SMOOTHNESS OF INVERSE LAPLACE TRANSFORMS OF FUNCTIONS UNIVALENT IN A HALF-PLANE

Albert Baernstein II

Dedicated to George Piranian

We denote by $\mathcal{H}$ the set of functions f univalent and analytic in the right half plane $\operatorname{Re} z > 0$ which satisfy the condition

$$\limsup_{x\to\infty} x^2|f(x)| \leqq 1.$$

This class was introduced by Hayman [9], who proved that each function $f \in \mathcal{H}$ is the Laplace transform of a function $a(t)$:

$$f(z) = \int_0^\infty a(t)e^{-tz}\,dt, \quad \operatorname{Re} z > 0.$$

The "Koebe function" for $\mathcal{H}$ is $k(z) = z^{-2}$, the corresponding inverse transform being $a(t) = t$. Hayman [9, p. 6] showed that

$$(1) \qquad \int_{-\infty}^{\infty} |f(1+iy)|\,dy \leqq \int_{-\infty}^{\infty} |k(1+iy)|\,dy = \pi, \quad f \in \mathcal{H}.$$

Set $a(t) = 0$ for $t \leqq 0$. The inverse Fourier transform of $f(1+iy)$ is $a(t)e^{-t}$. Hence $a(t) \in C(\mathbf{R})$ and

$$K_0 = \sup_{f\in\mathcal{H}} |a(1)|$$

is finite. If $f \in \mathcal{H}$ and $\lambda > 0$ then $\lambda^2 f(\lambda z) \in \mathcal{H}$ and the inverse transform of $\lambda^2 f(\lambda z)$ is $\lambda a(t/\lambda)$. We deduce that

$$|a(t)| \leqq K_0 t, \quad 0 < t < \infty, \quad f \in \mathcal{H}.$$

Such "homogeneity" arguments will appear frequently in this paper.

One of the main results of [9] is the relation

$$K_0 = \lim_{n\to\infty} \frac{1}{n} \sup\{|a_n| : f \in S\},$$

where S is the usual class of functions $f(z) = z + \sum_{n=2} a_n z^n$ univalent in $|z| < 1$. Hayman's "asymptotic Bieberbach conjecture" $K_0 = 1$ remained unproved until recently, the best known estimate having been Horowitz's $K_0 \leqq 1.066$ [11]. Nehari [12], and later Bombieri [3], proved that $K_0 = 1$ implies Littlewood's conjecture $|a_n| \leqq 4n|a_0|$ for the coefficients of non-vanishing univalent functions. Conversely, Hamilton [6] showed that the truth of Littlewood's conjecture implies

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$K_0 = 1$. Since De Branges [4] has now proved Bieberbach's conjecture, it follows that the conjectures of Hayman and Littlewood are true too.

In [7] Hamilton proved that $a(t)$ is continuously differentiable on $(0, \infty)$. The purpose of this note is to give a sharper result of this kind. We shall prove that in fact a' satisfies a Lipschitz condition of order $1/2 + \eta$ on every compact sub-interval $[\epsilon, 1/\epsilon]$ of $(0, \infty)$. Here η is a positive constant which we fix at $\eta = 1/632$. We do not know what the largest possible η is.

In order to state our result precisely it is convenient to introduce a certain normalized subclass of $\mathcal{H}$.

DEFINITION. $f \in \mathcal{H}_0$ if $f \in \mathcal{H}$ and $\sup_{y \in \mathbf{R}} |f(1+iy)| = |f(1)|$.

For $f \in \mathcal{H}$ we have $\lim_{y \to \pm\infty} f(1+iy) = 0$. This follows from (1) above and (1.2) in §1. Thus the sup above is attained at some $y_0 \in \mathbf{R}$. Now $f(z+iy_0) \in \mathcal{H}$ when $f \in \mathcal{H}$ [9, p. 6]. Thus, for each $f \in \mathcal{H}$ there exists $y_0 \in \mathbf{R}$ such that $f(z+iy_0) \in \mathcal{H}_0$. If $a(t)$ is the inverse transform of $f(z)$, then the inverse transform of $f(z+iy_0)$ is $a(t)e^{-ity_0}$.

Suppose that $\alpha > 0$. Write $\alpha = N + \beta$ where N is an integer and $0 \leqq \beta < 1$. We say that a function φ belongs to $C^{\alpha}(I)$ if φ is N times continuously differentiable on the interval I and

$$|\varphi^{(N)}(t_1) - \varphi^{(N)}(t_2)| \leqq C|t_1 - t_2|^{\beta}, \quad t_1, t_2 \in I,$$

for some C.

