# HOMOLOGY PRODUCTS AND THE ECKMANN-HILTON GROUPS

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Let $R$ be a fixed ring with unit; we shall work in the category of left $R$-modules. Of course, all results below are equally valid in any abelian category with enough injectives and projectives.

The Eckmann-Hilton groups, introduced by those authors in [2] and [3], comprise the left derived functors of Hom. Our purpose is to define product operations among the various (both left and right) derived functors of Hom. In particular, for any $A$ and $B$ let

$$H_n(A,B)=\begin{cases}\mathrm{Ext}^n(A,B) & n>0\\ \underline{\Pi}_{-n}(A,B) & n\leq 0\end{cases} \quad\text{and}\quad K_n(A,B)=\begin{cases}\mathrm{Ext}^n(A,B) & n>0\\ \overline{\Pi}_{-n}(A,B) & n\leq 0.\end{cases}$$

We obtain bilinear maps natural in $A$ and $C$ and commuting with the connecting homomorphisms:

$$H_p(B,C)\times H_q(A,B)\to H_{p+q}(A,C) \quad \text{unless } p\leq 0 \text{ and } q>0$$

$$K_p(B,C)\times K_q(A,B)\to K_{p+q}(A,C) \quad \text{unless } p>0 \text{ and } q\leq 0.$$

Our approach encompasses Yoneda products, certain products of Eckmann and Hilton, and some new products as well.

Hilton and Rees [4] considered the group of natural transformations from $\mathrm{Ext}^p(B,-)$ to $\mathrm{Ext}^q(A,-)$. In a similar vein we apply the products above to describe natural transformations involving all the derived functors of Hom. Specifically, we show that the group of natural transformations from $\{H_{p+q}(B,-)\}_{p\in\mathbf{Z}}$ to $\{H_p(A,-)\}_{p\in\mathbf{Z}}$ is isomorphic to $\underline{\Pi}_q(A,B)$, and dually for $\overline{\Pi}_q(A,B)$.

Of particular interest is the case $R=\mathbf{Z}G$, $G$ a finite group, for then we calculate $\underline{\Pi}_n(A,B)\cong \mathrm{Tor}_{n-1}^{\mathbf{Z}G}(A,B)$ and the products above give a product operation on the homology (and cohomology) of $G$ with coefficients in $A$.

## 1. Complete resolutions.

DEFINITION. A *complete resolution* $X$ is an exact complex, indexed by all the integers, with differential raising degree and such that $X^i$ is projective for $i\leq 0$, $X^i$ injective for $i>0$.

We may avoid negative indices (in some part) by setting $X^i=X_{-i}$, all integers $i$.

DEFINITION. A *complete resolution for* $A$ is a complete resolution $X$ admitting a factorization by a surjective $\epsilon$ and an injective $j$ as shown below, with $j\epsilon=d_0$.

$$\cdots\to X_2\to X_1\to X_0\to X^1\to X^2\to\cdots$$
$$X_0 \xrightarrow{\epsilon} A \xrightarrow{j} X^1$$

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DEFINITION. If $X$ is a complete resolution for $A$ and $n \geq 0$, then $\underline{\Pi}_n(B,A) = H_{n-1}\operatorname{Hom}(B,X)$ and $\bar{\Pi}_n(A,B) = H^n \operatorname{Hom}(X,B)$.

Notice that $\underline{\Pi}_0(B,A)$ is the group of homomorphisms from $B$ to $A$, modulo those factoring through a projective mapping onto $A$. In particular, we can describe $\underline{\Pi}(B,A)$ solely by reference to a projective resolution of $A$.

Likewise, $\bar{\Pi}_0(A,B)$ is the group of homomorphisms from $A$ to $B$, modulo those which factor through an injective containing $A$. And we may compute $\bar{\Pi}(A,B)$ solely by means of an injective resolution.

Of course for $n>0$, $H^{n+1}\operatorname{Hom}(B,X) = \operatorname{Ext}^n(B,A)$ and $H_n \operatorname{Hom}(X,B) = \operatorname{Ext}^n(A,B)$.

## 2. Comparison theorems.

DEFINITION. If $f: d^\alpha(X^\alpha) \to d^\beta(Y^\beta)$ for some $\alpha$ and $\beta$ in $\mathbf{Z}$, a *left ladder over f* is a collection of homomorphisms $\{f^i\}_{i \leq \alpha}$ with degree $\beta - \alpha$ forming a morphism of complexes. That is, such that $f^{i+1}d^i = d^{i+\beta-\alpha}f^i$ all $i < \alpha$, and $fd^\alpha = d^\beta f^\alpha$.

DEFINITION. A *homotopy of left ladders* $\{f^i\}_{i\leq\alpha}$ and $\{g^i\}_{i\leq\alpha}$ over

$$f: d^\alpha(X^\alpha) \to d^\beta(X^\beta)$$

is a collection of homomorphisms $\{\sigma^i\}_{i\leq\alpha+1}$ with $\sigma^i: X^i \to Y^{\beta-\alpha-1+i}$, $i \leq \alpha$, and $\sigma^{\alpha+1}: d^\alpha(X^\alpha) \to Y^\beta$, such that $f^i - g^i = d^{\beta-\alpha-1+i}\sigma^i + \sigma^{i+1}d^i$, all $i \leq \alpha$.

The homotopy type of a left ladder over $f$ is determined by $f$; but we shall find it useful to work with the entire ladder. Now if $X \to A$ is a projective resolution of $A$ and $Y \to B$ is an acyclic complex, then any $f: A \to B$ induces a morphism of complexes $F: X \to Y$. Any two such $F$ are homotopic.

PROPOSITION 1. *For any $f: d^\alpha(X^\alpha) \to d^\beta(Y^\beta)$ with $\alpha \leq 0$, there exists a left ladder over f, unique up to homotopy.*

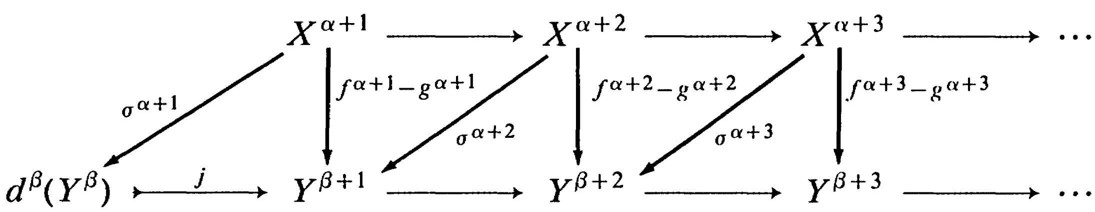
*Proof.* Since $\alpha \leq 0$, $X^i$ is projective for $i \leq \alpha$.

$$\begin{array}{ccccccc} \cdots & \to & X^{\alpha-1} & \to & X^\alpha & \twoheadrightarrow & d^\alpha(X^\alpha) \\ & & & & & & \downarrow f \\ \cdots & \to & Y^{\beta-1} & \to & Y^\beta & \twoheadrightarrow & d^\beta(Y^\beta). \end{array}$$

