ON THE RADIAL LIMITS OF FUNCTIONS WITH HADAMARD GAPS

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To George Piranian, on the occasion of his retirement

1. Introduction and results. We consider functions f with *Hadamard gaps,* i.e.

$$f(z)=\sum_{k=0}^{\infty} a_k z^{n_k}, \quad \frac{n_{k+1}}{n_k} \geq \lambda > 1 \quad (k=0,1,\ldots), \tag{1.1}$$

that are analytic in the unit disk $\mathbf{D}$. Let

$$M(r)=\max_{|z|=r}|f(z)| \quad (0 \leq r<1) \tag{1.2}$$

and let $\dim E$ denote the Hausdorff dimension, i.e.

$$\dim E=\inf\{\delta : E \text{ has } \delta\text{-dimensional Hausdorff measure } 0\}.$$

It is clear that $0 \leq \dim E \leq 1$ for $E \subset \partial\mathbf{D}$.

If (a_k) is bounded then f is a normal function. Hence angular limits, radial limits and asymptotic values are the same by the Lehto–Virtanen theorem [14, p. 268]. On the other hand, if (a_k) is unbounded then f is not a normal function [15], and Murai [13] (see also [6]) has proved that f has the asymptotic value ∞ at every point of $\partial\mathbf{D}$.

We shall consider the radial behaviour at points ζ of $\partial\mathbf{D}$. If $\sum |a_k|=\infty$ then

$$\operatorname{Re} f(r\zeta) \to +\infty \quad \text{as } r \to 1-0 \tag{1.3}$$

holds on a set E with $\dim E>0$ if $\lambda>3$ and with $\dim E=1$ if $n_{k+1}/n_k \to \infty$; see MacLane [11] and Hawkes [7, p. 28].

On the other hand, Csordas, Lohwater and Ramsey [5] have shown that, for any $\lambda>1$,

$$\sum_k |a_k|=\infty, \quad (a_k) \text{ bounded} \tag{1.4}$$

implies that (1.3) holds on a set E of positive capacity which also has positive Hausdorff dimension. Their proof is based on results of Kahane, Weiss and Weiss [9], and the same is true of the following generalization.

THEOREM 1. *For $\lambda>1$, there are positive numbers α, β, γ with the following property: If f has the form* (1.1) *and if*

$$\sum_k |a_k|=\infty, \quad \frac{|a_k|}{|a_0|+\cdots+|a_k|} \leq \alpha \quad (k \geq l), \tag{1.5}$$

then there is a closed set $E \subset \partial\mathbf{D}$ with $\dim E \geq \beta$ such that

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$$\operatorname{Re} f(r\zeta) \geq \gamma M(r) \quad \textit{for } r_0 \leq r < 1, \ \zeta \in E. \tag{1.6}$$

Note that, by Sidon's theorem [16, p. 247],

$$\sum_k |a_k| = \infty \Leftrightarrow M(r) \to \infty \quad (r \to 1-0). \tag{1.7}$$

We shall also prove the following result which shows, in particular, that $\operatorname{Re} f(z)$ has the angular limit $+\infty$ on E under the assumption (1.4).

THEOREM 2. *Let f satisfy* (1.1) *and let*

$$\sum_k |a_k| = \infty, \quad \frac{|a_k|}{|a_0| + \cdots + |a_k|} \to 0 \quad (k \to \infty). \tag{1.8}$$

Then there is a set $E \subset \partial\mathbf{D}$ *with* $\dim E \geq \beta > 0$ *such that,*

$$\operatorname{Re} f(z) \geq \gamma M(|z|) \quad \textit{for } \zeta \in E, \ z \in \zeta\Delta, \ r_0 < |z| < 1 \tag{1.9}$$

for any Stolz angle Δ *at* 1 *and* $r_0 = r_0(\Delta)$. *The constants* β *and* γ *depend only on* λ.

