CERTAIN CLASSES OF STARLIKE FUNCTIONS

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Dedicated to George Piranian

1. Introduction. Let A denote the class of functions $f(z)$ analytic in $U=\{z: |z|<1\}$ with $f(0)=f'(0)-1=0$. For a given α, $0\le\alpha<1$, $S^*(\alpha)\subset A$ is the class of functions f, starlike of order α, i.e. $\operatorname{Re}(zf'(z)/f(z))>\alpha$, $z\in U$. We write $S^*=S^*(0)$. We denote by K the class of functions f in S^* such that $f(U)$ is a convex domain. If $f\in A$ then $f\in K$ if and only if $zf'(z)\in S^*$.

In 1969 Mocanu [3] showed that if $f\in A$, $f(z)f'(z)\neq 0$ for $0<|z|<1$, and if for a given $\beta\ge 0$, $f(z)$ satisfies the inequality

$$\operatorname{Re}\left\{(1-\beta)\frac{zf'(z)}{f(z)}+\beta\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>0,\quad z\in U, \tag{1}$$

then $f\in S^*$. Under the same conditions but with the inequality (1) replaced by

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\left(z^2\{f,z\}+1+\frac{zf''(z)}{f'(z)}\right)\right]>0,\quad z\in U, \tag{2}$$

where $\{f,z\}$ denotes the Schwarzian derivative $(f''(z)/f'(z))'-\frac{1}{2}(f''(z)/f'(z))^2$, Miller and Mocanu [2] showed that again $f\in S^*$.

In this paper we consider classes of functions $f\in A$ with $f(z)\neq 0$ for $0<|z|<1$, and such that, for a given $\alpha>0$, $f(z)$ satisfies an inequality involving a suitably prescribed lower bound on $|z|=r<1$ for the real part of the analytic function

$$z^2\left\{\alpha\left(\frac{f'(z)}{f(z)}\right)^2+\left(\frac{f'(z)}{f(z)}\right)'\right\}=\frac{zf'(z)}{f(z)}\left\{(\alpha-1)\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)}\right\}. \tag{3}$$

This lower bound may have one of a variety of given forms from which we can conclude not only that $f\in S^*$ but that $\operatorname{Re}(zf'(z)/f(z))$ has a positive lower bound on $|z|=r<1$ with a pre-determined rate of growth as a function of r and the given parameter α. Because of the freedom of the parameter α and because of a wide choice for the given lower bound on the real part of the function (3), a large group of various classes of starlike functions emerges. The method of approach is quite different from that appearing in [2]. Rather, it depends fundamentally upon an extension of a lemma introduced by the author [4] in a study of second-order differential equations; see also [1].

2. The main theorem.

DEFINITION. Let Q denote the class of bounded real functions $q(r)$ having a continuous derivative $q'(r)$ for $0\le r<1$ and such that $0<q(r)\le q(0)=1$.

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LEMMA 1. *Let α be a given positive real number. Let $y(\rho)$ and its derivative $y'(\rho)$ be real functions that are continuous in ρ for $0<\rho<1$. Let $q(r)\in Q$. Then for all r_1 and r, $0<r_1<r<1$, the following inequality holds:*

$$(4)\quad \int_{r_1}^{r}[y'(\rho)]^2\,d\rho+\int_{r_1}^{r}y^2(\rho)\left[\left(\frac{\alpha q(\rho)}{\rho}\right)'+\left(\frac{\alpha q(\rho)}{\rho}\right)^2\right]d\rho\geq\left[\frac{\alpha q(\rho)}{\rho}y^2(\rho)\right]_{r_1}^{r}.$$

Equality holds if and only if

$$y(\rho)=k\rho^{\alpha}\exp\int_0^{\rho}\alpha(q(t)-1)\frac{dt}{t}$$

for $r_1<\rho<r$, where $k=k(r_1)$ is a real constant.