THEOREM 1. *For* $\eta = 1/632$ *we have* $a \in C^{3/2+\eta}([\epsilon, \epsilon^{-1}])$ *for every* $\epsilon > 0$ *and* $f \in \mathcal{H}$. *Moreover, if* $f \in \mathcal{H}_0$ *and* $|f(1)| \geqq A > 0$ *then*

$$|a'(t)| \leqq C(\epsilon), \quad \epsilon \leqq t \leqq \epsilon^{-1}, \tag{2}$$

and

$$|a'(t_1) - a'(t_2)| \leqq C(A, \epsilon)|t_1 - t_2|^{1/2+\eta}, \quad t_1, t_2 \in [\epsilon, \epsilon^{-1}]. \tag{3}$$

Here $C(\epsilon)$ and $C(A, \epsilon)$ are constants depending only on the parameters shown, not necessarily the same in different occurrences. Thus the inverse transforms of functions in $\mathcal{H}_0$ for which $|f(1)|$ is uniformly bounded away from zero form a bounded set in $C^{3/2+\eta}([\epsilon, \epsilon^{-1}])$, and the derivatives of these functions are uniformly equicontinuous on compact sub-intervals of $(0, \infty)$. The examples $f(z) = (z - iy_0)^{-2}$, $a(t) = te^{ity_0}$ show that there are no good uniform Lipschitz bounds in the full class $\mathcal{H}$.

Examples in §3 show that $C(\epsilon)$ in (2) cannot be chosen independently of ϵ and that $C(A, \epsilon)$ in (3) cannot be chosen independently of either A or ϵ. Concerning (2), the proof we give of Theorem 1 yields only $|a'(t)| \leqq C(A, \epsilon)$. To get rid of the dependence on A one needs the following.

THEOREM 2. *If* $f \in \mathcal{H}_0$ *then*

$$\int_{-\infty}^{\infty} |yf'(1+iy)|\, dy \leqq C.$$

Since the inverse Fourier transform of $yf'(1+iy)$ is $i(d/dt)(te^{-t}a(t))$, (2) follows from Theorem 2. From (2) along with homogeneity considerations follows the inequality

$$||a(t_1)|-|a(t_2)||\leqq C|t_1-t_2|, \quad t_1,t_2\in(0,\infty).$$

This is the $\mathcal{H}$ version of Hayman's inequality $||a_{n+1}|-|a_n||\leqq C$ for $f\in S$ ([10]). Theorem 2 can be proved by mimicking the proof in [10] that

$$\int_{|z|=r}|z-z_1||f'(z)||dz|\leqq Cn, \quad f\in S,$$

where $r=1-1/n$ and $|f(z_1)|=\max\{|f(z)|:|z|=r\}$. Since this is only an incidental part of our paper, we shall omit the details.

Returning now to the situation of Theorem 1, we state a more general version involving powers of $f\in\mathcal{H}$. Let $\eta=1/632$ again. For $p>0$ write $2p-\frac{1}{2}+\eta=N+\beta$ where N is an integer and $0\leqq\beta<1$.

THEOREM 3. *If $p\geq\frac{1}{4}-\frac{1}{2}\eta$ then for each $f\in\mathcal{H}$ there exists $b\in C(0,\infty)$ such that*

$$f^p(z)=\int_0^\infty b(t)e^{-tz}\,dt, \quad \operatorname{Re} z>0. \tag{4}$$

The functions b satisfy

$$|b(t)|\leqq Ct^{2p-1}, \quad 0<t<\infty, \tag{5}$$

$$b\in C^{2p-1/2+\eta}([\epsilon,\epsilon^{-1}]), \quad \epsilon>0. \tag{6}$$

Moreover, if $f\in\mathcal{H}_0$ and $|f(1)|\geqq A$ then

$$|b^{(j)}(t)|\leqq C(\epsilon,p,A), \quad t\in[\epsilon,\epsilon^{-1}],\ j\leqq N, \tag{7}$$

$$|b^{(N)}(t_1)-b^{(N)}(t_2)|\leqq C(A,\epsilon,p)|t_1-t_2|^\beta, \quad t_1,t_2\in[\epsilon,\epsilon^{-1}]. \tag{8}$$

Inequality (5) is the $\mathcal{H}$ analogue of the "Littlewood-Paley" theorem in S [13, p. 131]:

$$M(r,f)=O((1-r)^{-\alpha})\Rightarrow a_n=O(n^{\alpha-1}).$$

For $\alpha>\frac{1}{2}$ this is classical and for $\alpha=\frac{1}{2}$ and slightly smaller it is proved by the author in a forthcoming paper [2]. It is known [13] to be false for $\alpha\leqq .17$. The techniques of [2] are needed to prove (5) for $\frac{1}{4}-\frac{1}{2}\eta\leqq p\leqq\frac{1}{4}$.

