APPROXIMATE FIBRATIONS ON TOPOLOGICAL MANIFOLDS

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1. Introduction. The main results of this paper deal with parameterized families of approximate fibrations. Approximate fibrations were introduced by Coram and Duvall in [4] as a generalization of both Hurewicz fibrations and cell-like maps. These maps have been studied by many authors, not only because of their intrinsic interest, but also because they arise naturally in certain problems concerning topological manifolds. Our main results are applied herein to study the relationship of approximate fibrations to bundles and fibrations, the local connectivity of spaces of bundles and fibrations, and the homotopy relation in the space of controlled homotopy topological structures on a fibration.

Let E and B be locally compact separable metric ANRs, let $p: E \to B$ be a proper map (i.e., inverse images of compacta are compact), let α be an open cover of B, and let C be a subset of B. We say that p is an *α-fibration over C* provided that given any X and maps $F: X \times [0,1] \to C$ and $f: X \to E$ for which $F(x,0) = pf(x)$, then there exists a map $\tilde{F}: X \times [0,1] \to E$ such that $\tilde{F}(x,0) = f(x)$ and $p\tilde{F}$ is α-close to F. If $C = B$, then p is called an *α-fibration.* And if p is an α-fibration for every open cover α of B, then p is an *approximate fibration.* If $\epsilon > 0$, then ϵ also denotes the open cover of B by balls of diameter ϵ. Thus, we also speak of *ϵ-fibrations.*

In this paper a *fiber preserving* (*f.p.*) map is a map which preserves the obvious fibers over an n-simplex Δ. Specifically, if $\rho: X \to \Delta$, $\sigma: Y \to \Delta$, and $f: X \to Y$ are maps, then f is f.p. if $\sigma f = \rho$. Usually the maps ρ and σ will be understood to be some natural projections and will not be explicitly mentioned. If $p: E \times \Delta \to B \times \Delta$ is a f.p. map, then f is an approximate fibration if and only if $f_t: E \to B$ is an approximate fibration for each t in Δ. This can be derived from [5].

A *manifold* will be understood to mean a topological manifold which possesses a handlebody decomposition. It is now known that this includes all topological manifolds except nonsmoothable 4-manifolds (see [19]).

Our first main result is a parameterized version of a theorem of Chapman [2, Theorem 1]. It enables one to detect which parameterized families of maps can be deformed to a close-by parameterized family of approximate fibrations.

DEFORMATION THEOREM. *Let B be a polyhedron, let $m \geq 5$, let Δ be an n-simplex, and let α be an open cover of B. There exists an open cover β of B so that if M is an m-manifold without boundary and $f: M \times \Delta \to B \times \Delta$ is a f.p. map such that $f_t: M \to B$ is a β-fibration for each t in Δ and an approximate fibration for each t in $\partial\Delta$, then there is a f.p. approximate fibration $\tilde{f}: M \times \Delta \to B \times \Delta$ such that $\tilde{f}_t$ is α-close to f_t for each t in Δ and $\tilde{f} \mid M \times \partial\Delta = f \mid M \times \partial\Delta$.*

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The proof of this theorem is outlined in Sections 2–5 with the reader being referred to [2] and [11] for many details.

The following corollary of the Deformation Theorem appears in Section 5. In this paper spaces of maps are always given the compact-open topology.

COROLLARY 1. *If M is a closed m-manifold ($m \geq 5$) and B is a compact polyhedron, then both the space of approximate fibrations and the space of cell-like maps from M to B are locally n-connected for each $n \geq 0$.*

As will be pointed out in Section 5, both the Deformation Theorem and Corollary 1 remain true if B is replaced by a compact manifold (which has a handlebody decomposition by our convention). Haver has proved a theorem which implies that the space of cell-like maps from a closed m-manifold ($m \geq 5$) onto *itself* is weakly locally contractible [10].

Our second main result is a parameterized approximation theorem for approximate fibrations. It has previously been proved by Ferry in the special case $B = S^1$ [8].

APPROXIMATION THEOREM. *Let M and B be closed manifolds with* $\dim M \geq 5$. *Let Δ be an n-simplex with 0 in $\partial\Delta$ and let $p: M \times \Delta \to B \times \Delta$ be a f.p. approximate fibration. For every $\epsilon > 0$ there exists a f.p. homeomorphism* $H: M \times \Delta \to M \times \Delta$ such that $H_0 = \mathrm{id}$ *and pH is ϵ-close to* $p_0 \times \mathrm{id}_\Delta$.

The proof of this theorem is given in Sections 6 and 7. The key step is a handle lemma which is proved by means of a torus trick in Section 6.

Our first application of the Approximation Theorem concerns the local homotopy connectivity of certain subspaces of the space of approximate fibrations.

COROLLARY 2. *Let M and B be closed manifolds with* $\dim M \geq 5$. *Then both the space of bundle projections and the space of Hurewicz fibrations from M to B are locally n-connected for each $n \geq 0$.*

As far as the author knows this is the first result concerning the local connectivity of spaces of bundles or fibrations between topological manifolds. Actually our techniques apply to many more subspaces of the space of approximate fibrations than those mentioned above (see Section 7). Whether any of these spaces of maps are locally contractible remains an open question.

Our next application concerns the general question: *when can an approximate fibration be arbitrarily closely approximated by bundle projections?* This question has received the attention of several authors (e.g. Chapman [2], Chapman and Ferry [3], Goad [9], Husch [14], Quinn [17], [18]). The Deformation and Approximation Theorems imply the following result (see Section 7).

COROLLARY 3. *Let M and B be closed manifolds with* $\dim M \geq 5$. *An approximate fibration $p: M \to B$ can be approximated arbitrarily closely by bundle projections if and only if p is homotopic via approximate fibrations to a bundle projection.*

This should be compared with Husch's theorem [14] which says an approximate fibration $p: M \to S^1$ (where M is a closed manifold, $\dim M \geq 6$) can be approximated arbitrarily closely by bundle projections if and only if p is homotopic to a bundle projection. However, the examples of Chapman and Ferry [3] show Husch's theorem is false for approximate fibrations $p: M \to S^2$.

As a further application of our theorems to the relationship between approximate fibrations and bundles, we prove the following result in Section 7.

COROLLARY 4. *Let M and B be closed manifolds with* $\dim M \geq 5$. *The inclusion of the space of bundle projections from M to B into the space of approximate fibrations from M to B induces an isomorphism on homotopy groups and a monomorphism on path components.*

Our final application concerns controlled homotopy topological structures. To explain this further, let $p: E \to B$ be a Hurewicz fibration between closed manifolds ($\dim E \geq 5$). In Section 8 a semi-simplicial complex $\mathcal{S}(p: E \to B)$ is defined whose vertices are represented by maps $f: M \to E$ where M is a closed manifold, $\dim M = \dim E$, and f is a $p^{-1}(\epsilon)$-equivalence for every $\epsilon > 0$. (Another such map $f': M' \to E$ represents the same vertex if there is a homeomorphism $h: M \to M'$ such that $f'h = f$.) Here is a special case of the result in Section 8.

COROLLARY 5. *Let $f: M \to E$ and $f': M' \to E$ represent two vertices in $\mathcal{S}(p: E \to B)$. Then those vertices lie in the same path component of $\mathcal{S}(p: E \to B)$ if and only if for every $\epsilon > 0$ there exists a homeomorphism $h: M \to M'$ such that $f'h$ is $p^{-1}(\epsilon)$-homotopic to f.*

The theorems and proofs in this paper are heavily influenced by the author's Hilbert cube manifold results in [11] and [12]. Here are some of the differences and similarities between those papers and the present one. The proof in [11] of the Hilbert cube manifold version of the Deformation Theorem is based on a parameterized engulfing result which is proved by using a parameterized lifting property of parameterized approximate fibrations. In this paper, a parameterized engulfing result is again the basis for the Deformation Theorem. However, we have found a way to derive this parameterized engulfing result from a (non-parameterized) engulfing result of Chapman [2] by using a method found in the Kirby–Siebenmann book [15, Essay II]. This is carried out in Section 2. The rest of the proof of the Deformation Theorem follows [11] closely. This is outlined in Sections 3–5.

The Approximation Theorem contains a major improvement over the corresponding Hilbert cube manifold result in [12]. In [12] it was necessary to additionally hypothesize that $p_0: M \to B$ was a bundle projection. Using that assumption and some infinite-dimensional magic, the (weak) Approximation Theorem was proved by means of a handle lemma for 0-handles. In Section 6 of this paper we prove the corresponding handle lemma for arbitrary handles. This is the main new idea of the present paper.

Most of our notation is standard. Euclidean m-space is denoted by $\mathbf{R}^m$ and is given the box metric $d(x,y) = \max\{|x_i - y_i|\}$ where $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$. For $r > 0$ we use B_r^m to denote the (square) ball of radius r in $\mathbf{R}^m$ centered at the origin. Finally, S^1 denotes the unit circle in $\mathbf{R}^2$.

