MULTILINEAR CONVOLUTIONS AND TRANSFERENCE

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1. Introduction. In this paper, we present some theorems which are basic to the study of a wide class of nonlinear operators which arise in partial differential equations and many other parts of analysis. These results are part of a continuing program of nonlinear analysis developed over the past several years by R. R. Coifman, A. McIntosh, Y. Meyer, and others (see, for example, [1], [2], and [3]). A typical problem is that of obtaining L^p estimates for linear operators $T(b)$ which depend nonlinearly on a functional parameter b in a Banach space B. By analogy to the calculus of functions in finite dimensions, T is said to be *analytic at the origin in B* if and only if for all b with $\|b\|_B$ sufficiently small and for all $f \in L^p(X)$,

$$T(b)f = \sum_{k=0}^{\infty} M_{k+1}(b, \ldots, b, f), \tag{1.1}$$

where M_{k+1} is a bounded $(k+1)$-multilinear operator on $(B)^k \times L^p$ which satisfies an estimate of the form

$$\|M_{k+1}(b, \ldots, b, f)\|_p \leq C^k \|b\|_B^k \|f\|_p \tag{1.2}$$

for some absolute constant $C > 0$. The multilinear operator $k!\, M_{k+1}(b, \ldots, b, f)$ is, in fact, the action of f of the kth Fréchet differential of T at 0 in the direction b. In order to prove that T depends analytically on b, it suffices to find an explicit representation for T as a convergent "power series" of multilinear operators in a neighborhood of the origin in B. The problem of obtaining L^p estimates for T is thereby reduced to that of obtaining L^p estimates for the Taylor coefficients of T. Thus we are led to investigate certain broad classes of multilinear operators which arise naturally as the Fréchet differentials of nonlinear operators.

Of particular interest are the multilinear convolutions: multilinear operators which commute with the simultaneous action of a group of measure-preserving transformations of the underlying measure space. Specifically, let (X, μ) be a σ-finite measure space, and let $\{U_t\}$ be a group of measure-preserving transformations of X, indexed by $\mathbf{R}^n$. Let B be a Banach function space on X, and suppose that for $b \in B$, $\|b \circ U_t\|_B = \|b\|_B$; that is, the norm on B is invariant under the action of $\{U_t\}$. A k-multilinear operator M_k is called a *multilinear convolution* if and only if

$$M_k(f_1 \circ U_t, \ldots, f_k \circ U_t) = M_k(f_1, \ldots, f_k) \circ U_t. \tag{1.3}$$

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Multilinear convolutions arise naturally as the Fréchet differentials of operator-valued functions T which commute with the simultaneous action of U_t on B and $L^p(X)$; that is, for $b \in B$ and $f \in L^p(X)$,

$$T(b \circ U_t) f \circ U_t = (T(b)f) \circ U_t. \tag{1.4}$$

The classical example of such an operator T is the Cauchy integral for a Lipschitz curve. The Calderón commutators, which arise as the Fréchet differentials of the Cauchy integral, are multilinear operators which commute with simultaneous translations in $\mathbf{R}$ (see [1] and [3]).

In this paper, we shall examine the properties of multilinear convolutions, concentrating on the cases $X = \mathbf{R}^n$, $X = T^n$ (the n-torus), $B = L^\infty(X)$, in which the measure-preserving transformations are given by translations. We show that $L^p(\mathbf{R}^n)$ estimates can be transferred to other settings via measure-preserving actions of $\mathbf{R}^n$. As a consequence we are able to give necessary conditions for L^p boundedness of the tensor product of multilinear convolutions, and prove a deLeeuw-type theorem on L^p boundedness of periodized multilinear convolutions. As a partial converse to this last result, we give conditions under which it is possible to pass from L^p estimates on T^n to L^p estimates on $\mathbf{R}^n$.

2. Multilinear convolutions on T^n and $\mathbf{R}^n$. We begin with some notation. Euclidean space $\mathbf{R}^n$ is defined by

$$\mathbf{R}^n = \{x = (x_1, \dots, x_n) \colon x_i \in \mathbf{R},\ 1 \le i \le n\}; \tag{2.1}$$

the lattice of integer points in $\mathbf{R}^n$ is given by

$$Z^n = \{x \in \mathbf{R}^n \colon x_i \in Z,\ 1 \le i \le n\} \tag{2.2}$$

and will usually be denoted Λ_n. The *n-torus* is then defined by

$$T^n = \{x \in \mathbf{R}^n \colon x_i \in [0,1),\ 1 \le i \le n\} \tag{2.3}$$

and is identified with the quotient space $\mathbf{R}^n/\Lambda_n$. For each $x, \xi \in \mathbf{R}^n$, we define

$$e_x(\xi) = \exp(2\pi i x \cdot \xi) = \exp(2\pi i (x_1 \xi_1 + \cdots + x_n \xi_n)). \tag{2.4}$$

For $f \in L^1(\mathbf{R}^n)$, the Fourier transform and its inverse are defined according to the normalization