The lemma follows easily from the following calculation using partial integration:

$$\int_{r_1}^{r}\left[y'(\rho)-\frac{\alpha q(\rho)}{\rho}y(\rho)\right]^2 d\rho$$

$$=\int_{r_1}^{r}[y'(\rho)]^2\,d\rho-2\alpha\int_{r_1}^{r}y'(\rho)y(\rho)q(\rho)\frac{d\rho}{\rho}+\alpha^2\int_{r_1}^{r}y^2(\rho)q^2(\rho)\frac{d\rho}{\rho^2}$$

$$=\int_{r_1}^{r}[y'(\rho)]^2\,d\rho-\left[\frac{\alpha q(\rho)}{\rho}y^2(\rho)\right]_{r_1}^{r}+\int_{r_1}^{r}y^2(\rho)\left[\left(\frac{\alpha q(\rho)}{\rho}\right)'+\left(\frac{\alpha q(\rho)}{\rho}\right)^2\right]d\rho.$$

THEOREM 1. *Let $f(z)\in A$ and let $f(z)\neq 0$ for $0<|z|<1$. For a given $\alpha>0$ and a given $q(r)\in Q$ let $f(z)$ satisfy the inequality*

$$(5)\quad \operatorname{Re}\left[z^2\left\{\left(\frac{f'(z)}{f(z)}\right)'+\alpha\left(\frac{f'(z)}{f(z)}\right)^2\right\}\right]\geq r^2\left\{\left(\frac{q(r)}{r}\right)'+\alpha\left(\frac{q(r)}{r}\right)^2\right\},\quad |z|=r<1.$$

Then $f\in S^*$ and $\operatorname{Re}[zf'(z)/f(z)]\geq q(r)>0$, $|z|=r<1$.

Proof. Let $F(z)=[f(z)]^{\alpha}=z^{\alpha}\cdot\sum_{n=0}^{\infty}b_nz^n$, $b_0=1$, where α and f satisfy the conditions of Theorem 1. Then $z^{-\alpha}F(z)$ is analytic without zeros in U. The several branches of $F(z)$ differ by constant factors. Each branch of $F(z)$ satisfies the equation

$$\frac{z^2F''(z)}{F(z)}=\alpha z^2\left\{\left(\frac{f'(z)}{f(z)}\right)'+\alpha\left(\frac{f'(z)}{f(z)}\right)^2\right\}.$$

The function $z^2Q(z)=-[z^2F''(z)/F(z)]$ is analytic in U and, for $z\in U$, $W=F(z)$ satisfies the equation $W''(z)+Q(z)W(z)=0$. Moreover, by the hypothesis of Theorem 1,

$$\operatorname{Re}[-z^2Q(z)]\geq|z|^2\left[\left(\frac{\alpha q(r)}{r}\right)'+\left(\frac{\alpha q(r)}{r}\right)^2\right],\quad |z|=r<1.$$

The Green's Transform [4;1] of the equation $W''(z)+Q(z)W(z)=0$ may be written

$$(6)\qquad [\overline{W(z)}\,W'(z)]_{\rho=r_1}^{\rho=r}-\int_{r_1e^{i\theta}}^{re^{i\theta}}|W'(z)|^2\,\overline{dz}+\int_{r_1e^{i\theta}}^{re^{i\theta}}Q(z)|W(z)|^2\,dz=0,$$

where $z=\rho e^{i\theta}$, $\theta=$constant, $0<r_1\le\rho\le r<1$. We multiply (6) by $re^{i\theta}$ and equate the real part of the equation to zero. If $W(re^{i\theta})\neq 0$, $W(r_1e^{i\theta})\neq 0$, and $z=\rho e^{i\theta}$, then, by Lemma 1,

$$\left[|W(z)|^2\operatorname{Re}\left\{re^{i\theta}\frac{W'(z)}{W(z)}\right\}\right]_{\rho=r_1}^{\rho=r}$$

$$=r\int_{r_1}^{r}|W'(\rho e^{i\theta})|\,d\rho+r\int_{r_1}^{r}\operatorname{Re}[-z^2Q(z)]\left|\frac{W(z)}{z}\right|^2d\rho$$