2. Engulfing. This section contains the engulfing results needed for the proof of the main result. Only the proof of Proposition 2.2 is presented. It is the parameterized version of a result due to Chapman presented as Lemma 2.1 below.

Throughout this section Z will denote a compact polyhedron and B will denote an ANR which contains $Z\times\mathbf{R}$ as an open subset. Projection onto Z is denoted by p_1, projection onto $\mathbf{R}$ by p_2. Let Δ be the standard n-simplex for some fixed $n\geq 0$ and let C be a (possibly empty) closed subset of $\partial\Delta$.

Here is Chapman's lemma [2, Lemma 3.4].

LEMMA 2.1. *For every $m\geq 5$ and $\epsilon>0$ there exists a $\delta>0$ so that if M is an m-manifold, $\partial M=\varnothing$, and $f: M\to Z\times\mathbf{R}$ is a δ-fibration over $Z\times[-3,3]$, then there is a homeomorphism $h: M\to M$ such that*

(i) $f^{-1}(Z\times(-\infty,1])\subset hf^{-1}(Z\times(-\infty,0))$,

(ii) *there is an isotopy $h_s:\mathrm{id}\simeq h$, $0\leq s\leq 1$, which is a $(p_1f)^{-1}(\epsilon)$-homotopy and is supported on $f^{-1}(Z\times[-2,2])$.*

We are now ready for our key engulfing result.

Data for Proposition 2.2. Let $\alpha:\Delta\to[-1,1]$ and $\beta:\Delta\to[0,1]$ be maps such that $\alpha(t)<\beta(t)$ for all t in Δ and $\alpha^{-1}(-1)=C$. Let

$$\Gamma(\alpha)=\{(z,x,t)\in Z\times\mathbf{R}\times\Delta\mid x\leq\alpha(t)\}.$$

PROPOSITION 2.2. *For every $m\geq 5$ and $\epsilon>0$ there exists a $\delta>0$ so that if M is an m-manifold, $\partial M=\varnothing$, and $f: M\times\Delta\to B\times\Delta$ is a f.p. map such that $f_t: M\to B$ is a δ-fibration over $Z\times[-2,2]$ for each t in Δ, then there is a f.p. homeomorphism $h: M\times\Delta\to M\times\Delta$ such that*

(i) $f^{-1}(\Gamma(\alpha))\subset hf^{-1}(Z\times(-\infty,0)\times\Delta)$,

(ii) *there is a f.p. isotopy $h_s:\mathrm{id}\simeq h$, $0\leq s\leq 1$, which is a $(p_1f)^{-1}(\epsilon)$-homotopy over $Z\times\mathbf{R}\times\Delta$ and is supported on*

$$f^{-1}\{(z,x,t)\in Z\times\mathbf{R}\times\Delta\mid -1\leq x\leq\beta(t),\ t\notin C\}.$$

Proof. Choose maps $\alpha=\alpha_0,\hat\alpha_1,\alpha_1,\hat\alpha_2,\alpha_2,\ldots,\hat\alpha_n,\alpha_n,\alpha_{n+1}=\beta:\Delta\to[-1,1]$ such that $\alpha_0(t)<\hat\alpha_1(t)<\alpha_1(t)<\hat\alpha_2(t)<\cdots<\alpha_n(t)<\alpha_{n+1}(t)$ for each t in Δ and $\alpha_n(t)<0$ for each t in C. Choose a partition $-1=x_{-1}<x_0<x_1<\cdots<x_n=0$ such that $\sup\{\alpha_n(t)\mid t\in C\}<x_0$. For each i, $0\leq i\leq n$, let $r_i=\inf\{\hat\alpha_{i+1}(t)-\alpha_i(t)\mid t\in\Delta\}$ (where $\hat\alpha_{n+1}=\alpha_{n+1}$).

Once these choices have been made and $\epsilon>0$ is given, choose $\delta_i>0$ for $0\leq i\leq n$ by Lemma 2.1 so that the following statement is true: if M is an m-manifold, $\partial M=\varnothing$, $f: M\to Z\times\mathbf{R}$ is a δ_i-fibration over $Z\times[-2,2]$, and b is any number with $x_i\leq b\leq 1-r_i$, then there is a homeomorphism $h: M\to M$ such that

(i) $f^{-1}(Z\times(-\infty,b))\subset hf^{-1}(Z\times(-\infty,x_i))$,

(ii) there is an isotopy $h_s:\mathrm{id}\simeq h$, $0\leq s\leq 1$, which is a $(p_1f)^{-1}(\epsilon/n+1)$-homotopy and is supported on $f^{-1}(Z\times[\frac{1}{2}(x_{i-1}+x_i),\ b+r_i])$.

It is important to realize that δ_i is independent of b in the statement above. To see that this is possible one needs to examine Chapman's proof of Lemma 2.1. Of course, δ_i does depend on r_i.

Let $\delta=\min\{\delta_i \mid 0\le i\le n\}$ and let $f: M\times\Delta\to B\times\Delta$ be given as in the hypothesis. By our choices, for each t in Δ and $0\le i\le n$ there is a homeomorphism $h^{t,i}: M\to M$ such that

(i) $f^{-1}(Z\times(-\infty,\alpha_i(t)]\times\{t\})\subset h^{t,i}f^{-1}(Z\times(-\infty,x_i)\times\{t\})$,

(ii) there is an isotopy

$$h_s^{t,i}:\mathrm{id}\simeq h^{t,i},\quad 0\le s\le 1,$$

which is a $(p_1 f)^{-1}(\epsilon/n+1)$-homotopy over $Z\times\mathbf{R}$ and is supported on $f^{-1}(Z\times[\frac{1}{2}(x_{i-1}+x_i),\hat{\alpha}_{i+1}(t)]\times\{t\})$.

For t in C we take $h^{t,i}=\mathrm{id}$ and $h_s^{t,i}=\mathrm{id}$.

Given any map $\sigma:\Delta\to[0,1]$, t in Δ, and $0\le i\le n$, define $\hat{h}_s^{t,i}: M\times\Delta\to M\times\Delta$ for $0\le s\le 1$ by $\hat{h}_s^{t,i}(y,u)=(h^{t,i}_{\sigma(u)s}(y),u)$. Then the f.p. homeomorphism $\hat{h}^{t,i}=\hat{h}_1^{t,i}$ will be called a *fundamental homeomorphism associated to σ, t, and i*, and will simply be denoted by $\hat{h}$ (and the isotopy by $\hat{h}_s$) when t and i are understood. If $t\notin C$ then we will always assume that $\sigma(C)=0$.

ASSERTION 2.2.1. *For each t in Δ there exists an open neighborhood V_t of t in Δ such that if $0\le i\le n$ and $\sigma:\Delta\to[0,1]$ is a map with $\sigma^{-1}((0,1])\subset V_t$, then the fundamental homeomorphism $\hat{h}$ (and the corresponding isotopy $\hat{h}_s$) associated to σ, t, and i has the following properties:*

(i)

$$f^{-1}\{(z,x,u)\in Z\times\mathbf{R}\times\Delta \mid x\le\alpha_i(u),\ \sigma(u)=1\}\subset\hat{h}f^{-1}(Z\times(-\infty,x_i)\times\sigma^{-1}(1)),$$

(ii) *$\hat{h}_s$ is a $(p_1 f)^{-1}(\epsilon/n+1)$-homotopy over $Z\times\mathbf{R}\times\Delta$ and is supported on* $f^{-1}\{(z,x,u)\in Z\times\mathbf{R}\times\Delta \mid x_{i-1}\le x\le\alpha_{i+1}(u)\}$.

The proof of this assertion is left as an exercise in continuity. One first constructs a V_t which depends on i and then takes a finite intersection.

This assertion gives us an open cover $\mathcal{V}=\{V_t \mid t\in\Delta\}$ of Δ. For $t\notin C$ require further that $V_t\cap C=\varnothing$. Now write $\Delta=\bigcup_{i=0}^n D_i$ where each D_i is closed and the finite union $\bigcup D_{ij}$ of disjoint closed sets refining $\mathcal{V}$. (This comes from a standard handle decomposition of Δ by small handles.) For each D_{ij} choose V_{ij} in $\mathcal{V}$ containing D_{ij}.

Choose maps $\sigma_{ij}:\Delta\to[0,1]$ such that $\sigma_{ij}^{-1}(1)=D_{ij}$ and $\sigma_{ij}^{-1}((0,1])\subset V_{ij}$. Do this in such a way that $\sigma_{ij}(C)=0$ if $D_{ij}\cap C=\varnothing$ and $\sigma_{ij}^{-1}((0,1])\cap\sigma_{ik}^{-1}((0,1])=\varnothing$ if $j\ne k$.