Probably the most interesting cases of Theorem 3 are $p=1$, $\frac{1}{2}$, and $\frac{1}{4}$. The $p=\frac{1}{2}$ case is related to the unsolved problem about whether $||b_{2n+1}|-|b_{2n-1}||\leqq Cn^{-1/2}$ for odd univalent functions in $|z|<1$ ([5, p. 177]).

In §1 we prove some lemmas which easily give the proof of Theorem 1 in §2. The proof of Theorem 3 is similar to that of Theorem 1, and we will say just a little bit about it. Then in §3 we give the examples already mentioned in connection with Theorem 1.

1. Preliminary results. For any univalent functions g in $\operatorname{Re} z>0$ and any two points z_1,z_2 in the right half plane with $|z_1-z_2|\leqq(1-\epsilon)\operatorname{Re} z_1$ $(\epsilon>0)$ we have

(1.1) $$C(\epsilon)^{-1}|g'(z_2)| \leqq |g'(z_1)| \leqq C(\epsilon)|g'(z_2)|.$$

To prove this, note that it suffices to consider $z_1 = 1$. (Look at $g_1(z) = g(\lambda z + i\mu)$ for suitable $\lambda > 0$ and $\mu \in \mathbf{R}$.) Then define h in $|z|<1$ by $h(z) = g((1-z)/(1+z))$. This function satisfies $|h''(z)| \leqq C(1-|z|^2)^{-1}|h'(z)|$, [13, p. 21, Lemma 1.3], and (1.1) follows by integration and change of variable.

Now assume also that g is zero free. If z_1, z_2 are as in (1.1) then

(1.2) $$C(\epsilon)^{-1}|g(z_2)| \leqq |g(z_1)| \leqq C(\epsilon)|g(z_2)|.$$

This is proved the same way, making use of the inequality

(1.3) $$|h'(z)| \leqq 4(1-|z|^2)^{-1}|h(z)|,$$

valid for zero free univalent functions in $|z|<1$ ([8, p. 95]). (1.3) also shows that

(1.4) $$(\operatorname{Re} z)|g'(z)| \leqq 2|g(z)|, \quad \operatorname{Re} z > 0.$$

Integration of (1.4) gives

$$x_1^2|g(x_1)| \leqq x_2^2|g(x_2)|, \quad 0 < x_1 < x_2 < \infty.$$

If $f \in \mathcal{H}$ then $x^2|f(x)| \leqq 1$ [9, p. 6, (c)]. Also, f is zero free in $\operatorname{Re} z > 0$, since $\lim_{x\to\infty} f(x) = 0$. Thus, if $f \in \mathcal{H}$,

(1.5) $$x_1^2|f(x_1)| \leqq x_2^2|f(x_2)| \leqq 1, \quad 0 < x_1 < x_2 < \infty.$$

Our first lemma gives a Hardy–Littlewood type of sufficient condition for a function to be Lipschitz.

LEMMA 1. *Suppose that* $H \in L^1(\mathbf{R})$, *that* $h(y) = \int_{-\infty}^{\infty} H(t)e^{-ity}\,dt$, *and that for some* $\alpha > 0$, $\alpha \notin \mathbf{Z}$,

(1.6) $$\int_{s \leqq |y| \leqq 2s} |h(y)|\,dy \leqq s^{-\alpha}, \quad s \geqq 1,$$

and

(1.7) $$\int_{-1}^{1} |h(y)|\,dy \leqq 1.$$

Then $H \in C^\alpha(\mathbf{R})$ *and*

(1.8) $$\|H^{(j)}\|_\infty \leqq C(\alpha), \quad j = 0, 1, \ldots, N,$$

(1.9) $$|H^{(N)}(t_1) - H^{(N)}(t_2)| \leqq C(\alpha)|t_1 - t_2|^\beta, \quad t_1, t_2 \in \mathbf{R}.$$

Here $\alpha = N + \beta$ *with* $N \in \mathbf{Z}$ *and* $0 < \beta < 1$.

Proof. For $j = 0, \ldots, N$ we have

$$\int_{-\infty}^{\infty} |y|^j |h(y)|\,dy \leqq \int_{-1}^{1} |h(y)|\,dy + \sum_{k=0}^{\infty} \int_{2^k \leqq |y| \leqq 2^{k+1}} |y|^j |h(y)|\,dy \leqq C(\alpha),$$

which implies (1.8).