Hawkes [7, p. 32] has proved that, if $n_k = 2^k$ and

$$a_k > 0, \quad \frac{a_{k+1}}{a_k} \to 1, \quad \frac{a_k^{1+\delta}}{a_0 + \cdots + a_k} \to 0 \quad \text{as } k \to \infty \tag{1.10}$$

for some $\delta > 0$, then

$$\lim_{r \to 1-0} \frac{f(r\zeta)}{M(r)} \neq 0 \text{ exists for } \zeta \in E \tag{1.11}$$

where E is a set with $\dim E = 1$. The methods used to prove the above theorems seem to yield only a set with $\dim E > 0$, but not with $\dim E = 1$.

The next theorem shows, however, that either some condition on the exponents (like $\lambda > 3$) or some condition on the coefficients (like (1.5)) is necessary for any of the above assertions to hold for any ζ.

THEOREM 3. *There exists a function f of the form* (1.1) *with* $\lambda = 33/32$ *such that, for every* $\zeta \in \partial\mathbf{D}$,

$$\liminf_{r \to 1} \operatorname{Re} f(r\zeta) = -\infty, \quad \limsup_{r \to 1} \operatorname{Re} f(r\zeta) = +\infty, \tag{1.12}$$

$$\liminf_{r \to 1} \operatorname{Im} f(r\zeta) = -\infty, \quad \limsup_{r \to 1} \operatorname{Im} f(r\zeta) = +\infty. \tag{1.13}$$

Our last result is connected with the following conjecture of Anderson [1]: If g is analytic and univalent in $\mathbf{D}$ then there exists $\zeta \in \partial\mathbf{D}$ such that

$$\int_0^1 |g''(r\zeta)| \, dr < \infty. \tag{1.14}$$

THEOREM 4. *Let g be analytic and univalent in* $\mathbf{D}$. *If*

$$\log g'(z) = \sum_{k=0}^{\infty} a_k z^{n_k}, \quad \frac{n_{k+1}}{n_k} \geq \lambda > 1 \quad (k = 0, 1, \ldots), \tag{1.15}$$

then there is a set $E \subset \partial\mathbf{D}$ *with* $\dim E \geq \beta > 0$ *such that* (1.14) *holds for all* $\zeta \in E$.

Univalent functions for which $\log g'$ has Hadamard gaps are often useful as counter-examples; see for example [10, p. 274] and [14, p. 304]. Hence Theorem 4 makes Anderson's conjecture more plausible.

In the final section, we discuss some open problems about radial limits.

2. Some lemmas. The following lemmas will be used to prove Theorems 1 and 2. Let $|J|$ denote the length of the arc $J\subset\partial\mathbf{D}$.

LEMMA 1 (Kahane, Weiss, Weiss [9, p. 6]). *For $\lambda>1$ there are positive constants δ and γ with the following property: If*

$$g(z)=\sum_{k=j}^{j'}a_k z^{n_k},\quad \frac{n_{k+1}}{n_k}\geq\lambda, \tag{2.1}$$

then every arc $J\subset\partial\mathbf{D}$ with $|J|\geq\delta/n_j$ contains a subarc J' with $|J'|\geq 2\gamma/n_{j'}$ such that

$$\operatorname{Re} g(\zeta)\geq 4\gamma\sum_{k=j}^{j'}|a_k|\quad \text{for } \zeta\in J'. \tag{2.2}$$

The next lemma is also due to Kahane, Weiss and Weiss [9, p. 17]. Our formulation is somewhat different and makes its structure perhaps clearer. We therefore present a proof.

LEMMA 2. *Let $s=1,2,\ldots$ and $m=3,4,\ldots$ and let*

$$S_\nu=\{k\in\mathbf{N}: \nu sm-(m-1)s<k\leq\nu sm\}\quad(\nu=0,1,\ldots). \tag{2.3}$$

Let k_ν denote an integer in S_ν for which the sum $\sum_{j=k-s}^{k-1}|a_j|$ assumes its minimal value A_ν^* and write

$$A_\nu=\sum_{j=k_\nu}^{k_{\nu+1}-s-1}|a_j|\quad(\nu=0,1,\ldots). \tag{2.4}$$

Then

$$A_\nu^*=\sum_{j=k_\nu-s}^{k_\nu-1}|a_j|\leq(A_{\nu-1}+A_\nu)/(m-2)\quad(\nu=1,2,\ldots). \tag{2.5}$$