Using the assertion construct for each $0\le i\le n$ a f.p. homeomorphism $h^i: M\times\Delta\to M\times\Delta$ such that

(i) $f^{-1}\{(z,x,t) \mid x\le\alpha_i(t),\ t\in D_i\}\subset h^i f^{-1}(Z\times(-\infty,x_i)\times\Delta)$,

(ii) there is a f.p. isotopy

$$h_s^i:\mathrm{id}\simeq h^i,\quad 0\le s\le 1,$$

which is a $(p_1 f)^{-1}(\epsilon/n+1)$-homotopy over $Z\times\mathbf{R}\times\Delta$ and is supported on $f^{-1}\{(z,x,t) \mid x_{i-1}\le x\le\alpha_{i+1}(t),\ t\notin C\}$. Set $h=h^0\circ h^1\circ\cdots\circ h^n$ and $h_s=h_s^0\circ h_s^1\circ\cdots\circ h_s^n$.

Then h and h_s satisfy the conclusions of the proposition. That $f^{-1}(\Gamma(\alpha))\subset hf^{-1}(Z\times(-\infty,0)\times\Delta)$ follows from the fact that

$$f^{-1}\left\{(z,x,t)\mid x\le\alpha_i(t),\, t\in\bigcup_{k=1}^{n}D_k\right\}\subset h^i\circ\cdots\circ h^n f^{-1}(Z\times(-\infty,0)\times\Delta)$$

for $0\le i\le n$. □

The next theorem can be derived from Proposition 2.2 by a stacking procedure. See [11, Theorem 4.3] or [2, Lemma 3.5] for a proof.

Data for Theorem 2.3. Let $\Theta:\mathbf{R}\times\Delta\to\mathbf{R}\times\Delta$ be a f.p. homeomorphism with the following properties:
(i) $\Theta\mid\mathbf{R}\times C$ is the identity;
(ii) $x\le p_1\Theta(x,t)$ for each x in $\mathbf{R}$ and t in Δ;
(iii) Θ is supported on $[-1,1]\times\Delta$.
Let $\Theta':B\times\Delta\to B\times\Delta$ denote the f.p. homeomorphism which extends $\mathrm{id}_Z\times\Theta$ via the identity.

THEOREM 2.3. *For every $m\ge5$ and $\epsilon>0$ there exists a $\delta>0$ so that if M is an m-manifold, $\partial M=\varnothing$, and $f:M\times\Delta\to B\times\Delta$ is a f.p. map such that $f_t:M\to B$ is a δ-fibration over $Z\times[-2,2]$ for each t in Δ, then there is a f.p. homeomorphism $\tilde\theta:M\times\Delta\to M\times\Delta$ such that*
(i) *$f\tilde\theta$ is ϵ-close to $\theta'f$,*
(ii) *there is a f.p. isotopy $\tilde\theta_s:\mathrm{id}\simeq\tilde\theta$, $0\le s\le1$, which is a $(p_1f)^{-1}(\epsilon)$-homotopy over $Z\times\mathbf{R}\times\Delta$ and is supported on $f^{-1}(Z\times[-1,1]\times(\Delta\backslash C))$.*

3. Wrapping up. In this section we state without proof a result on wrapping up families of δ-fibrations around S^1. This result can be derived from Theorem 2.3 and we refer the reader to [11, Section 5] for a proof.

For notation let Z denote a compact polyhedron and let B denote an ANR which contains $Z\times\mathbf{R}$ as an open subset. Let Δ be a fixed n-simplex. Finally, let $e:\mathbf{R}\to S^1$ be the covering map defined by $e(x)=\exp(\pi ix/4)$ (thus e has period 8).

THEOREM 3.1. *For every $m\ge5$ and $\epsilon>0$ there exists a $\delta>0$ so that if M is an m-manifold, $\partial M=\varnothing$, and $f:M\times\Delta\to B\times\Delta$ is a f.p. map such that $f_t:M\to B$ is a δ-fibration over $Z\times[-3,3]$ for each t in Δ, then there is a closed m-manifold $\tilde M$, a f.p. map $\tilde f:\tilde M\times\Delta\to Z\times S^1\times\Delta$ such that $\tilde f_t:\tilde M\to Z\times S^1$ is an ϵ-fibration for each t in Δ, and a f.p. open embedding $\psi:f^{-1}(Z\times(-1,1)\times\Delta)\to\tilde M\times\Delta$ for which the following diagram commutes:*

$$\begin{array}{ccc}
\tilde M\times\Delta & \xrightarrow{\tilde f} & Z\times S^1\times\Delta\\
\psi\uparrow & & \uparrow \mathrm{id}\times e\times\mathrm{id}\\
f^{-1}(Z\times(-1,1)\times\Delta) & \xrightarrow{f|} & Z\times(-1,1)\times\Delta
\end{array}$$

4. Handle lemmas. In this section we state without proof two handle lemmas needed for the main results of the next section. The first handle lemma (Proposition 4.1) can be derived from the results of the previous sections. The second handle lemma (Proposition 4.2) can be derived from straightforward generalizations of those results. The reader is referred to [11, Section 6] and [2, Section 5] for more details.

For notation B will denote an ANR, Δ will be a fixed n-simplex, and C will be a (possibly empty) closed subset of $\partial\Delta$ which is collared in Δ.

PROPOSITION 4.1. *Suppose $p>0$ and $\mathbf{R}^p \to B$ is an open embedding. For every $m \geq 5$ and $\epsilon>0$ there exists a $\delta>0$ so that if $\mu>0$, M is an m-manifold, $\partial M=\emptyset$, and $f: M\times\Delta \to B\times\Delta$ is a f.p. map such that $f_t: M\to B$ is a δ-fibration for each t in Δ and an approximate fibration for each t in C, then there is a f.p. map $\tilde{f}: M\times\Delta\to B\times\Delta$ such that $\tilde{f}_t: M\to B$ is a μ-fibration over $\mathring{B}_1^p$ for each t in Δ and such that $\tilde{f}$ is f.p. ϵ-homotopic to f rel $[(M\times\Delta)\backslash f^{-1}(\mathring{B}_3^p\times\Delta)]\cup[M\times C]$.*

For more notation, X will be a compact polyhedron and $\mathring{c}(X)$ will be its open cone. That is, $\mathring{c}(X)=X\times[0,+\infty)/\sim$ where $\sim$ denotes the equivalence relation generated by $(x,0)\sim(x',0)$ for all x,x' in X. For any $r\geq 0$ let $c_r(X)=X\times[0,r]/\sim$ and $\mathring{c}_r(X)=X\times[0,r)/\sim$.

PROPOSITION 4.2. *Suppose $p\geq 0$ and $\mathring{c}(X)\times\mathbf{R}^p \hookrightarrow B$ is an open embedding. For every $m\geq 5$ and $\epsilon>0$ there exists a $\delta>0$ such that for every $\mu>0$ there exists a $\nu>0$ so that the following statement is true: if M is an m-manifold, $\partial M=\emptyset$, and $f: M\times\Delta\to B\times\Delta$ is a f.p. map such that $f_t: M\to B$ is a δ-fibration over $c_3(X)\times B_3^p$ and a ν-fibration over $[c_3(X)\backslash\mathring{c}_{1/3}(X)]\times B_3^p$ for each t in Δ and an approximate fibration for each t in C, then there is a f.p. map $\tilde{f}: M\times\Delta\to B\times\Delta$ such that $\tilde{f}_t: M\to B$ is a μ-fibration over $c_1(X)\times B_1^p$ for each t in Δ and such that $\tilde{f}$ is f.p. ϵ-homotopic to f rel $[(M\times\Delta)\backslash f^{-1}(\mathring{c}_{2/3}(X)\times\mathring{B}_3^p\times\Delta)]\cup[M\times C]$.*

5. The Deformation Theorem. In this section we state our main result on deforming a parameterized family of ϵ-fibrations to a parameterized family of approximate fibrations (Theorem 5.1). It will follow from this that the space of approximate fibrations from a closed m-manifold ($m\geq 5$) to a compact polyhedron is uniformly locally n-connected for every $n\geq 0$.

THEOREM 5.1. *Let B be a polyhedron, $m\geq 5$, Δ an n-simplex, and C a closed subset of ∂D which is collared in Δ. For every open cover α of B there exists an open cover β of B so that if M is an m-manifold, $\partial M=\emptyset$, and $f: M\times\Delta\to B\times\Delta$ is a f.p. map such that $f_t: M\to B$ is a β-fibration for each t* in *Δ and an approximate fibration for each t in C, then there is a f.p. map $\tilde{f}: M\times\Delta\to B\times\Delta$ such that $\tilde{f}: M\to B$ is an approximate fibration α-close to f_t for each t in Δ and $\tilde{f}_t=f_t$ for each t in C.*

The reader is referred to [2, Section 6] to see how to derive Theorem 5.1 from the handle lemmas of Section 4. One should notice that Theorem 5.1 remains true if it is assumed that C only has a radial neighborhood in Δ instead of a collar (for example, if C is a point). In addition, Theorem 5.1 remains true if B is assumed to be a manifold with a handle decomposition.