Dually, define a *right ladder over $f: d^\alpha(X^\alpha) \to d^\beta(Y^\beta)$* to be $\{f^i\}_{i\geq\alpha+1}$ making the following diagram commute.

$$\begin{array}{ccccccc} d^\alpha(X^\alpha) & \rightarrowtail & X^{\alpha+1} & \to & X^{\alpha+2} & \to & \cdots \\ \downarrow f & & \downarrow f^{\alpha+1} & & \downarrow f^{\alpha+2} & & \\ d^\beta(Y^\beta) & \rightarrowtail & Y^{\beta+1} & \to & Y^{\beta+2} & \to & \cdots \end{array}$$

DEFINITION. A *homotopy of right ladders* $\{f^i\}_{i\geq\alpha+1}$ and $\{g^i\}_{i\geq\alpha+1}$ over $f: d^\alpha(X^\alpha) \to d^\beta(Y^\beta)$ *is* $\{\sigma^i\}_{i\geq\alpha+1}$, $\sigma^i: X^i \to Y^{i+\beta-\alpha-1}$, for $i \geq \alpha+2$, and $\sigma^{\alpha+1}: X^{\alpha+1} \to d^\beta(Y^\beta)$ such that $f^i - g^i = d^{i+\beta-\alpha-1}\sigma^i + \sigma^{i+1}d^i$ all $i > \alpha+1$ and $f^{\alpha+1} - g^{\alpha+1} = j\sigma^{\alpha+1} + \sigma^{\alpha+2}d^{\alpha+1}$ in the following diagram.




Dual to Proposition 1 we have the following.

PROPOSITION 1′. *For any* $f: d^\alpha(X^\alpha) \to d^\beta(Y^\beta)$, $\beta \geq 0$, *there exists a right ladder over f, unique up to homotopy.*

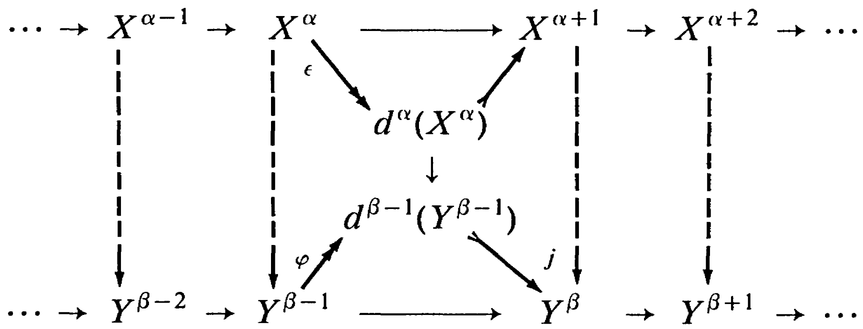
*Proof.* Since $\beta \geq 0$, $d^\beta(Y^\beta) \rightarrowtail Y^{\beta+1} \to Y^{\beta+2} \to \cdots$ is an injective resolution.

Now it is appropriate to apply both Propositions 1 and 1′ to a map $f: d^\alpha(X^\alpha) \to d^\beta(Y^\beta)$; for the composition $X^\alpha \to d^\alpha(X^\alpha) \to d^\beta(Y^\beta) \to Y^{\beta+1}$ represents an element of $\mathrm{Ext}^{\beta-\alpha}(A, B)$.

In particular we define a *complete ladder* of degree $n$ from $X$ to $Y$ to be $\{f^i\}_{i \in \mathbf{Z}}$, $f^i: X^i \to Y^{n+i}$, commuting with the differentials of $X$ and $Y$.

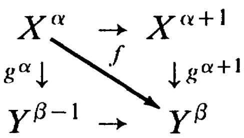
PROPOSITION 2. *For complete resolutions X and Y, and* $f: X^\alpha \to Y^\beta$ *such that* $fd^{\alpha-1} = 0 = d^\beta f$, $\alpha \leq 0$ *and* $\beta \geq 1$, *there is a complete ladder from X to Y with* $d^{\beta-1} f^\alpha = f = f^{\alpha+1} d^\alpha$ *and degree* $\beta - \alpha - 1$, *unique up to homotopy.*

*Proof.* $fd^{\alpha-1} = 0 = d^\beta f$ implies that for some $F: d^\alpha(X^\alpha) \to d^{\beta-1}(Y^{\beta-1})$ the composition $X^\alpha \twoheadrightarrow d^\alpha(X^\alpha) \to d(Y^\beta) \rightarrowtail Y^\beta$ is $f$.




By Propositions 1 and 1′, $f$ determines a complete ladder with degree $\beta - \alpha - 1$, $\{f^i\}_{i \in \mathbf{Z}}$. Clearly $d^{\beta-1} f^\alpha = f = f^{\alpha+1} d^\alpha$.

To show uniqueness, suppose that $\{g^i\}_{i \in \mathbf{Z}}$ is a ladder from $X$ to $Y$ such that both triangles in the square below commute.




Now $d^{\beta-1} g^\alpha = f$ so by the uniqueness statement of Proposition 1 $\{f^i\}_{i \leq \alpha}$ and $\{g^i\}_{i \leq \alpha}$ are homotopic left ladders. Similarly, by Proposition 1′, $\{f^i\}_{i \geq \alpha}$ is homotopic to $\{g^i\}_{i \geq \alpha+1}$.

Let $(\sigma^i\}_{i \leq \alpha+1}$ be a homotopy of $\{f^i\}_{i \leq \alpha}$ and $\{g^i\}_{i \leq \alpha}$, $\{\tau^i\}_{i \geq \alpha+1}$ a homotopy of $\{f^i\}_{i \geq \alpha+1}$ and $\{g^i\}_{i \geq \alpha+1}$. The union of $\{\sigma^i\}_{i \leq \alpha}$ and $\{\tau^i\}_{i \geq \alpha+2}$ would be a homotopy of complete ladders—if there existed a suitable map $X^{\alpha+1} \to Y^{\beta-1}$.

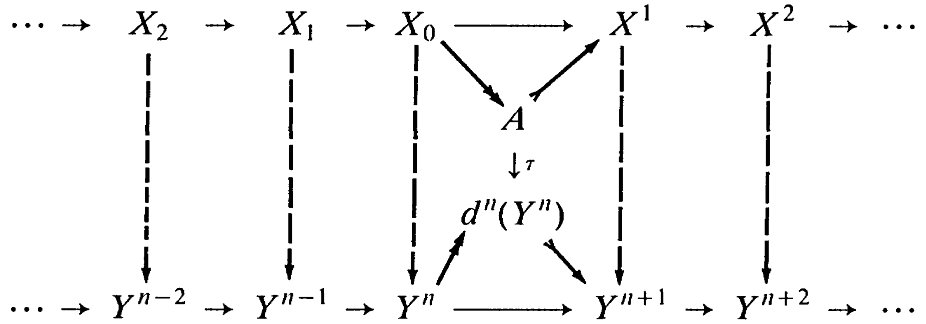
We contend that $\sigma^{\alpha+1}\epsilon = 0 = j\tau^{\alpha+1}$, for a suitable choice of $\sigma^i$, $i \le \alpha+1$ and $\tau^i, i \ge \alpha+2$; hence it is possible to choose $X^{\alpha+1} \to Y^{\beta-1}$ to be the zero map.