Proof. It follows from the definition that

$$(m-1)sA_\nu^*\leq\sum_{k\in S_\nu}\sum_{j=k-s}^{k-1}|a_j|\leq s\sum_{j=\nu sm-ms}^{\nu sm-1}|a_j|.$$

Since $k_{\nu-1}\leq(\nu-1)sm$ and $k_{\nu+1}-s-1>\nu sm-1$ by (2.3), we obtain, after division by s, that

$$(m-1)A_\nu^*\leq A_{\nu-1}+A_\nu^*+A_\nu$$

because of (2.4). This proves our assertion (2.5). □

LEMMA 3 (Beardon [4, p. 683]). *For $\nu=1,2,\ldots,$ let*

$$E_\nu=\bigcup_{j_1,\ldots,j_\nu=1,2}I_{j_1\ldots j_\nu}, \tag{2.6}$$

where $I_{j_1\dots j_\nu}$ $(j_1,\dots,j_\nu=1,2)$ *are disjoint arcs on* $\partial\mathbf{D}$ *such that, for* $j=1,2,$

$$I_{j_1\dots j_\nu j}\subset I_{j_1\dots j_\nu},\quad |I_{j_1\dots j_\nu j}|\geq q|I_{j_1\dots j_\nu}| \tag{2.7}$$

$$\operatorname{dist}(I_{j_1\dots j_\nu 1},I_{j_1\dots j_\nu 2})\geq c|I_{j_1\dots j_\nu}|, \tag{2.8}$$

where q *and* c *are constants with* $0<q<1$ *and* $c>0$. *Then*

$$\dim\left(\bigcap_{\nu=1}^{\infty}E_\nu\right)\geq\frac{\log 2}{\log(1/q)} \tag{2.9}$$

The final lemma connects partial sums and radial limits.

LEMMA 4. *Let* f *be of the form* (1.1) *and let*

$$\tau_k=\sum_{j=0}^{k}|a_j|,\quad |a_k|\leq\alpha\tau_k\quad(k\geq h) \tag{2.10}$$

where $0<\alpha\leq 1/2$. *If* $\zeta\in\partial\mathbf{D}$ *and* $1-1/n_k\leq r\leq 1-1/n_{k+1}$, *then*

$$\left|f(r\zeta)-\sum_{j=0}^{k}a_j\zeta^{n_j}\right|\leq\tau_h+\alpha K_1\tau_k\quad(k\geq h) \tag{2.11}$$

$$M(r)\leq(1+\alpha K_2)\tau_k\quad(k\geq h) \tag{2.12}$$

where the constants K_1,K_2,K_3 *depend only on* λ; *furthermore,*

$$(1-r)|f'(r\zeta)|\leq\alpha K_3\tau_k\quad(k\geq h') \tag{2.13}$$

where $h'\geq h$ *depends only on* λ, α *and* h.

Proof. It follows from (2.10) that

$$|a_{k+1}|\leq\frac{\alpha}{1-\alpha}\tau_k\leq 2\alpha\tau_k,\quad \tau_{k+1}\leq\tau_k+|a_{k+1}|\leq 2\tau_k$$

for $k\geq h$ and hence by induction that

$$|a_j|\leq 2^{j-k}\alpha\tau_k,\quad \tau_j\leq 2^{j-k}\tau_k\quad(j\geq k\geq h). \tag{2.14}$$

Since $n_j/n_k\leq\lambda^{j-k}$ for $j\leq k$, we obtain from (2.10) that

$$\begin{aligned}\sum_{j=0}^{k}|a_j|(1-r^{n_j})&\leq\sum_{j=0}^{k}|a_j|\frac{n_j}{n_k}\\&\leq\tau_h+\alpha\tau_k\sum_{j=h+1}^{k}\lambda^{j-k}\leq\tau_h+\frac{\alpha\tau_k}{1-1/\lambda}.\end{aligned} \tag{2.15}$$

Since $n_j/n_{k+1}\geq\lambda^{j-k-1}$ for $j>k$, we see from (2.14) that, for $k\geq h$,

$$\begin{aligned}\sum_{j=k+1}^{\infty}|a_j|r^{n_j}&\leq\sum_{j=k+1}^{\infty}|a_j|\left(1-\frac{1}{n_{k+1}}\right)^{n_j}\\&\leq\alpha\tau_k\sum_{j=k+1}^{\infty}2^{j-k}\exp(-\lambda^{j-k-1})=\alpha K_2\tau_k\end{aligned} \tag{2.16}$$

where $K_2<\infty$ depends only on λ.