If X is a space and $n\geq 0$ is an integer, then X is said to be *locally n-connected* if for each x in X and each open subset U of X containing x, there exists an open subset V of X containing x such that $V\subset U$ and if Δ is an $(n+1)$-simplex then any map $f: \partial\Delta\to V$ extends to a map $\tilde{f}: \Delta\to U$.

The following corollary follows immediately from Theorem 5.1. For a proof see [11, Section 7].

COROLLARY 5.2. *If M is a closed m-manifold ($m \geq 5$) and B is a compact polyhedron, then the space of approximate fibrations from M to B endowed with the compact-open topology is locally n-connected for each $n \geq 0$.*

A metric space X is said to be *uniformly* locally n-connected if for every $\epsilon > 0$ there exists a $\delta > 0$ such that every map $f: \partial\Delta \to X$, where Δ is an $(n+1)$-simplex, with the diameter of $f(\partial\Delta)$ less than δ extends to a map $\tilde{f}: \Delta \to X$ with the diameter of $\tilde{f}(\Delta)$ less than ϵ. If in the statement of Corollary 5.2 we fix a metric for B, then the proof shows that the space of approximate fibrations from M to B endowed with the uniform topology is uniformly locally n-connected for each $n \geq 0$.

If the proof of Theorem 5.1 is examined, it will be seen that we can replace B by $\mathbf{R}^p$ with the standard metric and replace the open covers by positive numbers so that the statement remains true. Then the proof of Corollary 5.2 shows that the space of approximate fibrations from an m-manifold M ($\partial M = \emptyset$, $m \geq 5$) to $\mathbf{R}^p$ endowed with the uniform topology (induced by the standard metric on $\mathbf{R}^p$) is uniformly locally n-connected for each $n \geq 0$.

As mentioned in [11, Section 7] a theorem of Ferry [7] can be combined with Corollary 5.1 to show that spaces of approximate fibrations with a specified fiber are locally n-connected. We conclude with one example of this.

COROLLARY 5.3. *If M is a closed m-manifold ($m \geq 5$) and B is a compact polyhedron, then the space of cell-like maps from M to B endowed with the compact-open topology is locally n-connected for each $n \geq 0$.*

6. Another handle lemma. This section contains the key ingredients for the proof of the Approximation Theorem in Section 7. In this section Δ will denote the standard n-simplex in $\mathbf{R}^n$ so that 0 (the origin) is in $\partial\Delta$. Let N_1 and N_2 denote two convex neighborhoods of 0 in Δ such that $N_1 \subset \operatorname{int} N_2$. Lemma 6.1 treats handles of highest index while Theorem 6.2 treats handles of lower index.

LEMMA 6.1. *Let B be an ANR which contains $\mathbf{R}^i$ as an open subset and let $m \geq 5$ be given. For every $\epsilon > 0$ there exists a $\delta > 0$ so that if M is an m-manifold, $\partial M = \emptyset$, $p: M \times \Delta \to B \times \Delta$ is a f.p. approximate fibration, and $f: M \times \Delta \to M \times \Delta$ is a f.p. map such that $f | M \times N_2 = \mathrm{id}$, $(p_0 \times \mathrm{id}) f$ is δ-close to p, and f is a homeomorphism over $(p_0 \times \mathrm{id})^{-1}((\mathbf{R}^i \backslash \mathring{B}_1^i) \times \Delta)$, then f is homotopic to a map $\tilde{f}$ via a f.p. $(p_0 \times \mathrm{id})^{-1}(\epsilon)$-homotopy rel*

$$(M \times \{0\}) \cup f^{-1}(p_0 \times \mathrm{id})^{-1}((B \backslash \mathring{B}_2^i) \times \Delta)$$

such that $\tilde{f}$ is a homeomorphism over $(p_0 \times \mathrm{id})^{-1}(\mathbf{R}^i \times \Delta)$.

Proof. Let q denote the map $p_0 \times \mathrm{id}$. The first step of the proof is to show that f is f.p. homotopic to a map k rel $(M \times N_1) \cup (qf)^{-1}((B \backslash \mathring{B}_{1.5}^i) \times \Delta)$ where k is a homeomorphism over $q^{-1}(\mathbf{R}^i \times \Delta)$. To this end use the isotopy extension theorem ([6], [20]) to produce a f.p. homeomorphism $H: M \times \Delta \to M \times \Delta$ so that $H | M \times N_1 = \mathrm{id}$ and $H | q^{-1}((B_2^i \backslash \mathring{B}_{1.5}^i) \times \Delta) = f |^{-1}$.

Define k by setting $k\,|\,(qf)^{-1}(B^i_2\times\Delta)=H^{-1}\,|$ and $k\,|\,(qf)^{-1}((B\backslash \mathring{B}^i_{1.5})\times\Delta)=f\,|$. The homotopy $k_s\colon f\simeq k$ comes from observing that $fk^{-1}\,|\colon q^{-1}(B^i_{1.5}\times\Delta)\to q^{-1}(B^i_{1.5}\times\Delta)$ is f.p. homotopic to the identity rel $q^{-1}(B^i_{1.5}\times N_1)\cup q^{-1}(\partial B^i_{1.5}\times\Delta)$.

The next step is to use engulfing to turn k into a map with the desired control. This type of modification has been used by Chapman [2, p. 47]. Let $r>0$ be small and let $\bar\theta\colon \mathbf{R}^i\to\mathbf{R}^i$ be a radially defined homeomorphism supported on $B^i_{1.8}\backslash B^i_r$ such that $\bar\theta(B^i_{3r})=B^i_{1.7}$. Extend $\bar\theta$ to all of B via the identity. Use $\bar\theta$ to define a f.p. homeomorphism $\theta\colon B\times\Delta\to B\times\Delta$ by setting $\theta_t=\bar\theta$ for t in $\Delta\backslash N_1$ and for t in N_1, phase $\bar\theta$ out so that $\theta_0=\mathrm{id}$.

By engulfing (Section 2) there are f.p. homeomorphisms $\Gamma,\Lambda\colon M\times\Delta\to M\times\Delta$ such that $\Gamma\,|\,M\times\{0\}=\mathrm{id}=\Lambda\,|\,M\times\{0\}$, Γ is supported on $p^{-1}((B^i_{1.9}\backslash B^i_{2r})\times\Delta)$ and Λ is supported on $q^{-1}((B^i_{1.9}\backslash B^i_{2r})\times\Delta)$, and $p\Gamma$ is close to θp and $q\Lambda^{-1}$ is close to $\theta^{-1}q$. Then set $\tilde f=\Lambda^{-1}k\Gamma$. It can be seen that $q\tilde f$ is close to qf.

In order to get the desired homotopy from f to $\tilde f$, first recall that there are controlled f.p. isotopies $\Gamma_s\colon\mathrm{id}\simeq\Gamma$ and $\Lambda_s\colon\mathrm{id}\simeq\Lambda$ coming from Section 2. Then $\Lambda_s^{-1}f\Gamma_s$ is a controlled f.p. homotopy from f to $\Lambda^{-1}f\Gamma$. Finally, $\Lambda^{-1}k_s\Gamma$ is a controlled f.p. homotopy from $\Lambda^{-1}f\Gamma$ to $\tilde f$. □

For notation in the next theorem, let B be an ANR which contains $\mathbf{R}^p$ as an open subset. Write $\mathbf{R}^p=\mathbf{R}^{p-i}\times\mathbf{R}^i$ for some fixed i, $0\le i<p$. Thus, we are thinking of $\mathbf{R}^p$ as a handle in B of index i (with the $i=p$ case discussed in Lemma 6.2). Finally, let $m\ge 5$ be given.

THEOREM 6.2. *For every $\epsilon>0$ there exists a $\delta>0$ so that if M is an m-manifold, $\partial M=\varnothing$, $p\colon M\times\Delta\to B\times\Delta$ is a f.p. approximate fibration, and $f\colon M\times\Delta\to M\times\Delta$ is a f.p. map such that $f\,|\,M\times N_2=\mathrm{id}$, $(p_0\times\mathrm{id})f$ is δ-close to p, and f is a homeomorphism over $(p_0\times\mathrm{id})^{-1}(\mathbf{R}^{p-i}\times(\mathbf{R}^i\backslash\mathring{B}^i_1)\times\Delta)$, then f is homotopic to a map $\tilde f$ via a f.p. $(p_0\times\mathrm{id})^{-1}(\epsilon)$-homotopy rel*

$$(M\times\{0\})\cup f^{-1}(p_0\times\mathrm{id})^{-1}((B\backslash(\mathring{B}^{p-i}_4\times\mathring{B}^i_2)\times\Delta)$$

such that $\tilde f$ is a homeomorphism over $(p_0\times\mathrm{id})^{-1}(B^p_3\times\Delta)$.

Proof. The proof is a torus trick and is divided into three parts: wrapping up, unwrapping, and final improvements. We often assume that $i>0$; the $i=0$ case is similar but easier (it is the case treated in [12]).