Indeed $\sigma^{\alpha+1}\epsilon = 0$ for some homotopy, since $\sigma^{\alpha+1}\epsilon = (f^\alpha - g^\alpha) - d^{\beta-2}\sigma^\alpha$ which becomes zero on composition with $\varphi$. Hence $X^\alpha$ projective and $\sigma^{\alpha+1}\epsilon(X^\alpha) \subseteq d(Y^{\beta-2})$ implies $\sigma^{\alpha+1}\epsilon$ factors through $Y^{\beta-2}$, say by $s: X^\alpha \to Y^{\beta-2}$. Replace $\sigma^\alpha$ by $(\sigma^\alpha + s)$ and $\sigma^{\alpha+1}$ by zero. Dually $j\tau^{\alpha+1}$ factors through $X^{\alpha+2}$ so $\{\tau^i\}_{i \ge \alpha+1}$ may be replaced by a homotopy $\{\hat{\tau}^i\}_{i > \alpha+1}$ with $j\hat{\tau}^{\alpha+1} = 0$.

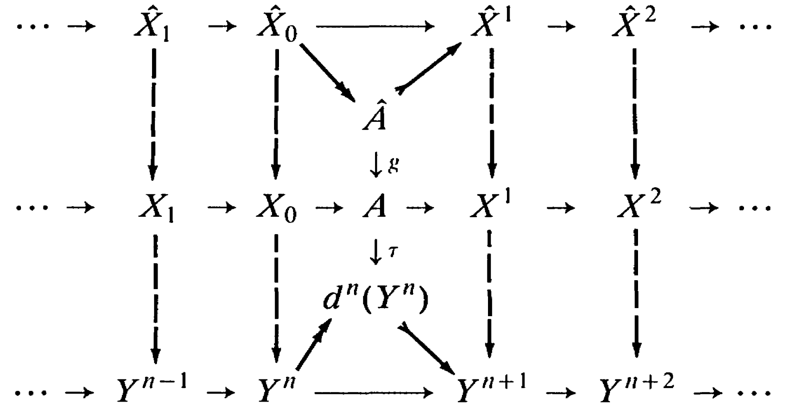
Now we characterize $\mathrm{Ext}(A, B)$ by means of Proposition 2.

THEOREM 3. *Let $X$ be a complete resolution for $A$, $Y$ a complete resolution for $B$. Then for $n > 0$, $\mathrm{Ext}^n(A, B)$ is isomorphic to the set of homotopy classes of complete ladders from $X$ to $Y$ with degree $n$.*

*Proof.* Let $S^n(X, Y)$ be the set of homotopy classes of complete ladders from $X$ to $Y$ with degree $n$. Use Proposition 2 to construct $\theta: \mathrm{Ext}^n(A, B) \to S^n(X, Y)$. To wit, for any cocycle $\tau: A \to Y^{n+1}$ let $\theta([\tau])$ be that element of $S^n(X, Y)$ determined by taking $f$ equal to $X_0 \twoheadrightarrow A \to Y^{n+1}$ in Proposition 2.




Notice also that $S^n(X, Y)$ is a functor of $A$ and $B$, for each $n \ge 0$. The functorial map induced by $\hat{A} \to A$ is described by the following commutative diagram. Let $\hat{X}$ be a complete resolution for $\hat{A}$ and $[\tau] \in \mathrm{Ext}^n(A, B)$.



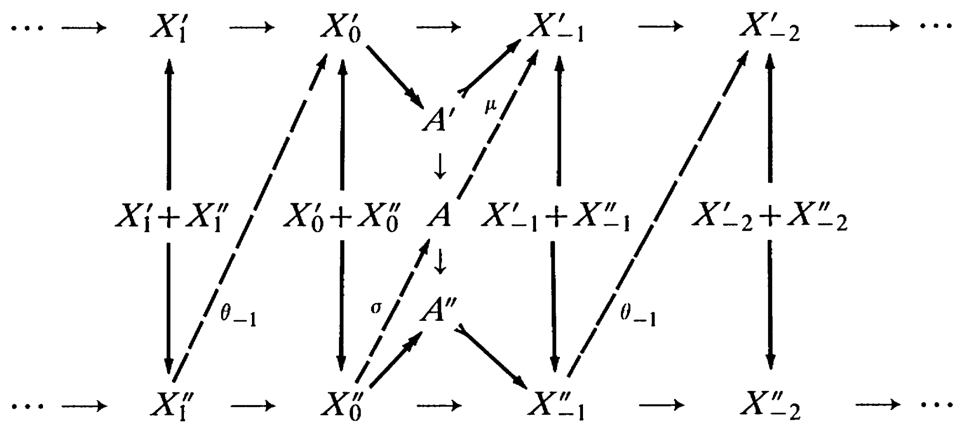

If $\{f^i\}_{i \in \mathbf{Z}}$ belongs to $\theta[\tau]$ and $\{g^i\}_{i \in \mathbf{Z}}$ is constructed by appeal to Proposition 2, we define the functorial map $S^n(X, Y) \to S^n(\hat{X}, Y)$ by sending $[\{f^i\}_{i \in \mathbf{Z}}]$ to $[\{f^i g^i\}_{i \in \mathbf{Z}}]$. This makes $\theta$ natural in the first variable because $\{f^i g^i\}_{i \in \mathbf{Z}}$ certainly belongs to $\theta([\tau \circ g]) = \mathrm{Ext}^n(g, B)([\tau])$.

Likewise in the second variable, if $B \to \hat{B}$, and $\hat{Y}$ is a complete resolution for $\hat{B}$, define $S^n(X, Y) \to S^n(X, \hat{Y})$ as composition with a ladder of degree zero induced by $h$, from $Y$ to $\hat{Y}$.

We inspect the connecting homomorphisms; let $0 \to A' \to A \to A'' \to 0$ be an exact sequence.

LEMMA 1. *For any two complete resolutions $X'$ of $A'$ and $X''$ of $A''$, $\{X_i' + X_i''\}_{i \in \mathbf{Z}}$ forms a complete resolution of $A$ with the differential given below.*

*Proof.* We must choose the differentials.




The maps $\sigma$ and $\mu$ result from projectivity of $X_0''$ and injectivity of $X_{-1}'$. The maps $\theta_i$, $i \in \mathbf{Z}$ are constructed inductively by projectivity of $X_i''$, $i \geq 0$ and injectivity of $X_i'$ for $i \leq -2$. Choosing $d_n = d_n' + \theta_n + d_n''$, then by [1] the middle row is a splice of an injective with a projective resolution of $A$, hence a complete resolution for $A$. [Note that $\theta_0 = 0$.]