Let $u(t+i\sigma)$, $\sigma > 0$, be the Poisson integral of $H^{(N)}$. We will show that

(1.10) $$\left|\frac{\partial u}{\partial \sigma}(t+i\sigma)\right| \leqq C(\alpha)\sigma^{\beta-1}, \quad \sigma > 0.$$

Then (1.9) follows from [14, p. 142, Proposition 7].

Write $P_\sigma(t)=(1/\pi)\sigma/(t^2+\sigma^2)$. Then

$$\frac{\partial u}{\partial \sigma}(t+i\sigma)=\int_{-\infty}^{\infty}\frac{\partial P_\sigma}{\partial \sigma}(\tau)H^{(N)}(t-\tau)\,d\tau.$$

From [14, p. 142, (50)] and (1.8) we have

$$\left|\frac{\partial u}{\partial \sigma}(t+i\sigma)\right|\leqq \|H^{(N)}\|_\infty C\sigma^{-1}\leqq C(\alpha)\sigma^{-1},\quad \sigma>0. \tag{1.11}$$

Next, the Fourier transform of $(\partial u/\partial\sigma)(t+i\sigma)$ is

$$\frac{\partial}{\partial\sigma}((iy)^N h(y)e^{-\sigma|y|})=-|y|(iy)^N h(y)e^{-\sigma|y|}.$$

Fix $\sigma\in(0,1)$, and denote the function on the right by $h_1(y)$. Let m be the integer with $2^{-m-1}<\sigma\leqq 2^{-m}$. By (1.6) we have, for $k\geqq 0$,

$$\int_{2^k\leqq|y|\leqq 2^{k+1}}|h_1(y)|\,dy\leqq 2^{(N+1)k}e^{-\sigma 2^k}2^{-k\alpha}=e^{-\sigma 2^k}2^{k(1-\beta)}.$$

Hence

$$\int_{|y|>1}|h_1(y)|\,dy\leqq \sum_{k=0}^{m}2^{k(1-\beta)}+\sum_{k=m+1}^{\infty}e^{-2^{k-m-1}}2^{k(1-\beta)}$$
$$\leqq C(\alpha)\sigma^{\beta-1},\qquad 0<\sigma<1.$$

By (1.7) the same inequality holds for integration over $(-\infty,\infty)$. This inequality together with (1.11) for $\sigma\geqq 1$ yields (1.10). □

The next lemma is essentially due to Hayman (cf. [9, p. 11]).

LEMMA 2. *If $f\in\mathcal{H}_0$ then*

$$|f(1+iy)|\leqq y^{-2}\exp\left\{(C+\log^+|y|)^{1/2}\left(C+4\log\frac{1}{|f(1)|}\right)^{1/2}\right\}. \tag{1.12}$$

Proof. By (1.2), it suffices to prove this with $|f(1+4iy)|$ instead of $|f(1+iy)|$. By (1.5) we may assume $y>1$.

Fix $y>1$ and let $R_2=|f(1+4iy)|$. Since $|f(1)|\geqq|f(1+4iy)|$, there exists $s\in[1,\infty)$ such that $|f(s)|=R_2$. If $s\geqq y$ then by (1.5)

$$|f(1+4iy)|=|f(s)|\leqq s^{-2}\leqq y^{-2},$$

and (1.12) holds.

Assume $1\leqq s\leqq y$. The circles centered at y and $y+4iy$ with radius y do not intersect, and contain the points s and $1+4iy$ respectively. Define $R_1=y^{-2}$. Then $R_1\geqq\max(|f(y)|,|f(y+4iy)|)$, by (1.5) and the fact that $f(z+iy_0)\in\mathcal{H}$ for $y_0\in\mathbf{R}$ ([8, p. 6]). If $R_2\leqq e^2R_1$ then (1.12) holds. If $R_2>e^2R_1$ then, since f is univalent and zero free in $\operatorname{Re}z>0$, Theorem 2.6 of [8] asserts that

$$\left(C+\log\frac{y}{s}\right)^{-1}+(C+\log y)^{-1}\leqq 2\left(\log\frac{R_2}{R_1}-1\right)^{-1}. \tag{1.13}$$

Here $C>0$. Since $R_2=|f(s)|\geqq s^{-2}|f(1)|$ (by (1.5)) and $R_1=y^{-2}$, we have $R_2\geqq R_1|f(1)|(y/s)^2$, so that

$$\log\frac{y}{s}\leqq\frac{1}{2}\left(\log\frac{1}{|f(1)|}+\log\frac{R_2}{R_1}\right).$$