It follows from (2.15) and (2.16) that

$$\left|f(r\zeta)-\sum_{j=0}^{k} a_j\zeta^{n_j}\right| = \left|\sum_{j=0}^{k} a_j(r^j-1)\zeta^{n_j}+\sum_{j=k+1}^{\infty} a_j(r\zeta)^{n_j}\right| \le \tau_h+\alpha K_1\tau_k$$

which proves (2.11), and (2.12) follows from (2.16) because

$$M(r)\le \sum_{j=0}^{k}|a_j|r^{n_j}+\sum_{j=k+1}^{\infty}|a_j|r^{n_j}\le \tau_k+\alpha K_2\tau_k.$$

Finally we see from (1.1) that

$$(1-r)|f'(r\zeta)|\le \sum_{j=0}^{\infty} n_j|a_j|(1-r)r^{n_j-1},$$

and, for $k\ge h$, it follows from (2.14) that this is bounded by

$$\frac{n_1}{n_k}\tau_h+\alpha\tau_k\left[\sum_{j=h}^{k}\frac{n_j}{n_k}+\sum_{j=k+1}^{\infty}\frac{n_j}{n_{k+1}}2^{j-k}\left(1-\frac{1}{n_{k+1}}\right)^{n_j}\right].$$

Hence we obtain (2.13) by an argument similar to the one used above; note that $n_h/n_k<\alpha$ if $k\ge h'$ for suitable $h'\ge h$. □

3. Proofs of the positive results. *Proof of Theorem* 1 (compare [5]). Let δ and γ be the constants of Lemma 1; we may assume that $\gamma<1/2$. We choose integers s and m such that

$$\lambda^{s+1}>\frac{3\delta}{2\gamma},\quad \frac{6}{m-2}<\gamma. \tag{3.1}$$

Then m and s depend only on λ. By adding exponents n_k with $a_k=0$ if necessary, we may assume that $\lambda\le n_{k+1}/n_k\le\lambda^2$. We now choose k_ν according to Lemma 2, define $k_0=0$, and use the notation (2.4) and (2.5).

For $\nu=1,2,\dots$ we choose systems of arcs $I_{j_1\dots j_\nu}\subset\partial\mathbf{D}$ $(j_1,\dots,j_\nu=1,2)$ recursively such that

$$|I_{j_1\dots j_\nu}|=\frac{3\delta}{n_{k_\nu}}\quad (\nu=1,2,\dots). \tag{3.2}$$

In order to obtain the next system of arcs, we divide $I_{j_1\dots j_\nu}$ into three equal subarcs J_1,J_0,J_2. By Lemma 1, there are arcs

$$I_{j_1\dots j_\nu 1}\equiv J_1'\subset J_1,\quad I_{j_1\dots j_\nu 2}\equiv J_2'\subset J_2$$

of lengths $\ge 2\gamma/n_{k_{\nu+1}-s-1}$ such that

$$\operatorname{Re}\sum_{k=k_\nu}^{k_{\nu+1}-s-1} a_k\zeta^{n_k}\ge 4\gamma A_\nu\quad \text{for } \zeta\in I_{j_1\dots j_\nu j},\ j=1,2. \tag{3.3}$$

It follows from (3.2) that

$$\operatorname{dist}(I_{j_1\dots j_\nu 1},I_{j_1\dots j_\nu 2})\ge|J_0|\ge\delta/n_{k_\nu}, \tag{3.4}$$

and (3.1) shows that

$$|I_{j_1\dots j_\nu j}| \ge \frac{2\gamma}{n_{k_{\nu+1}-s-1}} \ge \frac{2\gamma\lambda^{s+1}}{n_{k_{\nu+1}}} > \frac{3\delta}{n_{k_{\nu+1}}}.$$

Hence (3.2) holds for $\nu+1$ if we shorten $I_{j_1\dots j_\nu j}$ somewhat. This completes our construction.

