is well defined.

The function

$$(15)\qquad L(z,t)=h(z)+tQ(z)=\theta(q(z))+(1+t)Q(z)$$

is analytic in $U$ for all $t\geq 0$, and is continuously differentiable on $[0,\infty)$ for all $z\in U$. From (ii) we obtain

$$\frac{\partial L}{\partial z}(0,t)=\phi(q(0))q'(0)\left[\frac{\theta'(q(0))}{\phi(q(0))}+1+t\right]\neq 0,\quad\text{for } t\geq 0.$$

A simple calculation combined with (ii) yields

$$\operatorname{Re}\left[z\frac{\partial L}{\partial z}\bigg/\frac{\partial L}{\partial t}\right]=\operatorname{Re}\left[\frac{\theta'(q(z))}{\phi(q(z))}+(1+t)\frac{zQ'(z)}{Q(z)}\right]\geq\operatorname{Re}\frac{zh'(z)}{Q(z)}>0.$$

Hence by Lemma 2, $L(z,t)$ is a subordination chain and we have $L(z,s)\prec L(z,t)$, for $0\leq s\leq t$. From (15) we obtain $L(z,0)=h(z)$, and hence we must have

(16) $$L(\zeta, t) \notin h(U), \quad \text{for } |\zeta|=1 \text{ and } t \geq 0.$$

Now assume that $p \nprec q$. From Lemma 1, there are points $z_0 \in U$ and $\zeta_0 \in \partial U$, and an $m \geq 1$ such that

$$\theta(p(z_0)) + z_0 p'(z_0)\phi(p(z_0)) = \theta(q(\zeta_0)) + m\zeta_0 q'(\zeta_0)\phi(q(\zeta_0)).$$

Since $Q(z) = zq'(z)\phi(q(z))$ and $m \geq 1$, from (15) and (16) we obtain

$$\theta(p(z_0)) + z_0 p'(z_0)\phi(p(z_0)) = L(\zeta_0, m-1) \notin h(U).$$

But this contradicts (14), and hence we must have $p \prec q$. Since $p = q$ satisfies (14), the function $q$ is the best dominant of (14). □

REMARK. If we take $\theta(w) \equiv 0$ in Theorem 3, then condition (ii) reduces to (i) and we obtain

$$zp'(z)\phi(p(z)) \prec zq'(z)\phi(q(z)) \Rightarrow p(z) \prec q(z),$$

when $q$ is univalent and $zq'(z)\phi(q(z))$ is starlike.

We conclude this section by considering differential subordinations that are obtained from Theorem 3 by selecting different dominants $q$.

(A) *Dominant* $q(z) = z$. In this case, besides the requirements that $\theta$ and $\phi$ be analytic in a domain $D \supset U$, the conditions of Theorem 3 reduce to

(17) $$Q(z) = z\phi(z) \quad \text{is starlike in } U,$$

(18) $$\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re}\left[\frac{\theta'(z)}{\phi(z)} + 1 + \frac{z\phi'(z)}{\phi(z)}\right] > 0, \quad \text{and}$$

(19) $$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(z) + z\phi(z) = h(z).$$

If we take $\phi(z) = 1$ in (17), (18) and (19) then from Theorem 3 we obtain:

COROLLARY 3.1. *Let $\theta$ be analytic in a domain $D \supset U$ and suppose* $\operatorname{Re}\theta'(z) > -1$. *If $p$ is analytic in $U$, with $p(0) = 0$, $p(U) \subset D$ and*

$$\theta(p(z)) + zp'(z) \prec \theta(z) + z,$$

*then $p(z) \prec z$ and this is the best dominant.*

If we take $\phi(z) = \theta(z)/z$, $\theta(0) = 0$, in (17), (18) and (19), then from Theorem 3 we obtain:

COROLLARY 3.2. *Let $\theta$ be analytic in a domain $D \supset U$ and suppose $\theta$ is starlike in $U$. If $p$ is analytic in $U$, with $p(0) = 0$, $p(U) \subset D$ and*

$$\theta(p(z))[1 + zp'(z)/p(z)] \prec 2\theta(z),$$

*then $p(z) \prec z$, and this is the best dominant.*

If we take $\phi(z) = \theta'(z)$ in (17), (18) and (19) then from Theorem 3 we obtain:

COROLLARY 3.3. *Let $\theta$ be analytic in a domain $D \supset U$ and suppose $\theta$ is convex in $U$. If $p$ is analytic in $U$, with $p(0) = 0$, $p(U) \subset D$ and*

$$\theta(p(z))+zp'(z)\theta'(p(z)) \prec \theta(z)+z\theta'(z),$$

*then* $p(z) \prec z$ *and this is the best dominant.*

If we take $\theta(z)=z$ and $\phi(z)=(1+\lambda z)^{-1}$, $|\lambda|\le 1$, in (17), (18) and (19), then from Theorem 3 we obtain:

COROLLARY 3.4. *Let* $\lambda$ *be a complex number with* $|\lambda|\le 1$, *and let* $p$ *be analytic in* $U$, *with* $p(0)=0$. *If* $p$ *satisfies*

$$p(z)+\frac{zp'(z)}{1+\lambda p(z)} \prec \frac{z(2+\lambda z)}{1+\lambda z}, \tag{20}$$

*then* $p(z) \prec z$ *and this is the best dominant.*

An application of (20) in univalent function theory will be presented in the next section.

If we take $\theta(z)=z$ and $\phi(z)$ to be $1+\lambda z$, or $e^{\lambda z}$, or $(1+\lambda z)/(1-\lambda z)$, then from Theorem 3 we obtain respectively:

$$p(z)+zp'(z)(1+\lambda p(z)) \prec 2z+\lambda z \Rightarrow p(z) \prec z \quad \text{for } |\lambda|\le 1/2,$$

$$p(z)+zp'(z)e^{\lambda z} \prec z(1+e^{\lambda z}) \Rightarrow p(z) \prec z \quad \text{for } |\lambda|\le 1, \text{ and}$$

$$p(z)+zp'(z)\frac{(1+\lambda p(z))}{(1-\lambda p(z))} \prec \frac{2z}{1-\lambda z} \Rightarrow p(z) \prec z \quad \text{for } |\lambda|\le \sqrt{2}-1.$$

(B) *Dominant* $q(z)=(1+z)/(1-z)$. If we take $\theta(w)=w$ and $\phi(w)=1/w$ then conditions (i) and (ii) of Theorem 3 will be satisfied and so we obtain:

COROLLARY 3.5. *Let* $p$ *be analytic in* $U$ *with* $p(0)=1$. *If* $p$ *satisfies*

$$p(z)+\frac{zp'(z)}{p(z)} \prec \frac{1+z}{1-z}+\frac{2z}{1-z^2}=h(z),$$

*then* $p(z) \prec (1+z)/(1-z)$ *and this is the best dominant.*

An application of this result in univalent function theory will be presented in the next section.

(C) *Dominant* $q(z)=z/(1-z)^2$. If we take $\theta(w)=w$ and $\phi(w)=1/(1+4w)^{1/2}$ then conditions (i) and (ii) of Theorem 3 will be satisfied and so we obtain:

COROLLARY 3.6. *Let* $p$ *be analytic in* $U$ *with* $p(0)=0$. *If* $p$ *satisfies*

$$p(z)+\frac{zp'(z)}{\sqrt{1+4p(z)}} \prec \frac{2z}{(1-z)^2},$$

*then* $p(z) \prec z/(1-z)^2$ *and this is the best dominant.*

**5. Applications to univalent functions.** In [3] S. Kudryashov investigated the maximum value of $M$ for which

$$|f''(z)/f'(z)|\le M \tag{21}$$

implies $f$ is univalent in $U$. He showed that if $M = M_1 = 3.05\ldots$, where $M_1$ is the root of $8[x(x-2)^3]^{1/2} - 3(4-x)^2 = 12$, then any function $f$ analytic in $U$ that satisfies (21) will be univalent in $U$. The maximum value of $M$ cannot be larger than $\pi$. This follows from considering the functions $f(\lambda, z) = e^{\lambda z}$ for $|\lambda| \le \pi$ and $z \in U$. For these functions $|f''/f'| = |\lambda|$, whereas $f(\lambda, z)$ is univalent if and only if $|\lambda| < \pi$. The maximum value of $M$ for which (21) implies univalence is unknown, but this value satisfies $M_1 \le M \le \pi$.

A similar, but sharp result for convexity follows immediately since (21) is equivalent to $|(zf''(z)/f'(z)+1)-1| < M$, which implies

$$\operatorname{Re}[zf''(z)/f'(z)+1] > 1 - M.$$

Thus, letting $M = 1$ in (21) implies that $f$ is convex. Furthermore, $M = 1$ is the maximum value for which (21) implies convexity of $f$.