Now consider the complete ladders with degree zero $H = \{X_i' \to X_i' + X_i''\}_{i \in \mathbf{Z}}$ and $K = \{X_i' + X_i'' \to X_i''\}_{i \in \mathbf{Z}}$, and the ladder with degree one $\{\theta_i\}_{i \in \mathbf{Z}} = L$. Composition with $H, K$ and $L$ gives a triangle:

$$\begin{array}{ccc} \operatorname{Ext}(A'', C) & \to & \operatorname{Ext}(A, C) \\ \nwarrow & & \swarrow \\ & \operatorname{Ext}(A', C) & \end{array}$$

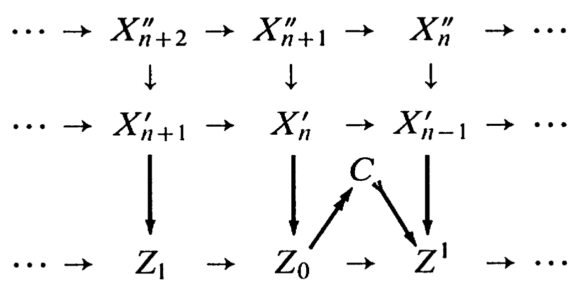
THEOREM 4. *The triangle above expresses the standard long exact sequence in* $\operatorname{Ext}(-, C)$ *induced by* $0 \to A' \to A \to A'' \to 0$.

*Proof.* Let $Z$ be a complete resolution for $C$; we show that the standard connecting homomorphism is composition with $L$.

If

$$[\tau] \in \operatorname{Ext}^n(A', C), \quad \tau : X_n' \to C, \quad j_{n+1}'' : X_{n+1}'' \to X_{n+1}' + X_{n+1}'',$$

and $\pi_n : X_n' + X_n'' \to X_n'$, then $\delta([\tau]) = [\tau \pi_n d_{n+1} j_{n+1}'']$ in $\operatorname{Ext}^{n+1}(A'', C)$. But $\pi_n d_{n+1} j_{n+1}' = \pi_n(d_{n+1}' + \theta_{n+1} + d_{n+1}'') j_{n+1}' = \theta_{n+1}$ so $\delta([\tau]) = [\tau \theta_{n+1}]$. Applying the $\theta$ of Theorem 3, $\theta[\tau] \mapsto \theta[\tau] \circ L$ expresses $S^n(X', Z) \to S^{n+1}(X'', Z)$ as claimed.



In view of the preceding discussion we will identify $S^n(X, Y)$ and $\mathrm{Ext}^n(A, B)$.

**3. Eckmann-Hilton groups.** Suppose that $0 \to A' \to A \to A'' \to 0$ is an exact sequence. Constructing complete resolutions $X'$, $X$ and $X''$ per the Lemma of the preceding section, we notice that $0 \to X' \to X \to X'' \to 0$ is a split exact sequence of complexes. Hence $0 \to \mathrm{Hom}(B, X') \to \mathrm{Hom}(B, X) \to \mathrm{Hom}(B, X'') \to 0$ and $0 \to \mathrm{Hom}(X'', B) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(X', B) \to 0$ are split exact sequences of complexes. Thus upon taking homology there result two long exact sequences, as follows.

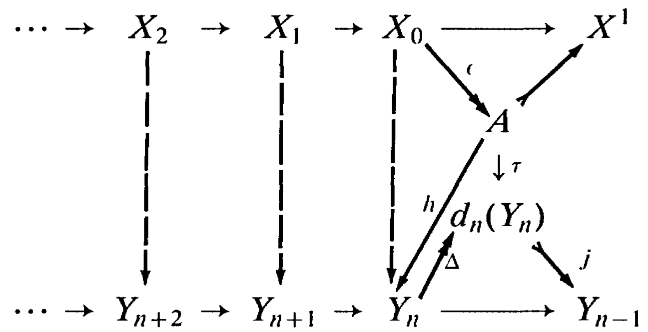
$$\cdots \to \underline{\Pi}_1(B, A'') \to \underline{\Pi}_0(B, A') \to \underline{\Pi}_0(B, A) \to \underline{\Pi}_0(B, A'') \to \mathrm{Ext}^1(B, A') \to \cdots \tag{1}$$

$$\cdots \to \overline{\Pi}_1(A', B) \to \overline{\Pi}_0(A'', B) \to \overline{\Pi}_0(A, B) \to \overline{\Pi}_0(A', B) \to \mathrm{Ext}^1(A'', B) \to \cdots \tag{2}$$

Since $\underline{\Pi}_n(B, P) = 0$ for $P$ projective, $n \geq 0$, we may conclude by [1, Theorem 5.1, p. 46] that the functors $H_n(B, -)$, $n \in \mathbf{Z}$, are satellites of $\underline{\Pi}_0(B, -)$. Dually the functors $K_n(-, B)$, $n \in \mathbf{Z}$, are satellites of $\overline{\Pi}_0(-, B)$.

THEOREM 5. *If $X$ is a complete resolution for $A$ and $Y$ is a complete resolution for $B$, and $n \geq 0$, then $\underline{\Pi}_n(A, B)$ is isomorphic to the set of homotopy classes of left ladders over maps $A \to d_n(Y^n)$.*

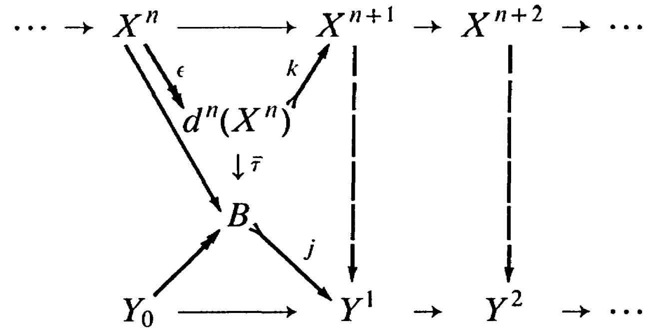
*Proof.* Given $t \in \underline{\Pi}_n(A, B)$, $t = [\tau]$ for some $\tau: A \to d_n(Y_n)$. Then Proposition 1 asserts that $\tau$ determines a homotopy class of left ladders over $\tau$ with degree $-n$, say $[\{f_i\}_{i \geq 0}]$.




To check that $[\tau] \mapsto [\{f_i\}_{i \geq 0}]$ is well-defined, suppose $[\tau] = 0$; so $\tau = \Delta h$ for some $h: A \to Y_n$. In this case we may choose $f_0 = h\epsilon$ where $\epsilon: X_0 \twoheadrightarrow A$. Then $f_0 d_1 = h\epsilon d_1 = 0$ implies that we may choose $f_i = 0$ all $i \geq 1$. Such an $\{f_i\}_{i \geq 0}$ is homotopic to zero. Additivity is clear.

Conversely, a left ladder $\{f_i\}_{i \geq 0}$ over some $f: A \to d_n(Y_n)$ gives an element of $\underline{\Pi}_n(A, B)$, namely $[A \to d_n(Y_n) \to Y_{n-1}]$. Suppose $\{f_i\}_{i \geq 0}$ is homotopic to zero, so in particular $f_0 = d_{n+1}\sigma_0 + \sigma\epsilon$ for some $\sigma_0: X_0 \to Y_{n+1}$ and $\sigma: A \to Y_n$. Then $jf = d_n\sigma$, for $jf\epsilon = d_n f_0 = d_n(d_{n+1}\sigma_0 + \sigma\epsilon) = d_n\sigma\epsilon$ and $\epsilon$ is onto. Thus $[\{f_i\}_{i \geq 0}] \mapsto [jf]$ is a well-defined inverse to $[\tau] \mapsto [\{f_i\}_{i \geq 0}]$ and the theorem is established. In precisely the same manner we find a characterization of $\overline{\Pi}_n(A, B)$.