Now $|f(1)|\leqq 1$ and $R_2\geqq e^2R_1$, so from (1.13) it follows that

$$(C+\log y)^{-1}\leqq 2\left(\log\frac{R_2}{R_1}-1\right)^{-1}-2\left(2C+\log\frac{1}{|f(1)|}+\log\frac{R_2}{R_1}\right)^{-1}$$
$$\leqq\left(8C+4\log\frac{1}{|f(1)|}+4\right)\left(\log\frac{R_2}{R_1}\right)^{-2}.$$

Hence

$$\log\frac{R_2}{R_1}\leqq(C+\log y)^{1/2}\left(C_1+4\log\frac{1}{|f(1)|}\right)^{1/2},$$

which gives (1.12). □

The next lemma is a slight generalization of an inequality of Clunie and Pommerenke (see [13, p. 132]). Suppose that f is univalent in $|z|<1$ and that $E\subset[0,2\pi]$. For fixed $r\in(0,1)$ define $M=\sup_E|f(re^{i\theta})|$, and for $\delta\in(0,1/40]$ define $\kappa>0$ by

$$\frac{1}{2}-\kappa=\frac{1-\delta+10\delta^2}{2-\delta}.$$

LEMMA 3. *With the situation specified above,*

$$\int_0^{2\pi}|f'(re^{i\theta})|\,d\theta\leqq C(\delta)|f'(0)|^{\delta/(2-\delta)}M^{(2-2\delta)/(2-\delta)}(1-r)^{-1/2+\kappa}.$$

Proof. It follows from Lemma 5.4 of [13, p. 129] that

$$(1.14)\qquad \int_0^{2\pi}|f'(re^{i\theta})|^\delta\,d\theta\leqq C(\delta)|f'(0)|^\delta(1-r)^{-10\delta^2}.$$

Also, Lemma 5.2 of [13] gives

$$(1.15)\qquad \int_E|f'(re^{i\theta})|^2\,d\theta\leqq C(1-r)^{-1}M^2.$$

That lemma is stated for $f\in S$, but for the special case considered here the proof there is valid for all univalent functions. For $B>0$ let

$$E_1=\{\theta\in E\colon |f'(re^{i\theta})|\leqq B\},\quad E_2=E\backslash E_1.$$

Then

$$\int_E|f'(re^{i\theta})|\,d\theta\leqq B^{1-\delta}\int_{E_1}|f'(re^{i\theta})|^\delta\,d\theta+B^{-1}\int_{E_2}|f'(re^{i\theta})|^2\,d\theta.$$

Choosing

$$B=\left(\int_E |f'(re^{i\theta})|^\delta\, d\theta\right)^{-1/(2-\delta)}\left(\int_E |f'(re^{i\theta})|^2\, d\theta\right)^{1/(2-\delta)}$$

and using (1.14) and (1.15), we obtain Lemma 3. □

Finally, we need a localized version of Lemma 3 for functions defined in a half plane. Let δ and κ have the same meaning as in Lemma 3.

LEMMA 4. *Suppose that f is univalent and zero free in* $\operatorname{Re} z>0$, *that $s>1$, and write* $M=\sup\{|f(1+iy)| : s\leqq y\leqq 2s\}$. *Then*

$$\int_s^{2s} |f'(1+iy)|\, dy \leqq C(\delta)|f(s)|^{\delta/(2-\delta)}M^{(2-2\delta)/(2-\delta)}s^{1/2-\kappa}. \tag{1.16}$$

Proof. Define $g(z)$ in $\operatorname{Re} z>0$ and $h(z)$ in $|z|<1$ by

$$g(z)=f(sz+is), \qquad g(z)=h\left(\frac{1-z}{1+z}\right),$$

and define $r\in(0,1)$ by $s^{-1}=(1-r)(1+r)^{-1}$. Let E be a subarc of the lower quarter-circle $re^{i\theta}$, $-\pi/2\leqq\theta\leqq 0$, to be specified later, and let E_1 be its image in $\operatorname{Re} z>0$ under the mapping $z\to(1-z)^{-1}(1+z)$. From Lemma 3 applied to h we deduce

$$\int_{E_1} |g'(z)||dz| \leqq C(\delta)|g'(1)|^{\delta/(2-\delta)}M_1^{(2-2\delta)/(2-\delta)}s^{1/2-\kappa} \tag{1.17}$$

where $M_1=\sup\{|g(z)| : z\in E_1\}$. Now E_1 is a subset of the circular arc running from $z=s^{-1}$ to $z=(1+r^2)^{-1}((1-r^2)+2ri)$ with increasing real and imaginary parts. Let E_2 denote the projection of E_1 onto the line $\operatorname{Re} z=s^{-1}$. Using (1.1), it is easily seen that