We now use Corollary 3.5 to obtain a similar result for starlike functions.

THEOREM 4. *If $f$ is analytic in $U$, $f(0) = 0$, and*

$$|f''(z)/f'(z)| \le 2, \tag{22}$$

*for $z \in U$, then $f$ is a starlike function.*

*Proof.* From Kudryashov's result we deduce that $f$ is univalent in $U$. If we set $p(z) = zf'(z)/f(z)$, then $p$ is analytic in $U$, $p(0) = 1$ and $p(z) \neq 0$ for $z \in U$. Using this $p$ and $h(z) = (z^2 + 4z + 1)/(1 - z^2)$ in Corollary 3.5 we obtain

$$\frac{zf''(z)}{f'(z)} + 1 \prec h(z) \Rightarrow \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}. \tag{23}$$

The function $w = h(z)$ maps $U$ onto the complex plane minus the half-lines $\operatorname{Re} w = 0$, $\operatorname{Im} w \ge \sqrt{3}$ and $\operatorname{Re} w = 0$, $\operatorname{Im} w \le -\sqrt{3}$. From (22) we have

$$|(zf''(z)/f'(z)+1)-1| < 2$$

for $z \in U$, and thus $zf''(z)/f'(z)$ is contained in the disc $|w-1| < 2$, for $z \in U$. It is easy to check that this disc is in $h(U)$, and so we have $zf''(z)/f'(z)+1 \prec h(z)$. Combining these results with (23) we deduce that (22) implies $zf'(z)/f(z) \prec (1+z)/(1-z)$. Hence $\operatorname{Re} zf'(z)/f(z) > 0$ and the function $f$ is a starlike function.

The function $f(z) = e^{\lambda z} - 1$ is starlike if and only if $|\lambda| < M_2$, where $M_2 = 2.8329\ldots$ [6, p. 338]. Since $|f''(z)/f'(z)| = |\lambda|$, we see that the constant $M$ in (21) which implies starlikeness of $f$ is at most $M_2$. Hence the maximum value of $M$ for which (21) implies starlikeness satisfies $2 \le M \le M_2$. The exact value of this $M$ remains an interesting open question.

The result (23) has a representation in terms of the Koebe function $k(z) = z/(1-z)^2$. This is the extremal function for many problems in univalent function theory. If $f$ is starlike, that is $zf'(z)/f(z) \prec (1+z)/(1-z) = zk'(z)/k(z)$, then it is well known that $f(z)/z \prec k(z)/z$ [7, p. 50]. A simple calculation shows that $zk''/k' + 1 = h$, and so from (23) we obtain the interesting chain

$$\frac{zf''}{f'}+1 \prec \frac{zk''}{k'}+1 \Rightarrow \frac{zf'}{f} \prec \frac{zk'}{k} \Rightarrow \frac{f}{z} \prec \frac{k}{z}.$$

For our last application we use Corollary 3.4 to obtain another inequality that implies starlikeness.

THEOREM 5. *If $f$ is analytic in $U$ with $f(0)=0$ and*

$$|zf''(z)/f'(z)+1|<2, \tag{24}$$

*then $f$ is starlike and $|zf'(z)/f(z)-1|<1$.*

*Proof.* If we let $h(z)=1+z+z/(1+z)$, $P(z)=p(z)+1$ and $\lambda=1$ in Corollary 3.4 we obtain

$$P(z)+\frac{zP'(z)}{P(z)} \prec h(z) \Rightarrow P(z)-1 \prec z, \tag{25}$$

when $P(z)$ is analytic in $U$ with $P(z)\neq 0$ and $P(0)=1$.

From (24) and Schwarz's lemma we obtain $|f''(z)/f'(z)|<3$ and hence, by Kudryashov's result, the function $f$ is univalent. If we set $P(z)=zf'(z)/f(z)$, then $P$ is analytic in $U$, $P(z)\neq 0$, for $z\in U$ and $P(0)=1$. Using this $P$ in (25) we obtain

$$zf''(z)/f'(z)+1 \prec h(z) \Rightarrow zf'(z)/f(z)-1 \prec z. \tag{26}$$

Using some simple calculus and analytic geometry, one can show that the disc $|w|<2$ is contained in $h(U)$. Thus if (24) is satisfied we have $zf''(z)/f'(z)+1 \prec h(z)$, which by (26) implies that $|zf'(z)/f(z)-1|<1$. Hence $\operatorname{Re} zf'(z)/f(z)>0$ and $f$ is a starlike function. □

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