We deduce from (2.3) that

$$k_{\nu+1}-k_\nu \le (\nu+1)sm-\nu sm+(m-1)s<2ms. \tag{3.5}$$

Hence we obtain from (3.2) that, for $j=1,2$,

$$\frac{|I_{j_1\dots j_\nu j}|}{|I_{j_1\dots j_\nu}|} = \frac{n_{k_\nu}}{n_{k_{\nu+1}}} \ge \left(\frac{1}{\lambda^2}\right)^{k_{\nu+1}-k_\nu} \ge \lambda^{-4ms}.$$

Thus (2.7) is satisfied with $q=\lambda^{-4ms}$, and (2.8) is satisfied with $c=1/3$, by (3.2) and (3.4). Hence we conclude from Lemma 3 that

$$\dim E \ge \frac{\log 2}{4ms\log\lambda} \equiv \beta, \quad E \equiv \bigcap_\nu E_\nu, \tag{3.6}$$

and β depends only on λ.

Let now $\zeta \in E$ and $k \ge l' \equiv l+2ms$. We choose μ such that $k_\mu \le k < k_{\mu+1}$; note that $k_\mu \ge l$ by (3.5). We write

$$\sum_{j=0}^{k} a_j \zeta^{n_j} = \sum_{\nu=0}^{\mu-1}\left(\sum_{j=k_\nu}^{k_{\nu+1}-s-1} + \sum_{j=k_{\nu+1}-s}^{k_{\nu+1}-1}\right) + \sum_{j=k_\nu}^{k} a_j\zeta^{n_j}. \tag{3.7}$$

Since $\zeta \in E_\nu$ for all ν, it follows from (2.6) and (3.3) that

$$\operatorname{Re}\sum_{j=0}^{k} a_j\zeta^{n_j} \ge \sum_{\nu=0}^{\mu-1}(4\gamma A_\nu - A^*_{\nu+1}) - \sum_{j=k_\mu}^{k}|a_j|.$$

Since $2\gamma \le 1$ we therefore see that, by (2.10) and (2.5),

$$\operatorname{Re}\sum_{j=0}^{k} a_j\zeta^{n_j} - 2\gamma\tau_k \ge \sum_{\nu=0}^{\mu-1}(2\gamma A_\nu - 2A^*_{\nu+1}) - 2\sum_{j=k_\mu}^{k}|a_j|$$

$$\ge \sum_{\nu=0}^{\mu-1}\left(2\gamma - \frac{6}{m-2}\right)A_\nu + \sum_{\nu=1}^{\mu}A^*_\nu - 3\sum_{j=k_\mu}^{k_{\mu+1}-1}|a_j|.$$

Using (3.1) and (2.14), we see that this expression is bounded from below by

$$\sum_{\nu=0}^{\mu-1}\gamma A_\nu + \sum_{\nu=1}^{\mu}A^*_\nu - 3\cdot 2^{k_{\mu+1}-k_\mu}\alpha\tau_{k_\mu}$$

and therefore by $(\gamma - 3\cdot 2^{2ms}\alpha - \alpha)\tau_{k_\mu}$ because of (2.4), (2.5), (2.10), and (3.5). This is non-negative if α is chosen small enough depending only on λ. We have thus proved that

$$\operatorname{Re}\sum_{j=0}^{k} a_j\zeta^{n_j} \ge 2\gamma\tau_k \quad \text{for } \zeta\in E,\ k\ge l'. \tag{3.8}$$

Let $\zeta \in E$ and $r_0' \le r < 1$ where $r_0' = 1-1/n_{l'}$. Then $1-1/n_k \le r < 1-1/n_{k+1}$ with $k \ge l'$. Hence we obtain from (3.8) and from Lemma 4 that

$$\begin{aligned}\operatorname{Re} f(r\zeta) &> 2\gamma\tau_k - \tau_l - \alpha K_1 \tau_k \\ &\geq \frac{2\gamma-\alpha K_1}{1-\alpha K_2} M(r) - \tau_l \geq \frac{3\gamma}{2} M(r) - \tau_l\end{aligned} \tag{3.9}$$

if we choose α (depending only on λ) sufficiently small. Since $M(r)\to\infty$ as $r\to 1-0$ by (1.5) and (1.7), we conclude that (1.6) holds if r_0 is suitably chosen with $r_0'\leq r_0<1$.