$$\int_{E_2} |g'(z)||dz| \leqq C\int_{E_1} |g'(z)||dz|. \tag{1.18}$$

If $s\leqq 3$ then (1.16) follows from (1.1), (1.2), and (1.4). Assume $s>3$. Then $r>1/2$ and $2r(1+r^2)^{-1}>4/5$. Choose E so that E_2 is the line segment from s^{-1} to $s^{-1}+i/2$. By (1.17), (1.18), (1.2), and (1.4)

$$\int_0^{1/2}\left|g'\left(\frac{1}{s}+iy\right)\right| dy \leqq C(\delta)|g(1)|^{\delta/(2-\delta)}M_2^{(2-2\delta)/(2-\delta)}s^{1/2-\kappa},$$

where $M_2=\sup\{|g(z)| : z\in E_2\}$. A change of variables gives

$$\int_s^{(3/2)s} |f'(1+iy)|\, dy \leqq C(\delta)|f(s+is)|^{\delta/(2-\delta)}M_3^{(2-2\delta)/(2-\delta)}s^{1/2-\kappa}$$

where $M_3=\sup\{|f(1+iy)| : s\leqq y\leqq \frac{3}{2}s\}$.

By (1.2), we can replace $|f(s+is)|$ by $|f(s)|$. Replacing s by $\frac{3}{2}s$ and then adding the integrals on the left, we obtain (1.16). □

2. Proof of Theorem 1. Take $\delta=1/40$ in Lemma 3. Then $\kappa=1/316$. Define $\eta=\frac{1}{2}\kappa=1/632$. Suppose $f\in\mathcal{H}_0$. By Lemma 2, $|f(1+iy)|\leqq C(A)|y|^{\eta-2}$.

Applying Lemma 4, we obtain for $s \geqq 1$

$$\int_s^{2s} |f'(1+iy)|\,dy \leqq C(A)\,|f(s)|^{\delta/(2-\delta)}(s^{\eta-2})^{(2-2\delta)/(2-\delta)}s^{1/2-\kappa}.$$

Since $|f(s)| \leqq s^{-2}$, by (1.5),

$$\int_s^{2s} |f'(1+iy)|\,dy \leqq C(A)s^{-3/2-\eta}, \quad s \geqq 1. \tag{2.1}$$

The same inequality holds for integration over $[-2s, -s]$. Also, by (1.1), (1.4), and (1.5), $\int_{-1}^{1} |f'(1+iy)|\,dy \leqq C$.

It follows from Lemma 1 that $te^{-t}a(t) \in C^{3/2+\eta}(\mathbf{R})$. Calculus shows that $a \in C^{3/2+\eta}([\epsilon, \epsilon^{-1}])$. The bounds (2) and (3) follow from (1.8) and (1.9). □

For general $f \in \mathcal{H}$ the conclusion $f \in C^{3/2+\eta}$ is obtained by consideration of $f(z+iy_0)$ for appropriate y_0.

Concerning Theorem 3, the smoothness statements are proved like those in Theorem 1, starting from the inequality

$$\int_s^{2s} |g'(1+iy)|\,dy \leqq C(A,p)s^{-(2p-1/2+\eta)}, \quad s \geqq 1. \tag{2.2}$$

Here $g(z) = f(z)^p$ and $f \in \mathcal{H}_0$. For $p \leqq 1$, (2.2) is obtained by replacing f by g in the argument used to prove (2.1). For $p > 1$ it follows from $g' = pf^{p-1}f'$, together with (2.1) (with η replaced by some $\eta_1 > \eta$) and Lemma 2. In the deduction of (6) from (2.2) there is a minor nuisance caused by the case $2p - \frac{1}{2} + \eta \in \mathbf{Z}$. To avoid this, note that (2.2) still holds for some $\eta_1 > \eta$ and then use the fact that $C^{\alpha}(I) \subset C^{\alpha'}(I)$ if $\alpha' < \alpha$ and I is compact.

For $p > \frac{1}{2}$ we have, for $f \in \mathcal{H}$,

$$\int_{-\infty}^{\infty} |f(1+iy)|^p\,dy \leqq \int_{-\infty}^{\infty} \frac{dy}{(1+y^2)^p} \leqq C(p), \tag{2.3}$$

where $C(p)$ is independent of p for $p \geqq \frac{3}{5}$, say. This inequality follows from Theorem 1 in [1] together with Hayman's argument to prove Theorem 2 in [9]. Homogeneity gives appropriate bounds for integrals on the line $\operatorname{Re} z = x$; existence of $b(t)$, along with the bound (5), is proved like that of $a(t)$ in the case $p = 1$.