I. Wrapping up. Let q denote the map $p_0\times\mathrm{id}$ and let $e\colon\mathbf{R}\to S^1$ be the covering map of period 16 defined by $e(x)=\exp(\pi ix/8)$ (this is different from the map e of Section 3). For any j, $T^j=S^1\times\cdots\times S^1$ (j times) and $e^j=e\times\cdots\times e\colon\mathbf{R}^j\to T^j$.

As in [6, Section 8] regard $\mathbf{R}\times T^{p-1}$ as an open subset of $\mathring{B}^p_7$ so that the composition $\mathrm{id}\times e^{p-1}\colon B^p_6=[-6,6]\times B^{p-1}_6\to[-6,6]\times T^{p-1}\subset\mathbf{R}\times T^{p-1}\subset B^p_7$ is the inclusion. Our first step is to produce the following commuting diagram (with explanations given below):

$$\begin{array}{ccc} \bar M\times\Delta & \xrightarrow{\bar p} & T^p\times\Delta \\ \Psi\uparrow & & \uparrow e\times\mathrm{id} \\ p^{-1}((-6,6)\times T^{p-1}\times\Delta) & \xrightarrow{p\,|} & (-6,6)\times T^{p-1}\times\Delta \end{array}$$

Let $\theta:\mathbf{R}\to\mathbf{R}$ be a homeomorphism supported on $[-7.5, 7.5]$ such that $\theta(x)=x+14$ for $-7.2\le x\le -6.8$. Let $\theta_s:\mathrm{id}\simeq\theta$, $0\le s\le 1$, be the obvious isotopy with the same support as θ. Let $C_1\supset C_2\supset\cdots\supset C_{12}$ be a decreasing sequence of compact subsets of T^{p-1} such that $C_{j+1}\subset\mathrm{int}\, C_j$ for each j,

$$C_1=e^{p-1}(B_6^{p-i-1}\times(B_6^i\backslash\mathring{B}_1^i)),$$

and each C_j contains $e^{p-1}(B_5^{p-i-1}\times(B_5^i\backslash\mathring{B}_{1.5}^i))$.

Let $\rho: T^{p-1}\to[0,1]$ be a map such that $\rho^{-1}(0)=T^{p-1}\backslash\mathring{C}_6$ and $\rho^{-1}(1)=C_7$. Define $\Sigma_s:\mathbf{R}\times T^{p-1}\to\mathbf{R}\times T^{p-1}$, $0\le s\le 1$, by $\Sigma_s(x,y)=(\theta_{s\rho(y)}(x),y)$. Write $\Sigma=\Sigma_1$. Extend Σ (and Σ_s) to all of B via the identity.

By engulfing (Section 2) there is a homeomorphism $\Omega: M\to M$ which is supported on $p_0^{-1}([-7.5,7.5]\times C_5)$ and has the property that $p_0\Omega$ is close to Σp_0 (as close as we want). Since f is a homeomorphism over $q^{-1}(\mathbf{R}\times C_1\times\Delta)$, we can define a f.p. homeomorphism $\Lambda: M\times\Delta\to M\times\Delta$ by first setting

$$\Lambda\mid p^{-1}([-8,8]\times C_3\times\Delta)=f^{-1}(\Omega\times\mathrm{id})f\mid$$

and then extending via the identity to all of $M\times\Delta$. Note that $p\Lambda$ is close to $(\Sigma\times\mathrm{id})p$, the closeness depending on δ.

By engulfing there is a f.p. homeomorphism $\Gamma: M\times\Delta\to M\times\Delta$ supported on $p^{-1}([-7.5,7.5]\times(T^{p-1}\backslash\mathring{C}_8)\times\Delta)$ such that $p\Gamma$ is as close as we want to $(\theta\times\mathrm{id})(\Sigma\times\mathrm{id})^{-1}p$. Then $p\Gamma\Lambda$ is close to $(\theta\times\mathrm{id})p$, the closeness depending on δ.

Now let $Y=\Gamma\Lambda p^{-1}((-\infty,-7]\times T^{p-1}\times\Delta)\backslash p^{-1}((-\infty,-7)\times T^{p-1}\times\Delta)$ and let Y_0 be the slice of Y over 0; that is,

$$Y_0=\Gamma\Lambda p^{-1}((-\infty,-7]\times T^{p-1}\times\{0\})\backslash p^{-1}((-\infty,-7)\times T^{p-1}\times\{0\}).$$

Let $\sim$ be the equivalence relation on Y generated by $y\sim\Gamma\Lambda(y)$ for each y in $p^{-1}(\{-7\}\times T^{p-1}\times\Delta)$. The obvious map $Y/\sim\ \to\Delta$ can be seen to be a submersion with closed manifold fibers, so $Y/\sim$ can be identified with $\bar{M}\times\Delta$ where $\bar{M}=Y_0/\sim$, a closed m-manifold. (This is the same wrapping up construction discussed in Section 3; see [11, Section 5] for more details.) There is also a f.p. map $\bar{p}:\bar{M}\times\Delta\to T^p\times\Delta$ such that each $\bar{p}_t$ is a $\bar{\delta}$-fibration (where $\bar{\delta}$ is small if δ is) and there is a f.p. open embedding $\Psi: p^{-1}((-6,6)\times T^{p-1}\times\Delta)\to\bar{M}\times\Delta$ which makes the diagram mentioned at the beginning of the proof commute. We can assume that $\bar{p}\mid\bar{M}\times N_2=\bar{p}_0\times\mathrm{id}$.

We now see how to wrap up the map f; this is something that was not done in Section 3. We first define an auxiliary map $\bar{\bar{f}}: Y\to q^{-1}(\mathbf{R}\times T^{p-1}\times\Delta)$. Define subsets E_- and E_+ of Y by $E_-=p^{-1}(\{-7\}\times T^{p-1}\times\Delta)$ and $E_+=\Gamma\Lambda(E_-)$. The map $\bar{\bar{f}}$ is to have the following two properties:

(i) $\bar{\bar{f}}\simeq f\mid Y\,\mathrm{rel}\,E_-\cup Y_0\cup(p^{-1}(\mathbf{R}\times C_9\times\Delta)\cap Y)\cup p^{-1}((-6,6)\times T^{p-1}\times\Delta)$ via a f.p. $q^{-1}(\delta')$-homotopy where the size of $\delta'>0$ depends on the size of δ.

(ii) $\bar{\bar{f}}\mid E_+=((\Gamma\Lambda\mid M\times\{0\})\times\mathrm{id}_\Delta)\circ f\circ(\Gamma\Lambda)^{-1}\mid E_+$.

Before defining $\bar{\bar{f}}$ we need to assert the existence of certain isotopies. First, along with the existence of Ω, engulfing gives us an isotopy $\Omega_s:\mathrm{id}_M\simeq\Omega$, $0\le s\le 1$, with the same support as Ω such that $p_0\Omega_s$ is close to $\Sigma_s p_0$. Then we can define a f.p. isotopy $\Lambda_s: \mathrm{id}_{M\times\Delta} \simeq \Lambda$, $0 \le s \le 1$, by setting $\Lambda_s \mid p^{-1}([-8,8]\times C_3\times\Delta) = f^{-1}(\Omega_s \times \mathrm{id}_\Delta)f \mid$ and then extending to all of $M\times\Delta$ via the identity. Finally, engulfing gives us a f.p. isotopy $\Gamma_s: \mathrm{id}_{M\times\Delta} \simeq \Gamma$, $0\le s\le 1$, with the same support as Γ such that $p\Gamma_s$ is close to $(\theta_s \times \mathrm{id})(\Sigma_s\times\mathrm{id})^{-1}p$. Note that $p\Gamma_s\Lambda_s$ is close to $(\theta_s\times\mathrm{id})p$.

Use these isotopies to define a f.p. homotopy

$$g_s: E_- \cup E_+ \cup Y_0 \cup (p^{-1}(\mathbf{R}\times C_q\times\Delta)\cap Y)\cup p^{-1}((-6,6)\times T^{p-1}\times\Delta) \to q^{-1}(\mathbf{R}\times T^{p-1}\times\Delta)$$

by setting $g_s \mid E_+ = ((\Gamma_s\Lambda_s \mid M\times\{0\})\times \mathrm{id}_\Delta)\circ f \circ (\Gamma_s\Lambda_s)^{-1} \mid E_+$ and setting $g_s = f$ elsewhere. Then qg_s is a small homotopy and g_0 extends to all of Y via f. By the estimated homotopy extension theorem, (i.e., the usual homotopy extension theorem with an estimate on control as developed by Chapman and Ferry), g_1 extends to the map $\bar{\bar{f}}$ with the desired properties.