THEOREM 6. *If* $n \geq 0$, $\bar{\Pi}_n(A, B)$ *is isomorphic to the set of homotopy classes of right ladders over maps* $d^n(X^n) \to B$, *for* $X$ *a complete resolution of* $A$ *and* $Y$ *a complete resolution of* $B$.



**4. Composition of ladders.** Fix three complete resolutions $X$, $Y$ and $Z$, complete resolutions for $A$, $B$ and $C$ respectively, $F: X \to Y$ a ladder with degree $m$, and $G: Y \to Z$ a ladder with degree $n$.

THEOREM 7. *If* $F = \{f_i\}_{i \in \mathbf{Z}}$ *and* $G = \{g_i\}_{i \in \mathbf{Z}}$ *are complete ladders of degrees* $m$ *and* $n$ *respectively, then* $GF$ *is a complete ladder of degree* $m+n$. *Moreover, for* $m > 0$ *and* $n > 0$, $([G], [F]) \mapsto [GF]$ *describes a natural transformation* $\mathrm{Ext}^n(B, C) \times \mathrm{Ext}^m(A, B) \to \mathrm{Ext}^{n+m}(A, C)$, *namely Yoneda product.*

*Proof.* Clearly $GF$ is a complete ladder. To show that composition of ladders respects homotopy equivalence, we show that $[G] = 0$ implies $[GF] = 0$ and $[F] = 0$ implies $[GF] = 0$.

$$\begin{array}{ccccccccc}
\cdots \to & X_{i+1} & \to & X_i & \to & X_{i-1} & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & Y_{i-m+1} & \to & Y_{i-m} & \to & Y_{i-m-1} & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & Z_{i-m-n+1} & \to & Z_{i-m-n} & \to & Z_{i-m-n-1} & \to \cdots
\end{array}$$

(a) Consulting the diagram, if $[G] = 0$ there is a homotopy $\{\tau_i\}_{i \in \mathbf{Z}}$ such that $g_j = d_{j-n+1}\tau_j + \tau_{j-1}d_j$ for all $j \in \mathbf{Z}$. In particular

$$g_{i-m}f_i = d_{i-m-n+1}(\tau_{i-m}f_i) + (\tau_{i-m-1}f_{i-1})d_i,$$

so $\{\tau_{i-m}f_i\}_{i \in \mathbf{Z}}$ shows $[GF] = 0$.

(b) Similarly, if there exists $\{\sigma_i\}_{i \in \mathbf{Z}}$ with $f_i = d_{i-m+1}\sigma_i + \sigma_{i-1}d_i$, all $i \in \mathbf{Z}$, then $\{g_{i-m+1}\sigma_i\}_{i \in \mathbf{Z}}$ is a homotopy showing $[GF] = 0$.

Thus $([G], [F]) \mapsto [GF]$ gives a function

$$Ext^n(B, C) \times \mathrm{Ext}^m(A, B) \to \mathrm{Ext}^{n+m}(A, C),$$

which we remark is additive in both $[G]$ and $[F]$.

To verify naturality, suppose $h: \hat{A} \to A$ and let $\hat{X}$ be a complete resolution for $\hat{A}$. By Proposition 2, $h$ induces a unique $[\{h_i\}_{i \in \mathbf{Z}}]$ with degree zero, and as noted earlier $\mathrm{Ext}(h, -)$ is composition with $\{h_i\}_{i \in \mathbf{Z}}$. Thus the following diagram illustrates naturality in $A$.

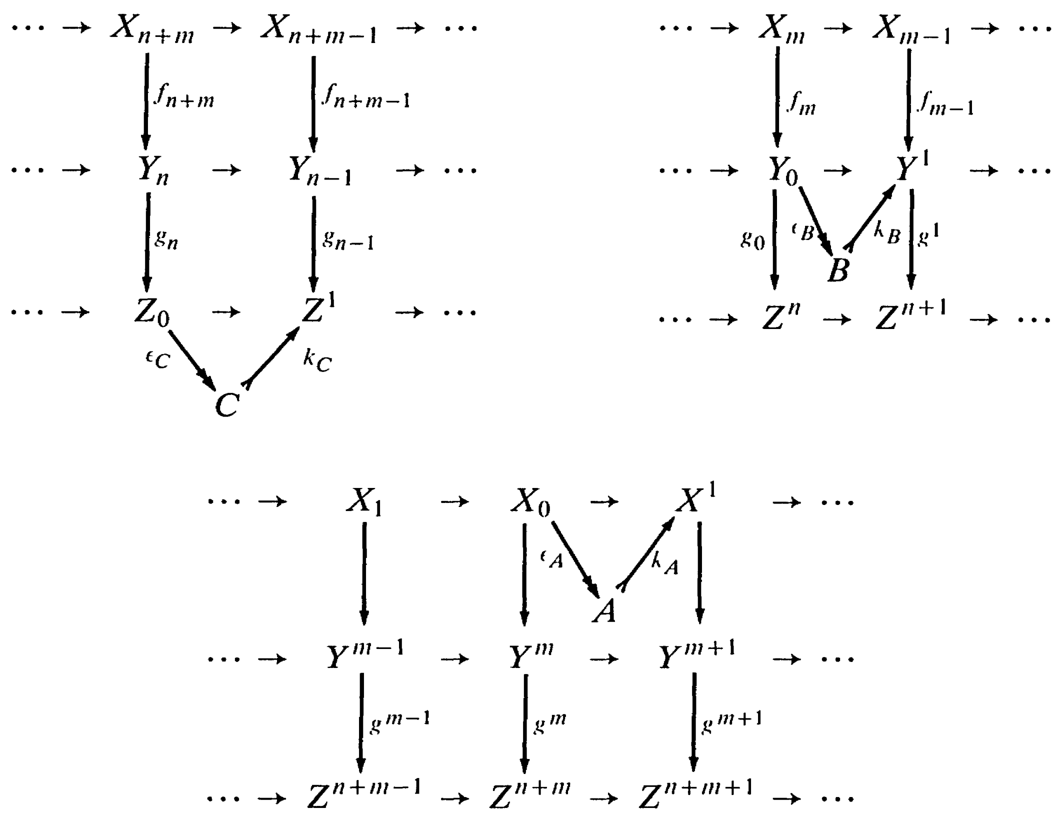
$$\begin{array}{ccccccccc}
\cdots \to & \hat{X}_{i+1} & \to & \hat{X}_i & \to & \hat{X}_{i-1} & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & X_{i+1} & \to & X_i & \to & X_{i-1} & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & Y_{i-m+1} & \to & Y_{i-m} & \to & Y_{i-m-1} & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & Z_{i-m-n+1} & \to & Z_{i-m-n} & \to & Z_{i-m-n-1} & \to \cdots
\end{array}$$

Naturality in $C$ follows in the same manner.