$$\ge r\int_{r_1}^{r}|W'(\rho e^{i\theta})|^2d\rho+r\int_{r_1}^{r}\left[\left(\frac{\alpha q(\rho)}{\rho}\right)'+\left(\frac{\alpha q(\rho)}{\rho}\right)^2\right]|W(\rho e^{i\theta})|^2d\rho$$

$$\ge r\left[|W(\rho e^{i\theta})|^2\frac{\alpha q(\rho)}{\rho}\right]_{r_1}^{r},$$

where $y(\rho)=|W(\rho e^{i\theta})|$ and $y'(\rho)=|W'(\rho e^{i\theta})|$. By hypothesis $f(z)$ and $F(z)$ have no zeros for $0<|z|<1$ so the condition $W(re^{i\theta})\,W(r_1e^{i\theta})\neq 0$ is satisfied. We now have

$$|W(re^{i\theta})|^2\operatorname{Re}\left[re^{i\theta}\frac{W'(re^{i\theta})}{W(re^{i\theta})}\right]-|W(r_1e^{i\theta})|^2\operatorname{Re}\left[re^{i\theta}\frac{W'(r_1e^{i\theta})}{W(r_1e^{i\theta})}\right]$$

(7)

$$\ge|W(re^{i\theta})|^2\alpha q(r)-\frac{r}{r_1}|W(r_1e^{i\theta})|^2\alpha q(r_1)$$

which is equivalent to

$$|F(re^{i\theta})|^2\left[\operatorname{Re}\left\{re^{i\theta}\frac{F'(re^{i\theta})}{F(re^{i\theta})}\right\}-\alpha q(r)\right]$$

(8)

$$\ge\frac{r}{r_1}|F(r_1e^{i\theta})|^2\left[\operatorname{Re}\left\{\frac{r_1e^{i\theta}F'(r_1e^{i\theta})}{F(r_1e^{i\theta})}\right\}-\alpha q(r_1)\right].$$

Since $F(z)=z^\alpha\sum_{n=0}^{\infty}b_nz^n$, $b_0=1$, we have

$$\operatorname{Re}\left[\frac{z_1F'(z_1)}{F(z_1)}\right]=\alpha+O(|z_1|),\quad z_1=r_1e^{i\theta}.$$

Since $q(0)=1$ and $q(r)$ has a continuous derivative $q'(r)$ for $0\le r<1$, we also have $\alpha q(r_1)=\alpha+O(r_1)$. Hence, as $r_1\to 0$,

$$\frac{r}{r_1}|F(r_1e^{i\theta})|^2\left[\operatorname{Re}\left\{\frac{r_1e^{i\theta}F'(r_1e^{i\theta})}{F(r_1e^{i\theta})}\right\}-\alpha q(r_1)\right]=O(r_1^{2\alpha}).$$

The functions $f(z)$ and $F(z)$ were assumed to have no zeros for $0<|z|<1$. We may now let $r_1\to 0$ and obtain for $z=re^{i\theta}$, $0\le\theta\le 2\pi$, the inequality

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}-q(r)\right]=\frac{1}{\alpha}\operatorname{Re}\left[\frac{re^{i\theta}F'(re^{i\theta})}{F(re^{i\theta})}-\alpha q(r)\right]\ge 0. \tag{9}$$

Thus $\operatorname{Re}[zf'(z)/f(z)]\ge q(|z|)>0$, $|z|<1$, and the proof is now complete. □

Since

$$|F(z)|=|f(z)|^{\alpha} \quad \text{and} \quad \frac{zF'(z)}{F(z)}=\frac{1}{\alpha}\frac{zf'(z)}{f(z)}$$

we easily obtain Corollary 1 from the proof of Theorem 1.

COROLLARY 1. *With the hypothesis of Theorem* 1 *the expression*

$$\frac{|f(z)|^{2\alpha}}{|z|}\left[\operatorname{Re}\frac{zf'(z)}{f(z)}-q(|z|)\right], \quad |z|<1, \tag{10}$$

represents an increasing function of $|z|$.