For small p we use instead of (2.3)

$$\int_{-\infty}^{\infty} |g'(1+iy)|\,dy \leqq C, \quad \tfrac{1}{4} - \tfrac{1}{2}\eta \leqq p \leqq 1. \tag{2.4}$$

This and homogeneity show that there exists $B(t)$ such that $|B(t)| \leqq Ct^{2p}$ and

$$g'(z) = \int_0^{\infty} e^{-tz}B(t)\,dt.$$

Then $b(t) = -t^{-1}B(t)$ is the function of Theorem 3.

For $p > \frac{1}{4}$, (2.4) can be obtained by adapting the argument used to prove Theorem 5.3 of [13], but the constant on the right obtained by this method blows up as $p \to \frac{1}{4}$. In [2] we explain how to prove (2.4) on the range indicated with an absolute constant. From (2.2), which we have proved in this paper, it follows that $g'(1+iy) \in L^1(\mathbf{R})$ for $p \geqq \frac{1}{4} - \frac{1}{2}\eta$, but the bound on the norm blows up when $\max_y |f(1+iy)| \to 0$.

3. Examples. Our first family of examples will show that the constant $C(A, \epsilon)$ of (3) in Theorem 1 cannot be chosen independently of A, no matter how small ϵ is. For $s>0$ define $g(z) = (z^2+s^2)^{-1}$. These functions belong to $\mathcal{H}$. For $s>1$ the maximum of $|g(1+iy)|$ is attained at $\pm y_0$, where $y_0^2 = s^2-1$. Thus $f(z) = g(z+iy_0)$ belongs to $\mathcal{H}_0$. The inverse transform of f is

$$a(t) = e^{-iy_0 t} \frac{e^{ist} - e^{-ist}}{2is},$$

so that

$$a'(t) = \tfrac{1}{2}(1+y_0 s^{-1}) e^{-i(s+y_0)t} + O(s^{-1}), \quad s \to \infty.$$

Let $t_1 = 1$ and $t_2 = \pi(s+y_0)^{-1}$. Then, as $s \to \infty$, $a'(t_1) \to 1$, $a'(t_2) \to -1$, and $t_2 - t_1 \to 0$. Thus the collection of functions $\{a' : s > 1\}$ is not equicontinuous on any neighborhood of $t=1$, and the constant $C(A, \epsilon)$ depends essentially on A.

The second family of examples will show that $C(A, \epsilon)$ depends essentially on ϵ. Let $\Omega = \mathbf{C} \backslash \Gamma$, where Γ is the union of the negative axis and a small symmetric circular arc $\{w = \mathrm{Re}^{i\varphi} : |\varphi - \pi| \leqq \tau\}$, and let g be a conformal mapping from $\operatorname{Re} z > 0$ onto Ω with $g(0) = \infty$, $g(\infty) = 0$. Then g has a double zero at $z = \infty$, a double pole at $z = 0$, and $g(z) = \bar{g}(\bar{z})$. Multiplying g by a suitable positive constant, we may assume $g \in \mathcal{H}$.

Now $|g(iy)|$ is a symmetric function on $(-\infty, \infty)$ which is non-increasing on $(0, \infty)$. It is not hard to show that $\log|g(z)|$ is the Poisson integral of $\log|g(iy)|$, and it follows from this that $|g(x+iy)|$ is a symmetric non-increasing function of y for every $x > 0$. Thus, the functions f defined by $f(z) = \lambda^2 g(\lambda z)$, $\lambda > 0$, all belong to $\mathcal{H}_0$.

Let $b(t)$ denote the inverse transform of g. If b' were a constant C on $(0, \infty)$ then, since $b(t) = O(t)$, it would follow that $b(t) = Ct$, and hence that $g(z) = cz^{-2}$, which is false. Thus there exist $s_1, s_2 \in (0, \infty)$ such that $|b'(s_1) - b'(s_2)| = \delta > 0$.

Denoting by $a(t)$ the inverse transform of f, we have $a'(t) = b'(\lambda^{-1} t)$, so that $|a'(\lambda s_1) - a'(\lambda s_2)| = \delta$. Letting $\lambda \to 0$, we see that the collection of functions $\{a' : 0 < \lambda \leqq 1\}$ is not equicontinuous on $(0, 1]$, even though $|f(1)| = \lambda^2 |g(\lambda)|$ tends to a positive limit as $\lambda \to 0$.