Proof of Theorem 2. Let Δ be a Stolz angle at 1. Then

$$\frac{|\vartheta|}{1-r} \leq q < \infty \quad \text{for } re^{i\vartheta}\in\Delta.$$

Hence we deduce from (2.13) in Lemma 4 that, for $re^{i\vartheta}\in\Delta$, $|\zeta|=1$,

$$\begin{aligned}|f(r\zeta)-f(r\zeta e^{i\vartheta})| &= \left|\int_0^{\vartheta} f'(r\zeta e^{it}) re^{it}\,dt\right| \\ &\leq \frac{\alpha K_3 \tau_k}{1-r}|\vartheta| \leq \alpha K_3 q \tau_k.\end{aligned} \tag{3.10}$$

The assumptions of Theorem 1 are satisfied where now α can be made arbitrarily small and l' depends on α. Let E be the set with $\dim E\geq\beta$ constructed in the proof of Theorem 1. It follows from (3.9) and (3.10) that, for $\zeta\in E$ and $re^{i\vartheta}\in\Delta$, $1-1/n_{l'}\leq r<1$,

$$\operatorname{Re} f(r\zeta e^{i\vartheta}) > (2\gamma - K_1\alpha - K_3 q\alpha)\tau_k - \tau_l,$$

and this is $>\frac{3}{2}\gamma\tau_k-\tau_l$ if $\alpha=\alpha(\Delta)$ is chosen small enough. Hence it follows from (2.12) that

$$\operatorname{Re} f(r\zeta e^{i\vartheta}) > \gamma M(r) \quad \text{if } r_0(\Delta)\leq r<1. \qquad \square$$

Proof of Theorem 4. By (1.15) the function

$$f(z) = \log g'(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \tag{3.11}$$

has Hadamard gaps. We see that

$$\int_0^1 |g''(r\zeta)|\,dr = \int_0^1 e^{\operatorname{Re} f(r\zeta)}|f'(r\zeta)|\,dr. \tag{3.12}$$

In the case that $\sum_k |a_k| = M<\infty$ it follows from (3.11) and (3.12) that, for every $\zeta\in\partial\mathbf{D}$,

$$\int_0^1 |g''(r\zeta)|\,dr \leq e^M \int_0^1 \sum_{k=0}^{\infty} n_k|a_k| r^{n_k-1}\,dr \leq Me^M<\infty.$$

Let now $\sum|a_k|=\infty$. Since g is univalent by assumption, it follows [14, p. 21] that

$$(1-|z|^2)|f'(z)| = (1-|z|^2)\left|\frac{g''(z)}{g'(z)}\right| \leq 6 \quad \text{for } z\in\mathbf{D}.$$

Hence (a_k) is bounded, and we obtain from the result of Csordas, Lohwater and Ramsey [5] (or from Theorem 2) that, with a constant $c_1>0$,

$$\operatorname{Re} f(r\zeta) < -c_1 M(r) \quad \text{for } \zeta \in E,\ r_0 \le r < 1$$

where $\dim E \ge \beta$. Thus it follows from Sidon's theorem [16, p. 247] applied to $f(rz)$ that

$$\operatorname{Re} f(r\zeta) < -c_2 \sum_{k=0}^{\infty} |a_k| r^{n_k} \quad \text{for } \zeta \in E,\ r_0 \le r < 1, \tag{3.13}$$

with $c_2>0$. We therefore conclude from (3.12) that, for $\zeta \in E$,

$$\int_{r_0}^{1} |g''(r\zeta)|\, dr \le \int_{r_0}^{1} \exp\Big(-c_2 \sum_k |a_k| r^{n_k}\Big) \sum_{k=0}^{\infty} n_k |a_k| r^{n_k-1}\, dr$$
$$< \int_0^\infty \exp(-c_2 \xi)\, d\xi < \infty.$$