3. Some examples of Theorem 1. We consider now some instructive special cases of Theorem 1. Let $q(r)=1-r/2\in Q$. Then, for any $\alpha>0$,

$$r^2\left[\left(\frac{q(r)}{r}\right)'+\alpha\left(\frac{q(r)}{r}\right)^2\right]=\alpha-1-\alpha\left(r-\frac{r^2}{4}\right).$$

With $|z|=r<1$ for the function $f(z)=ze^{-z/2}$ we have

(11)
$$\operatorname{Re}\left\{z^2\left[\left(\frac{f'(z)}{f(z)}\right)'+\alpha\left(\frac{f'(z)}{f(z)}\right)^2\right]\right\}=\alpha-1-\alpha\operatorname{Re}\left(z-\frac{z^2}{4}\right)\geq\alpha-1-\alpha\left(r-\frac{r^2}{4}\right).$$

By Theorem 1 we conclude that $f\in S^*$ and $\operatorname{Re}[zf'(z)/f(z)]\geq q(r)=1-r/2$. This last inequality may be readily verified directly since $zf'(z)/f(z)=1-z/2$.

For a second example, let $\{b_n\}$, $n=1,2,\ldots$ denote a sequence of real numbers $b_n\geq 0$ such that, for some $\alpha>0$, $\sum_{n=1}^{\infty}(n+\alpha)b_n\leq\alpha$. Let

$$q(r)=1-\frac{1}{\alpha}\frac{\sum_{n=1}^{\infty}nb_nr^n}{1-\sum_{n=1}^{\infty}b_nr^n}, \quad q(r)\in Q.$$

Let $f(z)=z(1-\sum_{n=1}^{\infty}b_nz^n)^{1/\alpha}$, $f'(0)=1$. For $|z|=r<1$ we have

$$\begin{aligned}\operatorname{Re}\left\{z^2\left[\left(\frac{f'(z)}{f(z)}\right)'+\alpha\left(\frac{f'(z)}{f(z)}\right)^2\right]\right\}&=\alpha-1-\frac{1}{\alpha}\operatorname{Re}\left[\frac{\sum_{n=1}^{\infty}n(n+2\alpha-1)b_nz^n}{1-\sum_{n=1}^{\infty}b_nz^n}\right]\\ &\geq\alpha-1-\frac{1}{\alpha}\frac{\sum_{n=1}^{\infty}n(n+2\alpha-1)b_nr^n}{1-\sum_{n=1}^{\infty}b_nr^n}\\ &=r^2\left[\left(\frac{q(r)}{r}\right)'+\alpha\left(\frac{q(r)}{r}\right)^2\right].\end{aligned} \tag{12}$$

Since $f(z)\neq 0$ for $0<|z|<1$, the conditions of Theorem 1 are satisfied. Hence

$$\operatorname{Re}\frac{zf'(z)}{f(z)}\geq q(r)=1-\frac{1}{\alpha}\frac{\sum_{n=1}^{\infty}nb_nr^n}{1-\sum_{n=1}^{\infty}b_nr^n}>0, \quad |z|=r, \tag{13}$$

and $f\in S^*$.

In the second example, take $b_1=\alpha/(\alpha+1)$, $\alpha>0$, $b_n=0$ for $n=2,3,\ldots$. Since

$$\operatorname{Re}\frac{zg'(z)}{g(z)}=(\alpha+1)\operatorname{Re}\left(\frac{1-z}{\alpha+1-\alpha z}\right)\geq q(r), \quad \text{where } q(r)=\frac{(\alpha+1)(1-r)}{\alpha+1-\alpha r}\in Q,$$

we see at once that

$$g(z)=z\left(1-\frac{\alpha z}{\alpha+1}\right)^{1/\alpha}\in S^*. \tag{14}$$

By Theorem 1, if $f(z)\in A$ and $f(z)\neq 0$ for $0<|z|<1$, and if

$$\begin{aligned}\operatorname{Re}\left\{z^2\left[\left(\frac{f'(z)}{f(z)}\right)'+\alpha\left(\frac{f'(z)}{f(z)}\right)^2\right]\right\}&\geq r^2\left[\left(\frac{q(r)}{r}\right)'+\alpha\left(\frac{q(r)}{r}\right)^2\right]\\ &=\frac{(\alpha+1)(\alpha-1-\alpha r)}{\alpha+1-\alpha r},\quad |z|=r<1,\end{aligned} \tag{15}$$