The last example will be an $f \in \mathcal{H}_0$ for which a' is not uniformly continuous on $(1, \infty)$ and $a' \notin L^\infty(1, \infty)$. Thus the interval $[\epsilon, \epsilon^{-1}]$ in (2) and (3) of Theorem 1 cannot be replaced by $[\epsilon, \infty)$. Let $\Omega = \mathbf{C} \backslash \Gamma$, where Γ is a polygonal path from $w = 0$ to $w = \infty$ with just one small bend. Let g be a conformal mapping from $\operatorname{Re} z > 0$ onto Ω with $g(0) = \infty$, $g(\infty) = 0$. Multiplying g by a suitable complex constant, we may achieve also $g(1) > 0$ and $g \in \mathcal{H}$.

There exists $s > 0$ such that $g'(s)/g(s) \notin \mathbf{R}$. To see this, observe that if $g'(x)/g(x)$ were everywhere real then so would be $\log g(x) - \log g(1)$, and hence

$g(x)$ would be real. Then Ω would be symmetric with respect to the real axis, which is impossible since Γ has a bend.

Fix an $s>0$ with $g'(s)/g(s)\notin \mathbf{R}$. Then

$$\frac{\partial}{\partial y}\log|g(s+iy)| = -\operatorname{Im}\frac{g'(s+iy)}{g(s+iy)}$$

does not vanish at $y=0$, and hence the maximum of $\log|g(s+iy)|$ occurs at some $y_0\neq 0$. Define $f(z)=s^2g(sz+iy_0)$. Then $f\in\mathcal{K}_0$.

Denote the inverse transforms of f and g by a and b respectively. Let $q=s^{-1}y_0$. Then $a(t)=sb(ts^{-1})e^{-iqt}$.

I claim that

(3.1) $$\lim_{t\to\infty} t^{-1}b(t)=\rho \text{ exists and is non-zero.}$$

Assume this for the moment. Then we can write

$$a(t)=\rho t e^{-iqt}\lambda(t),$$

where $\lim_{t\to\infty}\lambda(t)=1$. Define $t_n=2\pi nq^{-1}$, $t_n'=(2n+1)\pi q^{-1}$. Then $a(t_n)=\rho t_n\lambda(t_n)$, $a(t_n')=-\rho t_n'\lambda(t_n')$. If a' were uniformly continuous on $(1,\infty)$ there would exist a constant C_1 such that $|a'(t)-a'(t')|\leqq C_1$ whenever $t,t'\in[t_n,t_{n+1}]$, $n=1,2,3,\ldots$. We leave it to the reader to show that this is incompatible with the previously established relations $a(t_n)\sim\rho_1 n$, $a(t_n')\sim-\rho_1 n$ $(\rho_1=2\pi\rho q^{-1})$. These relations show also that $a'\notin L^\infty(1,\infty)$.

It remains to prove (3.1). By the Schwarz–Christoffel formula

$$g'(z)=Az^{-3}\left(\frac{z-iy_1}{z+iy_2}\right)^\gamma$$

where $A\in\mathbf{C}$, $y_1,y_2>0$, and γ is a real number with small absolute value. Thus, for $t>0$,

(3.2) $$-tb(t)=\frac{A}{2\pi i}\int_L e^{tz}z^{-3}\left(\frac{z-iy_1}{z+iy_2}\right)^\gamma dz,$$

where L is the vertical line from $x-i\infty$ to $x+i\infty$, x denoting any positive number. Deform L so that it becomes coincident with the imaginary axis except for a small semicircle centered at 0 and traced counterclockwise. Let L_1 denote the same path as L, except that the semicircle is in the left half plane and traced clockwise. From the dominated convergence theorem and Riemann–Lebesgue lemma it follows that

(3.3) $$\lim_{t\to\infty}\int_{L_1} e^{tz}z^{-3}\left(\frac{z-iy_1}{z+iy_2}\right)^\gamma dz=0.$$

Now $((z-iy_1)/(z+iy_2))^\gamma$ is analytic in the plane with the slits $\{iy: y\geqq y_1\}$ and $\{iy: y\leqq -y_2\}$ removed. Let $\sum_{n=0}^\infty c_nz^n$ be its power series expansion near the origin. Then

$$\frac{1}{2\pi i}\int_L - \int_{L_1} e^{tz}z^{-3}\left(\frac{z-iy_1}{z+iy_2}\right)^\gamma dz=\frac{1}{2}t^2c_0+tc_1+c_2.$$

With (3.2) and (3.3), this shows that

$$b(t) \sim -\tfrac{1}{2} A c_0 t, \quad t \to \infty.$$

Since $A \neq 0$ and $c_0 = (-y_1/y_2)^\gamma \neq 0$, we have (3.1).

We do not know any examples of $f \in \mathcal{H}_0$ for which a' is not uniformly continuous on $(0,1)$, or even for which $a' \notin L^\infty(0,1)$.

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Department of Mathematics
Washington University
St. Louis, Missouri 63130