4. Construction of the example. The proof of Theorem 3 is based on two elementary lemmas.

LEMMA 5. *For every* $\lambda>1$, *there exists* $\xi>0$ *such that, if* f *is of the form* (1.1) *and*

$$|a_k| = n_k^{\xi}, \quad r_k = e^{-\xi/n_k} \tag{4.1}$$

for $k=0,1,\dots,$ *then*

$$|f(z) - a_k z^{n_k}| < \frac{1}{12} |a_k| r_k^{n_k} \quad \textit{for } |z| = r_k. \tag{4.2}$$

Proof. It follows from (1.1) and (4.1) that, if $|z|=r_k$, then

$$\begin{aligned} \frac{|f(z)-a_k z^{n_k}|}{|a_k| r_k^{n_k}} &\le e^{\xi} \sum_{j\ne k} \left(\frac{n_j}{n_k} e^{-n_j/n_k}\right)^{\xi} \\ &\le \sum_{j \ne k} \left(\lambda^{j-k} e^{1-\lambda^{j-k}}\right)^{\xi} \end{aligned} \tag{4.3}$$

because $n_j/n_k \le \lambda^{j-k}$ $(j<k)$, and $n_j/n_k \ge \lambda^{j-k}$ $(j>k)$ and because xe^{-x} increases for $0<x<1$ and decreases for $1<x<\infty$. The last expression in (4.3) becomes $<1/12$ if we choose ξ sufficiently large. □

LEMMA 6. *Let* $q \in \mathbf{N}$. *For every* $\zeta \in \partial\mathbf{D}$, *there are infinitely many* $m \in \mathbf{N}$ *such that*

$$\max(\operatorname{Re}[i\zeta^{2^m q}], \operatorname{Re}[\zeta^{2^m 3q}]) \ge \sin\frac{\pi}{18}. \tag{4.4}$$

Proof. Let $\zeta = e^{i\vartheta}$ with $0<\vartheta \le 2\pi$ and $k \in \mathbf{N}$. We consider the binary expansion

$$2^k q\vartheta/\pi = \text{integer} + \sum_{n=1}^{\infty} d_n 2^{-n}, \quad d_n = 0,1$$

where we allow, contrary to the usual convention, that d_n is eventually 1 but not eventually 0. Let j be the first index such that $d_j = 1$ and let $m = j + k$. Then

$$2^m q\vartheta = 2\pi p + \pi + \tau, \quad p \in \mathbf{Z}, \; 0 < \tau \le \pi,$$

hence

$$\operatorname{Re}[\zeta^{2^{m+1}3q}] = \cos 6\tau \ge \cos\frac{\pi}{3} \quad \text{for } 0 < \tau \le \frac{\pi}{18},$$

$$\operatorname{Re}[i\zeta^{2^m q}] = \sin\tau \ge \sin\frac{\pi}{18} \quad \text{for } \frac{\pi}{18} < \tau \le \frac{17\pi}{18},$$

$$\operatorname{Re}[\zeta^{2^m 3q}] = -\cos 3\tau \ge \cos\frac{\pi}{6} \quad \text{for } \frac{17\pi}{18} < \tau \le \pi.$$

As k was arbitrary we see that (4.4) holds for infinitely many m. □

Proof of Theorem 3. We arrange the mutually distinct numbers

(4.5) $\qquad 2^m q \quad (m = 0, 1, 2, \ldots;\; q = 1, 3, 5, 7, 11, 15, 21, 33)$

into an increasing sequence (n_k) and write $\lambda_k = n_{k+1}/n_k$. Then either $\lambda_k = q/(2^p q')$ or $\lambda_k = (2^p q)/q'$, where p is a nonnegative integer and q and q' are integers from the finite list in (4.5). In the first case, the inequality $\lambda_k > 1$ implies that $2^p q' \ge 32$ and therefore $\lambda_k \ge 33/32$. In the second case, $\lambda_k \ge 40/33$ (if $q' = 33$) or $\lambda_k \ge 22/21$ (if $q' \le 21$). Therefore $\lambda_k \ge 33/32$ for all k.