then $f\in S^*$ and

$$\operatorname{Re}\frac{zf'(z)}{f(z)}\geq q(r)=\frac{(\alpha+1)(1-r)}{\alpha+1-\alpha r}>0,\quad |z|=r<1.$$

4. A related theorem by another method. Our results so far have depended upon the Green Transform method. We shall use another method in this section to prove the following result.

THEOREM 2. *Let $f(z)\in A$ with $f(z)f'(z)\neq 0$, for $0<|z|<1$, and let β be a constant, $\frac{1}{2}\leq\beta<1$. If*

$$\left|\frac{f''(z)}{f'(z)}\right|\leq\frac{2(1-\beta)}{\beta}\left|\frac{zf'(z)}{f(z)}\right|,\quad z\in U \tag{16}$$

holds, then $f\in S^*(\beta)$ and $zf'(z)/f(z)$ is subordinate to $\beta/(\beta-(1-\beta)z)$, $z\in U$. In particular, if $\frac{3}{4}\leq\beta<1$, then $f\in K$ and $|zf''(z)/f'(z)|\leq 1$, $z\in U$. The constant $\frac{3}{4}$ cannot be replaced by a smaller value.

Proof. The function

$$W(z)=\frac{\beta}{1-\beta}\left(1-\frac{f(z)}{zf'(z)}\right)$$

is analytic for $z\in U$, since $f(z)f'(z)\neq 0$, and it omits the value $\beta/(1-\beta)$. Moreover, $W(0)=0$ and

$$\frac{zf'(z)}{f(z)}=\frac{\beta}{\beta-(1-\beta)W(z)},\qquad (zW(z))'=\frac{\beta}{1-\beta}\frac{f(z)f''(z)}{(f'(z))^2}. \tag{17}$$

From (16) and (17) we obtain $|(zW(z))'|\leq 2|z|$, from which it follows that

$$|zW(z)|=\left|\int_0^z (tW(t))'\,dt\right|\leq\int_0^{|z|}2|t|\,d|t|=|z|^2<1.$$

Since $|W(z)|\leq|z|<1$ we conclude from (17) that $zf'(z)/f(z)$ is subordinate to $\beta/(\beta-(1-\beta)z)$. For $\frac{1}{2}\leq\beta<1$ the subordination implies that $\operatorname{Re}[zf'(z)/f(z)]\geq\beta$ and $f\in S^*(\beta)$.

Next suppose that

$$\tfrac{3}{4}\leq\beta<1.$$

The subordination of $zf'(z)/f(z)$ to $\beta/(\beta-(1-\beta)z)$ implies that

$$\left|\frac{zf'(z)}{f(z)}\right| \leq \frac{\beta}{2\beta-1}.$$

Then from (16) we have

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{2(1-\beta)}{\beta}|z|\frac{\beta}{2\beta-1} \leq |z| < 1,$$

from which we conclude that $f \in K$. For the function

$$f(z) = z\left[1-\frac{(1-\beta)}{\beta}z^3\right]^{-1/3}$$

we have

$$1+\frac{zf''(z)}{f'(z)} = \frac{\beta+3(1-\beta)z^3}{\beta-(1-\beta)z^3},$$

so that $f \in K$ if $\frac{3}{4} \leq \beta < 1$. If $0 < \beta < \frac{3}{4}$ and $z = -r$, where $\beta/(3(1-\beta)) < r^3 < 1$, then

$$\operatorname{Re}\left[1+\frac{zf''(z)}{f(z)}\right] = \frac{\beta-3(1-\beta)r^3}{\beta+(1-\beta)r^3} < 0,$$

which implies $f(z) \notin K$.

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