Since $q\bar{\bar{f}}$ is close to $p\mid Y$, we can assume that the image of $\bar{\bar{f}}$ lies in $[(\Gamma\Lambda\mid)^{-1}(Y_0)\cup Y_0\cup(\Gamma\Lambda\mid)(Y_0)]\times\Delta$. The quotient map $\pi: Y_0\times\Delta\to\bar{M}\times\Delta$ extends to $\bar{\pi}: [(\Gamma\Lambda\mid)^{-1}(Y_0)\cup Y_0\cup(\Gamma\Lambda\mid)(Y_0)]\times\Delta\to\bar{M}\times\Delta$ in the obvious way. Now $\bar{\pi}\bar{\bar{f}}$ factors through the appropriate equivalence classes to define a map $\bar{f}: \bar{M}\times\Delta\to\bar{M}\times\Delta$.

Here is a summary of the important properties of $\bar{f}$. First $(\bar{p}_0\times\mathrm{id})\bar{f}: \bar{M}\times\Delta\to T^p\times\Delta$ is $\bar{\delta}$-close to $\bar{p}$. Also $\bar{f}\mid\bar{M}\times N_2 = \mathrm{id}$ (actually our $\bar{f}$ was only constructed so that $\bar{f}\mid\bar{M}\times\{0\} = \mathrm{id}$, but is not hard to modify the arguments above to achieve this extra condition; we will not give the details here). The following diagram commutes:

$$\begin{array}{ccc} \bar{M}\times\Delta & \xrightarrow{\bar{f}} & \bar{M}\times\Delta \\ \Psi\mid\uparrow & & \uparrow\bar{\Psi}_0\mid\times\mathrm{id} \\ p^{-1}((-5,5)\times T^{p-1}\times\Delta) & \xrightarrow{f\mid} & q^{-1}((-6,6)\times T^{p-1}\times\Delta) \end{array}$$

Finally, $\bar{f}$ is a homeomorphism over $(\bar{p}_0\times\mathrm{id})^{-1}(S^1\times C_{10}\times\Delta)$. Achieving this last condition is what makes this wrapping up construction more delicate than the construction of [12, Section 2].

Before proceeding to the next step of the proof we will produce a homeomorphism $h: \bar{M}\times\Delta\to\bar{M}\to\Delta$. This homeomorphism will be the key to the construction of the map $\tilde{f}$ in the next two parts of the proof. Simply use the isotopy extension theorem ([6], [20]) to find a f.p. homeomorphism $h: \bar{M}\times\Delta\to\bar{M}\times\Delta$ such that $h\mid\bar{M}\times N_2 = \mathrm{id}$ and $h = \bar{f}$ over $(\bar{p}_0\times\mathrm{id})^{-1}(S^1\times C_{11}\times\Delta)$.

We will need a f.p. homotopy

$$\alpha_s: \bar{f}\simeq h, \quad 0\le s\le 1,$$

rel $(\bar{M}\times N_2)\cup\bar{f}^{-1}(\bar{p}_0\times\mathrm{id})^{-1}(S^1\times C_{12}\times\Delta)$. To this end let $\sigma:\bar{M}\to[0,1]$ be a map such that $\sigma^{-1}(0) = \bar{p}_0^{-1}(S^1\times(T^{p-1}\setminus\mathring{C}_{11}))$ and $\sigma^{-1}(1) = \bar{p}_0^{-1}(S^1\times C_{12})$. Define a homotopy $r_s: \bar{M}\times\Delta\to\bar{M}\times\Delta$ by $r_s(x,t) = (x, (1-s)t + \sigma(x)st)$ (r_s is not f.p.!). Finally, define $\alpha_s(x,t) = (p_{\bar{M}}\bar{f}h^{-1}r_sh(x,t), t)$ where $p_{\bar{M}}$ denotes projection to $\bar{M}$.

II. Unwrapping. In this part of the proof we unwrap everything by pulling-back via $e^p: \mathbf{R}^p \to T^p$. For notation we let $\hat{C}_j$ denote $(e^{p-1} \mid B_6^{p-1})^{-1}(C_j)$, so that $\hat{C}_1 = B_6^{p-i-1} \times (B_6^i \setminus \mathring{B}_1^i)$ and each $\hat{C}_j$ contains $B_5^{p-i-1} \times (B_5^i \setminus \mathring{B}_{1.5}^i)$. Also let $\bar{q} = \bar{p}_0 \times \mathrm{id}$.

Let F be a f.p. δ'-homotopy from $\bar{p}$ to $\bar{q}\bar{f}$ rel $\bar{M} \times \bar{N}_2$, where δ' is small if $\bar{\delta}$ is. Form the pull-back diagram:

$$\begin{array}{ccc} \hat{M} \times \Delta \times [0,1] & \xrightarrow{\hat{F}} & \mathbf{R}^p \times \Delta \\ \downarrow & & \downarrow e^p \times \mathrm{id} \\ \bar{M} \times \Delta \times [0,1] & \xrightarrow{F} & T^p \times \Delta \end{array}$$

Then $\hat{F}$ is a f.p. $\hat{\delta}$-homotopy from $\hat{p}$ to $(\hat{p}_0 \times \mathrm{id})\hat{f}$ where $\hat{p}: \hat{M} \times \Delta \to \mathbf{R}^p \times \Delta$ is the pull-back of $\bar{p}$ and $\hat{f}: \hat{M} \times \Delta \to \hat{M} \times \Delta$ is the pull-back of $\bar{f}$. Let $\hat{q} = (\hat{p}_0 \times \mathrm{id})$; this is the pull-back of $\bar{q}$. Note that $\hat{f} \mid \hat{M} \times N_2 = \mathrm{id}$ and $\hat{f}$ is a homeomorphism over $\hat{q}^{-1}(\mathbf{R} \times \hat{C}_{10} \times \Delta)$.

The homotopy $\alpha: \bar{f} \simeq h$ pulls back to a f.p. homotopy $\hat{\alpha}: \hat{f} \simeq \hat{h}$ rel $\hat{M} \times N_2$ where $\hat{h}$ is a homeomorphism and $\hat{h} = \hat{f}$ over $\hat{q}^{-1}(\mathbf{R} \times \hat{C}_{11} \times \Delta)$. Moreover, $\hat{q}\hat{\alpha}$ is a f.p. bounded homotopy.

III. Final Improvements. In this part of the proof we deform $\hat{f}$ to a map f^* which is a homeomorphism over $\hat{q}^{-1}(B_4^p \times \Delta)$. First choose $K > 5$ large (depending on the size of the homotopy $\hat{q}\hat{\alpha}$). Then choose $J > K$ so large that $(\hat{q}\hat{h})^{-1}((\mathbf{R}^p \setminus \mathring{B}_J^p) \times \Delta)$ is disjoint from $(\hat{q}\hat{f})^{-1}(B_K^p \times \Delta)$.

Let $u: \hat{M} \times \Delta \to [0,1]$ be a map such that $u^{-1}(0) = (\hat{q}\hat{f})^{-1}(B_K^p \times \Delta)$ and $u^{-1}(1) = (\hat{q}\hat{h})^{-1}((\mathbf{R}^p \setminus \mathring{B}_J^p) \times \Delta)$. Define $\hat{\hat{f}}: \hat{M} \times \Delta \to \hat{M} \to \Delta$ by setting $\hat{\hat{f}}(x,t) = \hat{\alpha}_{u(x,t)}(x,t)$.

If K is large enough, $\hat{\hat{f}} = \hat{f}$ over $\hat{q}^{-1}(B_5^p \times \Delta)$. For some large $L > J$, $\hat{\hat{f}} = \hat{h}$ over $\hat{q}^{-1}((\mathbf{R}^p \setminus \mathring{B}_L^p) \times \Delta)$. Moreover, $\hat{\hat{f}} \mid \hat{M} \times N_2 = \mathrm{id}$ and $\hat{\hat{f}} \mid (\hat{q}\hat{f})^{-1}(\mathbf{R} \times \hat{C}_{12} \times \Delta) = \hat{f} \mid$.

Let $\bar{\gamma}: \mathbf{R}^p \to \mathbf{R}^p$ be a radially defined homeomorphism which takes B_L^p to B_6^p and leaves B_5^p fixed. Use $\bar{\gamma}$ to define a f.p. homeomorphism $\gamma: \mathbf{R}^p \times \Delta \to \mathbf{R}^p \times \Delta$ by setting $\gamma_t = \bar{\gamma}$ for t in $\Delta \setminus N_2$, and then phase $\bar{\gamma}$ out for t in $N_2 \setminus N_1$ so that $\gamma \mid \mathbf{R}^p \times N_1 = \mathrm{id}$. Note that each $(\gamma\hat{q}\hat{\hat{f}})_t$ is a $\hat{\hat{\delta}}$-fibration provided K is large enough.