Let ξ be determined as in Lemma 5 and let $a_k = c_k n_k^{\xi}$ where

(i) $\qquad c_k = i$ for $n_k = 2^m$, $\quad c_k = 1$ for $n_k = 2^m 3$,

(ii) $\qquad c_k = -i$ for $n_k = 2^m 5$, $\quad c_k = -1$ for $n_k = 2^m 15$,

(iii) $\qquad c_k = -1$ for $n_k = 2^m 7$, $\quad c_k = i$ for $n_k = 2^m 21$,

(iv) $\qquad c_k = 1$ for $n_k = 2^m 11$, $\quad c_k = -i$ for $n_k = 2^m 33$.

Let now $\zeta \in \mathbf{D}$. It follows from (4.2) and (4.1) that

$$\operatorname{Re} f(r_k\zeta) > |a_k| r^{n_k}\left(\operatorname{Re}\zeta^{n_k} - \frac{1}{12}\right) = e^{-\xi} n_k^{\xi}\left(\operatorname{Re}[c_k\zeta^{n_k}] - \frac{1}{12}\right)$$

for all k. We choose m such that (4.4) holds and consider the value k for which $n_k = 2^m$ or $n_k = 2^m 3$; from (i) it follows that

$$\operatorname{Re} f(r_k\zeta) > e^{-\xi} 2^{\xi m}\left(\sin\frac{\pi}{18} - \frac{1}{12}\right) > \frac{1}{12} e^{-\xi} 2^{\xi m}$$

and the first of the assertions in (1.12) becomes obvious. To see that the first assertion in (1.13) follows from (iii), we consider the exponents $n_k = 2^m 7$ or $n_k = 2^m 11$. The second assertions in (1.12) and (1.13) follow similarly if we use (ii) and (iv). □

5. Some open problems. There remain a number of interesting open problems about the existence of radial limits. Let f be an unbounded function with Hadamard gaps and let

$$E_\infty = \{\zeta \in \partial \mathbf{D} : |f(r\zeta)| \to \infty \text{ as } r \to 1-0\}. \tag{5.1}$$

Anderson and Hornblower [3, p. 136] have asked whether E_∞ is always non-empty. If f is the function of Theorem 3 then Re f and Im f have no radial limit at any point. But this does not exclude the possibility that $E_\infty \neq \emptyset$ because $f(r\zeta)$ might "spiral" to ∞.

We could also ask whether it is always true that dim E_∞ is >0 or even $=1$. This is motivated by Theorem 1 and the result (1.11) of Hawkes [7].

Finally, under what conditions is it true that mes $E_\infty = 2\pi$? This would mean that

$$\lim_{r \to 1-0} |f(r\zeta)| = \infty \quad \text{for almost all } \zeta \in \partial \mathbf{D}. \tag{5.2}$$

By the Privalov uniqueness theorem [14, p. 325], this is impossible if (a_k) is bounded because then radial limits are also angular limits. Now let

$$t_k = \sum_{j=0}^{k} |a_j|^2 \quad (k = 0, 1, \ldots).$$

Anderson [2] has conjectured that (5.2) holds if

$$\frac{|a_k|^2}{t_k} \to 0 \;\; (k \to \infty), \quad \sum_k \frac{1}{t_k} < \infty. \tag{5.3}$$

This was suggested by results on random power series; see for example [8] for their connection to lacunary series.

The only known results seem to be for functions with stronger than Hadamard gaps. Murai has proved (in a paper [12, p. 143] submitted in 1976) that (5.2) holds if $(|a_k|)$ increases and if

$$\frac{\log n_{k+1}}{\log n_k} \geq \lambda' > 1, \quad \sum_k \frac{|a_k|}{t_k} < \infty.$$

Hawkes [7, p. 27] has shown that (5.2) holds for $a_k > 0$ if (5.3) is satisfied and if furthermore

$$\sum_k a_k \frac{n_k}{n_{k+1}} < \infty, \quad \sum_{k=1}^{\infty} \sum_{j=1}^{k-1} \frac{1}{t_j(t_k - t_j)} < \infty.$$

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