Finally to see that $([G],[F]) \mapsto [GF]$ expresses the Yoneda product, we prove commutativity in the following square, where the vertical isomorphisms come from Theorem 3 and $Y$ means Yoneda product.

$$\begin{array}{ccc}
S^n(Y, Z) \times S^m(X, Y) & \to & S^{n+m}(X, Z) \\
\downarrow & & \downarrow \\
\operatorname{Ext}^n(B, C) \times \operatorname{Ext}^m(A, B) & \xrightarrow{Y} & \operatorname{Ext}^{n+m}(A, C).
\end{array}$$

Consider the following portion of the complete ladders $G$ and $F$.



Under the isomorphism of Theorem 3, $([G],[F])$ corresponds to $([g^1 k_B],[f^1 k_A])$ which equals $((-1)^n[\epsilon_C g_n],(-1)^m[\epsilon_B f_m])$ in $\operatorname{Ext}^n(B,C) \times \operatorname{Ext}^m(A,B)$. By [5, Exercise 2, p. 91], the Yoneda product of these two classes is $(-1)^{n+m}[\epsilon_C g_n f_{n+m}]$, which equals $[g^{m+1} f^1 k_A]$ in $\operatorname{Ext}^{n+m}(A,C)$. But $[g^{m+1} f^1 k_A]$ maps to $[GF]$ under the isomorphism of Theorem 3 gives the result.

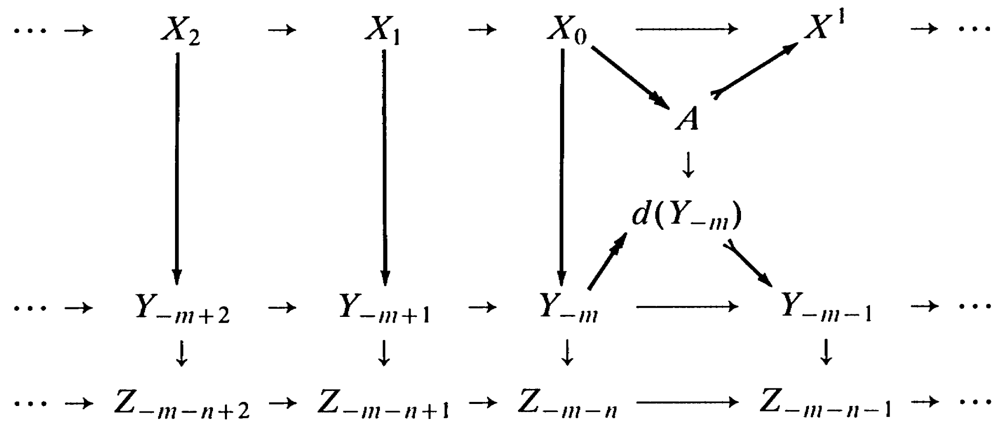
REMARKS. (i) Since the connecting homomorphism $\mathrm{Ext}^i \to \mathrm{Ext}^{i+1}$ is given by a composition of ladders, $([G],[F]) \mapsto [GF]$ is compatible with the connecting homomorphisms.

(ii) Theorem 7 constructs maps (natural with respect to $A$ and $C$) $S^n(Y,Z)\times S^m(X,Y)\to S^{n+m}(X,Z)$ for any integers $n$ and $m$ and any $X$, $Y$ and $Z$. We examine $([G],[F]) \mapsto [GF]$ for non-positive $m$ or $n$.

(I) Suppose $m\le 0$ and $n>0$. If $G=\{g_i\}_{i\in \mathbf{Z}}$, then $[G]$ is in $\mathrm{Ext}^n(B,C)$.

If $F=\{f_i\}_{i\ge z}$, some $z\le 0$, then $[\hat{F}]=[\{f_i\}_{i\ge 0}]$ belongs to $\underline{\Pi}_{-m}(A,B)$. Then $([G],[\hat{F}])\to[G\hat{F}]=[\{g_{i-m}f_i\}_{i\ge 0}]$ gives a new product

$$\mathrm{Ext}^n(B,C)\times \underline{\Pi}_{-m}(A,B)\to\begin{cases}\mathrm{Ext}^{n+m}(A,C) & \text{if } n+m>0\\ \underline{\Pi}_{-(n+m)}(A,C) & \text{if } n+m\le 0\end{cases}$$



Proof of invariance with respect to homotopy is omitted throughout.

REMARK. Had the $F$ above been a right ladder, $F=\{f^i\}_{i\ge -m+1}$, $[F]$ would have belonged to $\bar{\Pi}_{-m}(A,B)$. But $\{g^i f^{-m+i}: X^{-m+i}\to Y^i\to Z^{n+i}\}_{i\ge 1}$ cannot be extended to the left by means of Proposition 1. Hence the most we would have obtained is the unsatisfactory

$$\mathrm{Ext}^n(B,C)\times\bar{\Pi}_{-m}(A,B)\to\bar{\Pi}_{-m}(A,d^n(Z^n)).$$

Similarly, changing $G$ to a left ladder in Case II below is unprofitable.

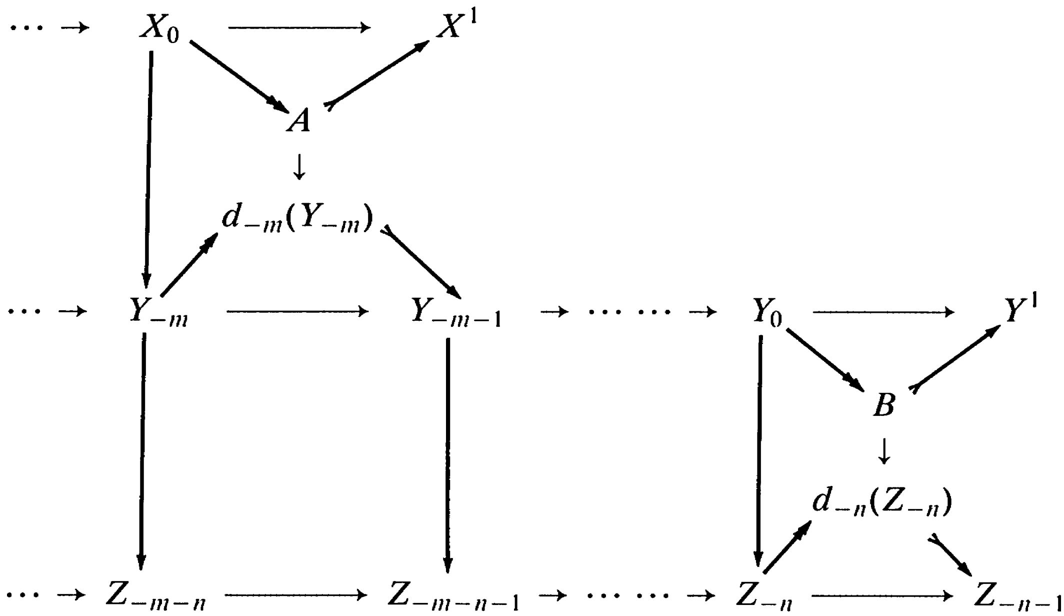
(II) Suppose $m>0$ and $n\le 0$.