The next step is to use a shuffle trick to produce f^* from $\hat{\hat{f}}$. Let $\Sigma: \mathbf{R}^p \times \Delta \to \mathbf{R}^p \times \Delta$ be a f.p. homeomorphism such that $\Sigma \mid \mathbf{R}^p \times N_1 = \mathrm{id}$, Σ only affects the first $\mathbf{R}$-coordinate of any point, Σ is supported on $[-12, 12] \times B_{12}^{p-i-1} \times B_{1.7}^i \times \Delta$, and $\Sigma(B_5^{p-i} \times B_{1.6}^i \times (\Delta \setminus N_2)) \subset (\mathbf{R}^p \setminus (B_6^{p-i} \times B_{1.5}^i) \times \Delta)$. By engulfing there are f.p. "covering" homeomorphisms $\Gamma, \Lambda: \hat{M} \times \Delta \to \hat{M} \times \Delta$ for Σ so that $\gamma\hat{q}\hat{\hat{f}}\Gamma$ is close to $\Sigma\gamma\hat{q}\hat{\hat{f}}$ and $\gamma\hat{q}\Lambda$ is close to $\Sigma\gamma\hat{q}$. The homeomorphisms Γ, Λ can be chosen to be supported on

$$(\gamma\hat{q}\hat{\hat{f}})^{-1}([-12, 12] \times B_{13}^{p-i-1} \times B_{1.8}^i \times \Delta) \text{ and } (\gamma\hat{q})^{-1}([-12, 12] \times B_{13}^{p-i-1} \times B_{1.8}^i \times \Delta),$$

respectively. Moreover, we can assume $\Gamma \mid \hat{M} \times N_1 = \mathrm{id} = \Lambda \mid \hat{M} \times N_1$.

Let $f^* = \Lambda^{-1}\hat{\hat{f}}\Gamma$. It can be seen that $\hat{q}f^*$ is close to $\hat{q}\hat{\hat{f}}$. Moreover, there is a f.p. homotopy from $\hat{\hat{f}}$ to f^* which is small when projected to $\mathbf{R}^p \times \Delta$ by $\hat{q}$. This homotopy comes from isotopies of the identity to Γ and Λ, and can be taken to be rel $(\hat{M}\times N_1)\cup(\hat{q}\hat{f})^{-1}(\mathbf{R}\times\hat{C}_{12}\times\Delta)$.

By our choice of notation we can identify $p^{-1}(B_5^p\times\Delta)$ with $\hat{p}^{-1}(B_5^p\times\Delta)$. Using this identification, the map f^*, and the homotopy $\hat{\hat{f}}\simeq f^*$, it is easy to define $\tilde{f}$. □

7. The Approximation Theorem and its corollaries. This section contains the proofs of the Approximation Theorem (7.1) and Corollaries 2, 3, and 4 of the introduction.

THEOREM 7.1. *Let M and B be closed manifolds with* $\dim M\geq 5$, *let Δ be a simplex with* 0 *in $\partial\Delta$, and let $p:M\times\Delta\to B\times\Delta$ be a f.p. approximate fibration. For every $\epsilon>0$ there exists a f.p. homeomorphism $H:M\times\Delta\to M\times\Delta$ such that $H_0=\mathrm{id}$ and pH is ϵ-close to $p_0\times\mathrm{id}_\Delta$.*

Proof. First notice that there is a f.p. approximate fibration $p':M\times\Delta\to B\times\Delta$ close to p so that for some small neighborhood N of 0 in Δ, $p'|M\times N=p_0\times\mathrm{id}_N$. In fact, p' and N can be chosen so that p' is f.p. homotopic to $p_0\times\mathrm{id}$ rel $M\times N$. Using the f.p. approximate homotopy lifting property [11, Section 2], there is a f.p. map $f:M\times\Delta\to M\times\Delta$ such that $f|M\times N=\mathrm{id}$ and $(p_0\times\mathrm{id})f$ is close to p' (and therefore close to p). Using the results of Section 6 working through a handle decomposition of B, we can deform f to a f.p. homeomorphism $\tilde{f}$ for which $(p_0\times\mathrm{id})\tilde{f}$ is close to $(p_0\times\mathrm{id})f$ and $\tilde{f}_0=\mathrm{id}$. Then $H=\tilde{f}^{-1}$ is the desired homeomorphism. □

Let M and B denote closed manifolds with $\dim M\geq 5$. Our next result is concerned with showing that certain spaces of maps from M onto B are locally n-connected. We now isolate the conditions that such a space must satisfy in order for our proof to show that it is locally n-connected. Let Γ be a space of maps from M onto B (endowed with the compact-open topology) such that

(1) every map in Γ is an approximate fibration, and
(2) if p is in Γ and $h:M\to M$ is a homeomorphism which is isotopic to the identity, then ph is in Γ.

The following is a partial list of possibilities for Γ:

(1) the space of bundle projections,
(2) the closure of the space of bundle projections,
(3) the space of Hurewicz fibrations,
(4) the closure of the space of Hurewicz fibrations,
(5) the space of Serre fibrations,
(6) the closure of the space of Serre fibrations.

See [5] for a proof that Serre fibrations are approximate fibrations.

Corollary 2 follows from the next result.

COROLLARY 7.2. Γ *is locally n-connected for each* $n\geq 0$.

Proof. This follows almost immediately from Corollary 5.2 and Theorem 7.1. See [12, p. 171] for more details. □

One consequence of this result is that two close bundle projections $p_0, p_1: M \to B$ can be connected by a small path of bundle projections. One might ask whether that path could be chosen to induce a bundle projection $p: M \times [0,1] \to B \times [0,1]$ which restricts to p_i over $B \times \{i\}$ for $i=0,1$. If this were the case, then p_0 and p_1 would have homeomorphic fibers. We now give an example to show that this need not be the case.

Let $f: S^3 \to X$ be a map such that X is not homeomorphic to S^3 but $f \times \mathrm{id}: S^3 \times S^1 \to X \times S^1$ can be approximated arbitrarily closely by homeomorphisms. For example, X could be the one-point compactification of Bing's dogbone space [1, Section 8]. For any $n \geq 1$, let $h: S^3 \times T^n \to X \times T^n$ be a homeomorphism close to $f \times \mathrm{id}$. Let $p_0: S^3 \times T^n \to T^n$ be h followed by projection. Then p_0 is a bundle projection with fiber X which is close to the trivial fiber bundle projection $p_1: S^3 \times T^n \to T^n$ with fiber S^3.

For our next result let M and B continue to denote closed manifolds with $\dim M \geq 5$ and let Γ be as above. Let AF denote the space of all approximate fibrations from M onto B. Corollary 4 follows immediately from the next result.

COROLLARY 7.3. *The inclusion map* $i: \Gamma \to \mathrm{AF}$ *induces an isomorphism* $i_*: \pi_k(\Gamma) \to \pi_k(\mathrm{AF})$ *for* $k > 0$ *and a monomorphism for* $k=0$.

Proof. The proof of Corollary 7.2 shows that $i_*: \pi_k(\Gamma) \to \pi_k(\mathrm{AF})$ is a monomorphism for all $k \geq 0$. It remains to show that it is an epimorphism for $k > 0$. We will illustrate this for $k=1$, the general case being similar. So given a loop of approximate fibrations based at a map $p_0: M \to B$ in Γ, we will show that there is a nearby loop of maps in Γ based at p_0. The given loop is represented by a f.p. approximate fibration $p: M \times [0,1] \to B \times [0,1]$ such that $p_0 = p_1$. By Theorem 7.1 there is a f.p. homeomorphism $H: M \times [0,1] \to M \times [0,1]$ such that $H_0 = \mathrm{id}$ and pH is close to $p_0 \times \mathrm{id}$. Thus each p_t is close to $p_0 H_t^{-1}$. By Corollary 7.2 there is a small path q in Γ connecting $p_0 H_1^{-1}$ to p_1. The paths $(p_0 \times \mathrm{id})H^{-1}$ and q fit together to form the desired loop in Γ based at p_0. □

Corollary 3 of the introduction follows immediately from the following result. We continue to use M, B, and Γ as above.

COROLLARY 7.4. *An approximate fibration* $p: M \to B$ *can be approximated arbitrarily closely by maps in* Γ *if and only if* p *is homotopic via approximate fibrations to a map in* Γ.

Proof. If p can be approximated arbitrarily closely by maps in Γ, then it follows from Corollary 5.2 that p is homotopic via approximate fibrations to a map in Γ.

On the other hand, if p is homotopic via approximate fibrations to a map in Γ, then there is a f.p. approximate fibration $q: M \times [0,1] \to B \times [0,1]$ such that $q_0 = p$ and q_1 is in Γ. By Theorem 7.1 there is a f.p. homeomorphism $H: M \times [0,1] \to M \times [0,1]$ such that $H_0 = \mathrm{id}$ and qH is close to $q_0 \times \mathrm{id}$. In particular, $q_1 H_1$ is a map in Γ close to $q_0 = p$. □

Finally, we state an embellishment of Theorem 7.1 which will be needed in the next section. The proof will not be given since it is only an obvious modification of the proof of Theorem 7.1. For notation, M and B will continue to denote closed manifolds with $\dim M \geq 5$, Δ will be a simplex, and I will denote the interval $[0,1]$. Also, f.p. will mean f.p. over $\Delta \times I$ (or $\partial\Delta \times I$).

