UNIVALENT HARMONIC MAPPINGS ONTO PARALLEL SLIT DOMAINS

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Zu Dim Geburtstag wünsched mer
Dir Glück und Gsundheit, Freud und Ehr!

(For George Piranian on his 70th)

1. Introduction. Let D be any domain of $\bar{\mathbf{C}}$ that contains the point at infinity. It is well known that for each $c \in \mathbf{C} \setminus \{0\}$ there is a (univalent) conformal mapping ϕ_c of D onto the complement of horizontal slits and points, normalized by

$$\phi_c(z) = cz + o(1) \quad \text{as } z \to \infty.$$

Such mappings can be obtained by solving the linear extremal problem $\max \operatorname{Re}\{cb_1\}$ over all conformal mappings f of D with expansion

$$f(z) = cz + \frac{b_1}{z} + \cdots$$

near infinity.

Many authors [1, 2, 4, 5, 6, 7, 8] have generalized this result to univalent, canonical slit mappings satisfying the partial differential equation

(1) $$f_{\bar{z}} = \mu f_z + \nu \overline{f_z} \text{ in } D,$$

where μ and ν satisfy the uniform ellipticity condition $\sup_D(|\mu| + |\nu|) < 1$ and where D is finitely connected.

In this article D may have arbitrary connectivity, and we are interested in the equation (1) with $\mu \equiv 0$. We shall assume that ν is an anti-analytic function and $|\nu| < 1$ in D, but we shall permit $|\nu|$ to approach one at the boundary. We shall obtain horizontal slit mappings which are locally quasiconformal, harmonic mappings.

2. Existence. Let a be analytic in D and satisfy $|a| < 1$. Then diffeomorphic solutions of

(2) $$f_{\bar{z}} = \bar{a}\overline{f_z}$$

will be locally quasiconformal in D, but the distortion as measured by the dilatation quotient $(|f_z| + |f_{\bar{z}}|)/(|f_z| - |f_{\bar{z}}|) = (1 + |a|)/(1 - |a|)$ may be unbounded at the boundary. In addition, since $f_{\bar{z}z} = \bar{a}\overline{f_{z\bar{z}}}$ where $|a| < 1$, the mapping satisfies $f_{z\bar{z}} = 0$ and thus is harmonic. Conversely, each univalent, orientation-preserving, harmonic mapping f of D satisfies (2) for some analytic function a with $|a| < 1$.

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If a univalent harmonic mapping f of D leaves infinity fixed, then f has the representation (e.g., see [3, Lemma 3.1])

$$f(z)=Az+B\bar{z}+\alpha\log|z|+\sum_{n=0}^{\infty}c_nz^{-n}+\overline{\sum_{n=1}^{\infty}d_nz^{-n}}$$

in a neighborhood of infinity. Furthermore, if f satisfies (2), then $B=\overline{a(\infty)A}$ and $\alpha=2(\bar{a}_1\bar{A}+\overline{a(\infty)}a_1A)/(1-|a(\infty)|^2)$, where $a_1=\lim_{z\to\infty}z[a(z)-a(\infty)]$.

THEOREM 1. *Let D be a domain containing ∞, let a be an analytic function in D with $|a|<1$, and let $A\in\mathbf{C}\setminus\{0\}$ be constant. Set $c=(1-a(\infty))A$, and denote by ϕ_c a conformal mapping of D onto a horizontal slit domain, normalized by $\phi_c(z)=cz+o(1)$ as $z\to\infty$. Assume that* $\mathrm{Re}\{(1+a)/(1-a)\,d\phi_c\}$ *is an exact differential in $D\setminus\{\infty\}$. Then there exists a univalent solution f of* (2) *that maps D onto a horizontal slit domain and is normalized so that*

$$f(z)=Az+\overline{a(\infty)Az}+\alpha\log|z|+o(1)\quad \text{as } z\to\infty, \tag{3}$$

where $\alpha=2(\bar{a}_1\bar{A}+\overline{a(\infty)}a_1A)/(1-|a(\infty)|^2)$ *and* $a_1=\lim_{z\to\infty}z[a(z)-a(\infty)]$.

Proof. Fix $z_0\in D$. Then

$$f(z)=\int_{z_0}^{z}\mathrm{Re}\left\{\frac{1+a}{1-a}\,d\phi_c\right\}+i\,\mathrm{Im}\,\phi_c(z) \tag{4}$$

is a single-valued harmonic function on D whose partial derivatives are

$$f_{\bar{z}}=\overline{a\phi_c'/(1-a)}\quad\text{and}\quad f_z=\phi_c'/(1-a). \tag{5}$$

Thus f satisfies (2), and since the Jacobian

$$J_f=|f_z|^2-|f_{\bar{z}}|^2=(1-|a|^2)|\phi_c'|^2/|1-a|^2$$

is positive, f is locally univalent and preserves orientation. Furthermore, since $\phi_c'(z)=(1-a(\infty))A+O(z^{-2})$ as $z\to\infty$, it follows from (5) that f has the normalization (3) except for an additive constant, which we may subtract.