An $F=\{f_i\}_{i\in\mathbf{Z}}$ represents $[F]$ in $\mathrm{Ext}^m(A,B)$. If $G=\{g^i\}_{i\ge z}$, $z\le -n+1$, then $[\hat{G}]=[\{g^i\}_{i\ge -n+1}]$ belongs to $\bar{\Pi}_{-n}(B,C)$. Thus $([\hat{G}],[F])\mapsto[\hat{G}F]=[\{g^{-n+i}f^{-n-m+i}\}_{i\ge 1}]$ gives a product

$$\bar{\Pi}_{-n}(B,C)\times\mathrm{Ext}^m(A,B)\to\begin{cases}\mathrm{Ext}^{m+n}(A,C) & \text{if } m+n>0\\ \bar{\Pi}_{-(m+n)}(A,C) & \text{if } m+n\le 0.\end{cases}$$

$$\begin{array}{ccccccccc}
\cdots\to & X^{-n-m-1} & \to & X^{-n-m} & \to & X^{-n-m+1} & \to\cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots\to & Y^{-n-1} & \to & Y^{-n} & \to & Y^{-n+1} & \to\cdots \\
& & & & & \downarrow & \\
& Z^{-1} & \to & Z^{0} & \to & Z^{1} & \to\cdots
\end{array}$$

(III) $F=\{f_i\}_{i\geq r}$ and $G=\{g_i\}_{i\geq s}$, $m\leq 0$ and $n\leq 0$. If $r$ and $s$ are non-positive consider $\{f_i\}_{i\geq 0}$ and $\{g_i\}_{i\geq 0}$.



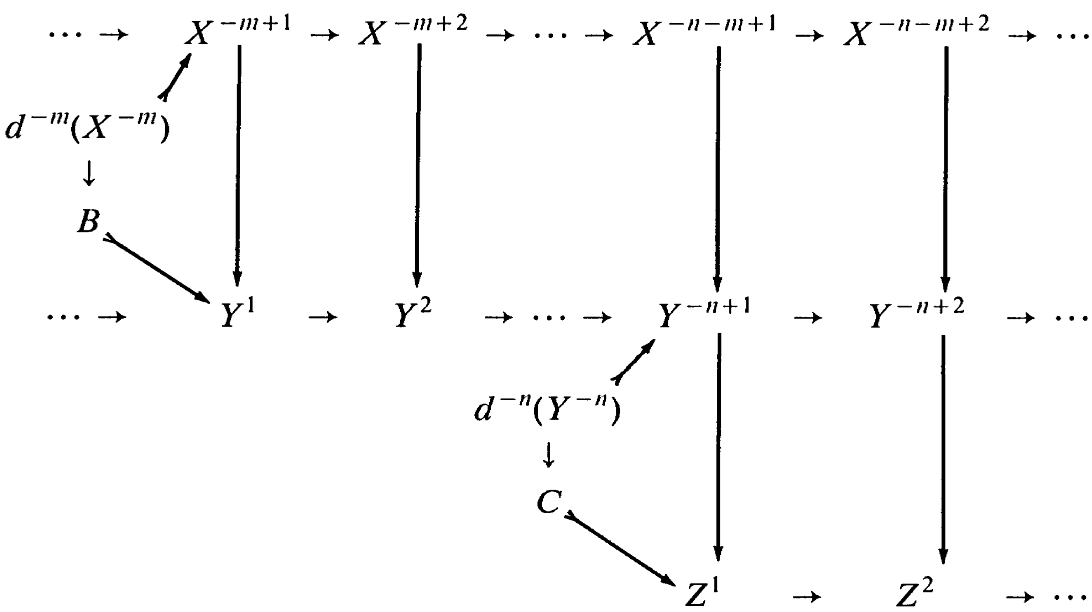

Assuming $m\leq 0$ and $n\leq 0$, the diagram above showing

$$([G],[F])\mapsto[\{g_{-m+i}f_i\}_{i\geq 0}]$$

illustrates

$$\underline{\Pi}_{-n}(B,C)\times\underline{\Pi}_{-m}(A,B)\to\underline{\Pi}_{-n-m}(A,C).$$

(IV) $F=\{f^i\}_{i\geq r}$ and $G=\{g^i\}_{i\geq s}$. If $r\leq -m+1$ and $s\leq -n+1$, $m\leq 0$ and $n\leq 0$, the following product exists.




$$\overline{\Pi}_{-n}(B,C)\times\overline{\Pi}_{-m}(A,B)\to\overline{\Pi}_{-n-m}(A,C).$$

We record the observations of (I)–(IV).

PROPOSITION 8. *$\underline{\Pi}(B,B)$ and $\bar{\Pi}(B,B)$ are given graded ring structures by the products of* (III) *and* (IV) *respectively.*

*Proof.* The identity for $\underline{\Pi}(B,B)$ is $1=[\{\mathrm{id}_{Y_i}\}_{i\geq 0}]$. Likewise $[\{\mathrm{id}_{Y_i}\}_{i<0}]$ is the identity for $\bar{\Pi}(B,B)$.

Recall that

$$H_n(B,C)=\begin{cases}\mathrm{Ext}^n(B,C) & \text{for } n>0\\ \underline{\Pi}_{-n}(B,C) & \text{for } n\leq 0.\end{cases}$$

LEMMA 9. *$H(B,C)$ is a left* $\mathrm{Ext}(C,C)$*-module and a right $\underline{\Pi}(B,B)$ module.*

LEMMA 9′. *$K(B,C)$ is a right* $\mathrm{Ext}(B,B)$ *module and a left $\bar{\Pi}(C,C)$ module.*

REMARK. The products (III) and (IV) were known to Eckmann and Hilton.

**5. Natural transformations.** We have established maps

$$H_p(B,C)\times H_q(A,B)\to H_{p+q}(A,C);$$

thus any $[\sigma]$ in $\underline{\Pi}_q(A,B)$ gives an element of $[\{H_{p+q}(B,-)\}_{p\in Z},\{H_p(A,-)\}_{p\in Z}]$.

THEOREM 10. *The only natural transformations of graded functors from $\{H_p(B,-)\}_{p\in Z}$ to $\{H_p(A,-)\}_{p\in Z}$ lowering degree by $q$ $[q\geq 0]$ are multiplication on the right by elements of $\underline{\Pi}_q(A,B)$. That is,*

$$[\{H_{p+q}(B,-)\}_{p\in Z},\{H_p(A,-)\}_{p\in Z}]\approx\underline{\Pi}_q(A,B)\quad \textit{for } q\geq 0.$$

*Proof.* (i) Let

$$\eta: H_0(B,-)\to H_{-q}(A,-)$$

be a natural transformation. We claim there is one and only one element $\{\eta_{p+q}\}_{p\in Z}$ belonging to $[\{H_{p+q}(B,-)\}_{p\in Z},\{H_p(A,-)\}_{p\in Z}]$ with $\eta_0=\eta$. Now $\{H_n(M,-)\}_{n\in Z}$ is an exact connected sequence of covariant functors for any module $M$. Also, $H_n(M,P)=0$ for $P$ projective and $n\leq 0$, and $H_n(M,Q)=0$ for $Q$ injective and $n>0$. Thus we may apply Proposition 5.2 of [1] with $\Phi_0=\eta$.