PROPOSITION 7.5. *Let $p: M \times \Delta \times I \to B \times \Delta \times I$ be a f.p. approximate fibration and let $G: M \times \partial\Delta \times I \to M \times \partial\Delta \times I$ be a f.p. homeomorphism such that $pG = (p \mid M \times \partial\Delta \times \{0\}) \times \mathrm{id}_I$ and $G \mid M \times \partial\Delta \times \{0\} = \mathrm{id}$. Then for every $\epsilon > 0$ there exists a f.p. homeomorphism $H: M \times \Delta \times I \to M \times \Delta \times I$ such that $H \mid M \times \Delta \times \{0\} = \mathrm{id}$, $H \mid M \times \partial\Delta \times I = G$, and pH is ϵ-close to $(p \mid M \times \Delta \times \{0\}) \times \mathrm{id}_I$.* □

8. Controlled homotopy topological structures. Let $p: E \to B$ be an Hurewicz fibration where E and B are closed manifolds with $\dim E = m \geq 5$. Define a semi-simplicial complex $\mathcal{S}(p: E \to B)$, called "the space of controlled homotopy topological structures on $p: E \to B$", as follows. An n-simplex of $\mathcal{S}(p: E \to B)$ is an equivalence class represented by a map $f: M \to E \times \Delta$ where M is an $(m+n)$-manifold, Δ is the standard n-simplex, $\rho = (\mathrm{proj}) \circ f: M \to \Delta$ is a bundle with closed manifold fiber, and f is a f.p. $(p \times \mathrm{id})^{-1}(\epsilon)$-equivalence for every $\epsilon > 0$. This latter condition can be rephrased by saying f is a f.p. map such that f is a homotopy equivalence and $(p \times \mathrm{id})f: M \to B \times \Delta$ is an approximate fibration (see [13, Lemma 2.1]). Another such map $g: N \to E \times \Delta$ is *equivalent* to f if there is a f.p. homeomorphism $h: M \to N$ such that $gh = f$.

For the remainder of this section we will simply use $\mathcal{S}$ to denote $\mathcal{S}(p: E \to B)$ because the fibration is understood. It is easy to see that $\mathcal{S}$ satisfies the Kan condition, so its homotopy groups are well-defined (see [16]). The "base" vertex is represented by the identity on E.

There are two special cases which might be enlightening. First, if $B = \{\text{point}\}$ then $\pi_0 \mathcal{S}$ is a "structure set" which is an object of study in surgery theory ([15, p. 265], [22, p. 102]). On the other hand, if $B = E$ and $p = \mathrm{id}$, then all vertices are represented by cell-like maps so Siebenmann's Approximation Theorem [21] shows that $\pi_0 \mathcal{S} = 0$.

In the following result, $n \geq 0$ and Δ is the standard n-simplex.

THEOREM 8.1. *Let $f: M \to E \times \Delta$ and $f': M' \to E \times \Delta$ represent n-simplices in $\mathcal{S}$ which determine elements $[f]$ and $[f']$ of $\pi_n \mathcal{S}$, respectively. Then $[f] = [f']$ in $\pi_n \mathcal{S}$ if and only if for every $\epsilon > 0$ there exists a f.p. homeomorphism $h: M \to M'$ such that $f'h \mid \rho^{-1}(\partial\Delta) = f \mid \rho^{-1}(\partial\Delta)$ and $f'h$ is f.p. $(p \times \mathrm{id})^{-1}(\epsilon)$-homotopic to f rel $\rho^{-1}(\partial\Delta)$. (Here $\rho: M \to \Delta$ is the bundle projection* $(\mathrm{proj}) \circ f$).

Proof. Suppose first that $[f] = [f']$ in $\pi_n \mathcal{S}$. Then there is a bundle $\tilde{\rho}: \tilde{M} \to \Delta \times I$ with closed m-manifold fibers and a f.p. map $\tilde{f}: \tilde{M} \to E \times \Delta \times I$ representing an $(n+1)$-simplex of $\mathcal{S}$ such that $\tilde{f} \mid \tilde{\rho}^{-1}(\Delta \times \{0\})$ is equivalent to f, $\tilde{f} \mid \rho^{-1}(\Delta \times \{1\})$ is equivalent to f', and $\tilde{f} \mid \tilde{\rho}^{-1}(\partial_i \Delta \times I)$ is equivalent to the base n-simplex of $\mathcal{S}$ for $0 \leq i \leq n$.

It follows that there is a f.p. homeomorphism $j: E \times \partial\Delta \times I \to \tilde{\rho}^{-1}(\partial\Delta \times I)$ such that $\tilde{f}j = \mathrm{id}$. Since $\tilde{\rho}$ is trivial there is a f.p. homeomorphism $\tilde{h}: E \times \Delta \times I \to \tilde{M}$. Set $q = (p \times \mathrm{id})\tilde{f}\tilde{h}: E \times \Delta \times I \to B \times \Delta \times I$ and note that q is a family of approximate fibrations parameterized by $\Delta \times I$.

Let $\tilde{h}_0=\tilde{h}\,|\,E\times\Delta\times\{0\}$ and $j_0=j\,|\,E\times\partial\Delta\times\{0\}$. Consider the f.p. homeomorphism $G\colon E\times\partial\Delta\times I\to E\times\partial\Delta\times I$ defined by

$$G=\tilde{h}^{-1}\circ j\circ(j_0^{-1}\times\mathrm{id}_I)\circ(\tilde{h}_0\times\mathrm{id}_I)\,|\,E\times\partial\Delta\times I.$$

By Proposition 7.5 there exists a f.p. homeomorphism $H\colon E\times\Delta\times I\to E\times\Delta\times I$ such that $H\,|\,E\times\Delta\times\{0\}=\mathrm{id}$, $H\,|\,E\times\partial\Delta\times I=G$, and qH is close to $(q\,|\,E\times\Delta\times\{0\})\times\mathrm{id}_I$.

By the choice of $\tilde{f}$ there exist f.p. homeomorphisms $\alpha\colon M\to\tilde{p}^{-1}(\Delta\times\{0\})$ and $\beta\colon M'\to\tilde{p}^{-1}(\Delta\times\{1\})$ such that $\tilde{f}\alpha=f$ and $\tilde{f}\beta=f'$. Define $h\colon M\to M'$ to be the composition

$$M\xrightarrow{\alpha}\tilde{p}^{-1}(\Delta\times\{0\})\xrightarrow{\tilde{h}_0^{-1}}E\times\Delta\times\{0\}\xrightarrow{\mathrm{id}}E\times\Delta\times\{1\}$$
$$\xrightarrow{H|}E\times\Delta\times\{1\}\xrightarrow{\tilde{h}|}\rho^{-1}(\Delta\times\{1\})\xrightarrow{\beta^{-1}}M'.$$

The homotopy from f to $f'h$ is given at time t by the composition

$$M\xrightarrow{\alpha}\tilde{p}^{-1}(\Delta\times\{0\})\xrightarrow{\tilde{h}_0^{-1}}E\times\Delta\times\{0\}\xrightarrow{\mathrm{id}}E\times\Delta\times\{t\}$$
$$\xrightarrow{H|}E\times\Delta\times\{t\}\xrightarrow{\tilde{h}|}\tilde{p}^{-1}(\Delta\times\{t\})\xrightarrow{\tilde{f}|}E\times\Delta\times\{t\}=E\times\Delta.$$

Conversely, suppose we are given a f.p. homeomorphism $h\colon M\to M'$ as in the hypothesis. Using the homotopy from f to $f'h$, we can define a f.p. map $F\colon M\times I\to E\times\Delta\times I$ such that $F|M\times\{0\}=f$, $F\,|M\times\{1\}=f'h$, $F\,|\,\rho^{-1}(\partial\Delta)\times I=f\,|\times\mathrm{id}_I$, and F is $(p\times\mathrm{id})^{-1}(\epsilon)$-close to $f\times\mathrm{id}_I$.

Consider the f.p. map $(p\times\mathrm{id})F\colon M\times I\to B\times\Delta\times I$. Each level of this map is an ϵ'-fibration (where $\epsilon'>0$ is small if ϵ is) and $(p\times\mathrm{id})F$ is an approximate fibration when restricted to $(\rho^{-1}(\partial\Delta)\times I)\cup(M\times\{0,1\})$. By Theorem 5.1 $(p\times\mathrm{id})F$ is f.p. homotopic rel $(\rho^{-1}(\partial\Delta)\times I)\cup(M\times\{0,1\})$ to a f.p. approximate fibration G. Since $p\times\mathrm{id}$ is a fibration, this homotopy can be lifted to show that F is f.p. homotopic rel $(\rho^{-1}(\partial\Delta)\times I)\cup(M\times\{0,1\})$ to a map $\tilde{F}$ such that $(p\times\mathrm{id})\tilde{F}=G$. It is easy to see that $\tilde{F}$ provides an $(n+1)$-simplex in $\mathcal{S}$ showing $[f]=[f'h]$. Since $[f']=[f'h]$ we are done. □

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