Next we show that f is globally univalent on D and that $f(D)$ is a horizontal slit domain. For that purpose, let $\zeta=\xi+i\eta$ belong to the horizontal slit domain $\Omega=\phi_c(D)$ and define

$$F(\zeta)=f\circ\phi_c^{-1}(\zeta)=\int_{\zeta_0}^{\zeta}\mathrm{Re}\left\{\frac{1+a\circ\phi_c^{-1}}{1-a\circ\phi_c^{-1}}\,d\zeta\right\}+i\eta,$$

where $\zeta_0=\phi_c(z_0)$. Denote by L_η the horizontal line $\{\xi+i\eta:\xi\in\mathbf{R}\}$. Then $F(\Omega\cap L_\eta)$ is contained in the same line L_η for each η.

Since $F(\infty)=\infty$ and

$$\frac{\partial}{\partial\xi}\,\mathrm{Re}\,F=\mathrm{Re}\left\{\frac{1+a\circ\phi_c^{-1}}{1-a\circ\phi_c^{-1}}\right\}>0,$$

every line L_η contained in Ω is mapped in a strictly increasing fashion onto itself. Each remaining line $L_{\hat{\eta}}$ intersects Ω in countably many open intervals. If I_1 and I_2 denote two such (possibly semi-infinite) intervals with I_1 to the left of I_2, then F

carries I_1 and I_2 each in a strictly increasing fashion into $L_{\hat{\eta}}$. It remains to show that $F(I_1)$ is entirely to the left of $F(I_2)$.

If $\zeta_1 = \xi_1 + i\hat{\eta} \in I_1$ and $\zeta_2 = \xi_2 + i\hat{\eta} \in I_2$, then since Ω is a horizontal slit domain, we can find closed intervals $J_n = [\xi_1 + i\eta_n, \xi_2 + i\eta_n]$ in Ω with $\eta_n \to \hat{\eta}$ as $n \to \infty$. Now $\operatorname{Re} F$ is increasing on each J_n and so $\operatorname{Re} F(\zeta_1) \le \operatorname{Re} F(\zeta_2)$ by continuity. Thus every point of $F(I_1)$ is to the left of every point of $F(I_2)$. Since these intervals are open, they are even disjoint.

Therefore F is univalent and $F(\Omega)$ is a horizontal slit domain. The same is true of $f = F \circ \phi_c$ and $f(D) = F(\Omega)$. □

REMARKS. (i) Theorem 1 is constructive. Except for an additive constant, a solution is given by formula (4). Other normalizations are possible. For example, by adding constants to (4) we may normalize $f(z_0) = 0$ or $f(z_0) = z_0$ for a fixed point $z_0 \in D \setminus \{\infty\}$.

(ii) The assumption that $\operatorname{Re}\{(1+a)/(1-a)\, d\phi_c\}$ is exact requires α to equal $2a_1 A/(1-a(\infty))$ and to be real, where $a_1 = \lim_{z\to\infty} z[a(z) - a(\infty)]$. If D is simple connected, then these are the only requirements.

(iii) One can obtain a normalized solution of (2) that maps D onto the complement of points and parallel slits with inclination θ. If we replace a by $e^{2i\theta}a$ and A by $e^{-i\theta}A$ in Theorem 1, then the function $e^{i\theta}f$ will have the desired properties.

3. Uniqueness. For arbitrary domains, even the conformal mappings ϕ_c (i.e., $a \equiv 0$) are not uniquely determined. Therefore we shall restrict D somewhat.

THEOREM 2. *Let D be a domain containing ∞ and having only countably many boundary components. Then a univalent solution of* (2) *that maps D onto a horizontal slit domain and has normalization* (3) *is unique.*

Proof. If f_1 and f_2 are two such mappings, then $g = f_1 - f_2$ satisfies (2), vanishes at ∞, and is uniformly bounded. Furthermore, $\operatorname{Im} g$ is constant on each boundary component of D.

If $\tilde{D}$ is any relatively compact subdomain of D, then $\sup_{\tilde{D}} |a|$ is less than one. By the similarity principle (cf. [2, Theorem 4.3]) the function g is either constant or open on $\tilde{D}$. Due to the arbitrary nature of $\tilde{D}$, the function g is either constant or open on D. If g is constant, then we are finished since g vanishes at ∞. If g is open, then $g(D)$ is a bounded domain which misses all but countably many horizontal lines. The latter is impossible. □

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