(ii)

$$\underline{\Pi}_q(A,B)\approx[\underline{\Pi}_0(B,-),\underline{\Pi}_q(A,-)].$$

For $[\sigma]\in\underline{\Pi}_q(A,B)$ and $[\tau]\in\underline{\Pi}_0(B,C)$, let $\eta_C^{[\sigma]}([\tau])=[\tau][\sigma]$. We show that $\eta\mapsto\eta_B([\mathrm{id}_B])$ is an inverse for $([\sigma]\mapsto\eta^{[\sigma]})$. First, since $[\mathrm{id}_B][\sigma]=[\sigma]$, $[\sigma]\mapsto\eta^{[\sigma]}\mapsto\eta_B^{[\sigma]}([\mathrm{id}_B])$ is the identity.

Conversely, we need $\eta^{\eta_B([\mathrm{id}_B])}=\eta$; that is, for any $[\tau]\in\underline{\Pi}_0(B,C)$ that

$$[\tau]\eta_B([\mathrm{id}_B])=\eta_C([\tau])$$

in $\underline{\Pi}_q(A,C)$. Now $\tau: B\to C$ induces the following commutative square.

$$\begin{array}{ccc}\underline{\Pi}_0(B,B) & \xrightarrow{\eta_B} & \underline{\Pi}_q(A,B)\\ \downarrow & & \downarrow\\ \underline{\Pi}_0(B,C) & \xrightarrow{\eta_C} & \underline{\Pi}_q(A,C)\end{array}$$

Chasing the diagram, $\eta_C \circ \underline{\Pi}_0(B,\tau)([\mathrm{id}_B]) = \eta_C([\tau]) = \underline{\Pi}_q(A,\tau)\eta_B([\mathrm{id}_B]) = [\tau]\eta_B([\mathrm{id}_B])$ as desired. Hence $[\sigma] \mapsto \{\eta_{p+q}^{[\sigma]}\}_{p\in\mathbf{Z}}$ is the isomorphism proving the theorem. □

REMARKS. (a) From the proof of (i) note that in fact

$$\underline{\Pi}_q(A,B) \simeq [\mathrm{Ext}^p(B,-), \underline{\Pi}_{q-p}(A,-)]$$

for any $p>0$ with $q-p\geq 0$. More generally, Hilton and Rees [4] proved that $[\mathrm{Ext}^p(B,-),G] \simeq S_pG(B)$, which we may rewrite as $[H^p(B,-),H^q(A,-)] \simeq H^{q-p}(A,B)$ for any $p>0$, $q\leq 0$.

(b) It is interesting to note that Hilton and Rees also obtain the isomorphism $[\mathrm{Ext}^p(B,-),\mathrm{Ext}^q(A,-)] \simeq S^{p-1}\underline{\Pi}_{q-1}(A,B)$.

THEOREM 11. *The only natural transformations of graded functors from* $\{K_p(-,A)\}_{p\in\mathbf{Z}}$ *to* $\{K_p(-,B)\}_{p\in\mathbf{Z}}$ *lowering degree by* $q$, $q\geq 0$, *are multiplications on the left by elements of* $\overline{\Pi}_q(A,B)$. *That is,*

$$[\{K_{p+q}(-,A)\}_{p\in\mathbf{Z}}, \{K_p(-,B)\}_{p\in\mathbf{Z}}] \approx \overline{\Pi}_q(A,B) \text{ by } [\sigma] \mapsto \{^{[\sigma]}\eta_{p+q}\}_{p\in\mathbf{Z}}.$$

*Proof.* Proof is entirely dual to that of Theorem 10.

**6.** $\underline{\Pi}_i(\mathbf{Z},A) \approx \mathrm{Tor}_{i-1}^{\mathbf{Z}G}(\mathbf{Z},A)$ for $i\geq 2$. A natural transformation $f: T\to U$ induces the following commutative diagram.

$$\begin{array}{ccccc} L_0T & \to & T & \to & R^0T \\ \downarrow & & \downarrow f & & \downarrow \\ L_0U & \to & U & \to & R^0U \end{array}$$

Let $\bar{f}$ be the map induced from $L_0T$ to $R^0U$. Then the sequence of functors

$$\cdots L_2T, L_1T, \mathrm{kernel}(\bar{f}), \mathrm{cokernel}(\bar{f}), R^1U, R^2U, \cdots$$

is the *derived sequence* of $f$. It is easily seen that the derived sequence of a map is an exact connected sequence of functors [1, Chapter V].

Henceforth let the ring $R$ be $\mathbf{Z}G$, for $G$ a finite group, and $I$ the augmentation ideal. If $N=\sum_{x\in G}x$, $N$ induces a map $N^*: A/IA \to A^G$ for each $G$-module $A$. Denote the derived sequence of $N^*: \mathrm{Tor}_0^{\mathbf{Z}G}(\mathbf{Z},A) \to \mathrm{Ext}^0_{\mathbf{Z}G}(\mathbf{Z},A)$ by $\{\hat{H}^i(G,A)\}_{i\in\mathbf{Z}}$. In particular we have

$$\hat{H}^i(G,A) = \begin{cases} \mathrm{Ext}^i_{\mathbf{Z}G}(\mathbf{Z},A) & \text{if } i>0 \\ \mathrm{Coker}(A/IA \to A^G) = A^G/NA & \text{if } i=0 \\ \mathrm{Ker}(A/IA\to A^G) = (N-\mathrm{torsion}(A))/IA & \text{if } i=-1 \\ \mathrm{Tor}_{i-1}^{\mathbf{Z}G}(\mathbf{Z},A) & \text{if } i<-1. \end{cases}$$

PROPOSITION 12. $\hat{H}^i = S_1\hat{H}^{i+1}$ *all integers* $i$.

*Proof.* Let $0\to R\to P\to A\to 0$ be a projective presentation of $A$. Then $S_1\hat{H}^{i+1}(G,A) = \mathrm{kernel}(\hat{H}^{i+1}(G,R)\to\hat{H}^{i+1}(G,P)) = \hat{H}^{i+1}(G,R) = \hat{H}^i(G,A)$.

However $\{H_i(\mathbf{Z},A)\}_{i\in\mathbf{Z}}$ is also a sequence of satellites with $H_i(\mathbf{Z},A) = \mathrm{Ext}^i_{\mathbf{Z}G}(\mathbf{Z},A)$ for $i>0$. So $\{\hat{H}^i(\mathbf{Z},A)\}_{i\in\mathbf{Z}} = \{H_i(\mathbf{Z},A)\}_{i\in\mathbf{Z}}$. In particular we see that $\underline{\Pi}_i(\mathbf{Z},A) = \mathrm{Tor}_{i-1}^{\mathbf{Z}G}(\mathbf{Z},A)$ for all $i\geq 2$.

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