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} e_{-\xi}(x) f(x)\, dx, \tag{2.5}$$

$$\check{f}(x) = \int_{\mathbf{R}^n} e_x(\xi) f(\xi)\, d\xi. \tag{2.6}$$

If $f \in \mathfrak{M}(T^n)$ — that is, f is measurable on T^n — then, for each $k = (k_1, \dots, k_n) \in \Lambda_n$, we define

$$\hat{f}(k) = \int_{T^n} e_{-k}(\xi) f(\xi)\, d\xi, \tag{2.7}$$

the Fourier coefficient corresponding to k. Any finite sum of the form

(2.8) $$\sum_{|k|\le m} c_k e_k \qquad (|k|=k_1+\cdots+k_n)$$

is called a trigonometric polynomial, and the set of all trigonometric polynomials on T^n is denoted $\mathcal{P}(T^n)$. We define a family of translations on $\mathbf{R}^n$ by setting, for each $\xi, x\in\mathbf{R}^n$,

(2.9) $$\tau_\xi(x)=x-\xi$$

which gives rise to a family of translations on T^n if we define, for $\xi\in\mathbf{R}^n$ and $x\in T^n$,

(2.10) $$\tilde\tau_\xi(x)=\tau_\xi(x) \bmod \Lambda_n.$$

We shall abuse this notation slightly by writing $\tau_\xi f=f\circ\tau_\xi$ and $\tilde\tau_\xi g=g\circ\tilde\tau_\xi$, where f, g are measurable functions on $\mathbf{R}^n$, T^n, respectively.

Now suppose that M_k is a k-multilinear operator acting on $(\mathcal{P}(T^n))^k$ which commutes with simultaneous translations. That is to say, $M_k:(\mathcal{P}(T^n))^k\to\mathfrak{M}(T^n)$ is linear in each variable separately; and moreover, if $t_1,\dots,t_k\in\mathcal{P}(T^n)$, $\xi\in\mathbf{R}^n$, then

(2.11) $$M_k(\tilde\tau_\xi t_1,\dots,\tilde\tau_\xi t_k)=\tilde\tau_\xi M_k(t_1,\dots,t_k).$$

Then the operator M_k has a special form.

PROPOSITION 2.1. *If $t_1,\dots,t_k\in\mathcal{P}(T^n)$ and $\theta\in T^n$, then*

(2.12) $$M_k(t_1,\dots,t_k)(\theta)=\sum_{m\in\Lambda_n}\left(\sum_{|l|=m}\sigma(l_1,\dots,l_k)\hat t_1(l_1)\dots\hat t_k(l_k)\right)e_m(\theta)$$

where, for $(l_1,\dots,l_k)\in(\Lambda_n)^k$, we have $|l|=l_1+\cdots+l_k$ and

(2.13) $$\sigma(l_1,\dots,l_k)=M_k(e_{l_1},\dots,e_{l_k})(0).$$

Proof. For any $\theta,\eta\in T^n$ and for any $l=(l_1,\dots,l_k)\in\Lambda_n$, we have

(2.14) $$\begin{aligned}M_k(e_{l_1},\dots,e_{l_k})(\theta+\eta)&=\tilde\tau_{-\theta}M_k(e_{l_1},\dots,e_{l_k})(\eta)\\&=M_k(\tilde\tau_{-\theta}e_{l_1},\dots,\tilde\tau_{-\theta}e_{l_k})(\eta)\\&=e_{|l|}(\theta)M_k(e_{l_1},\dots,e_{l_k})(\eta).\end{aligned}$$

Taking $\eta=0$ in (2.14) and defining σ according to (2.13), we obtain

(2.15) $$M_k(e_{l_1},\dots,e_{l_k})(\theta)=e_{|l|}(\theta)\sigma(l_1,\dots,l_k)$$

and (2.12) follows by multilinearity of M_k. □

COROLLARY 2.1.1. *Suppose that $1\le p_1,\dots,p_k,\ q\le\infty$, and*

(2.16) $$\|M_k(t_1,\dots,t_k)\|_q\le C\prod_{j=1}^k\|t_j\|_{p_j}.$$

Then σ is bounded by C on $(\Lambda_n)^k$.

Proof. By (2.15), we have $|\sigma(l_1,\dots,l_k)|=|M_k(e_{l_1},\dots,e_{l_k})(\theta)|$ for all $\theta\in T^n$ and all $l_1,\dots,l_k\in\Lambda_n$. Moreover,

$$C = C \prod_{j=1}^{k} \|e_{l_j}\|_{p_j} \geq \|M_k(e_{l_1}, \dots, e_{l_k})\|_q. \tag{2.17}$$

But it is easy to see that $|\sigma(l_1, \dots, l_k)| = \|M_k(e_{l_1}, \dots, e_{l_k})\|_q$, from which the result follows. □

As a consequence of these results, we see that any bounded multilinear convolution M_k on T^n has the form

$$M_k(f_1, \dots, f_k)(\theta) = \sum_{m \in \Lambda_n} \left(\sum_{|l|=m} \sigma_k(l_1, \dots, l_k) \prod_{j=1}^{k} \hat{f}_j(l_j) \right) e_m(\theta), \tag{2.18}$$

where

$$\sup_{(\Lambda_n)^k} |\alpha_k(l_1, \dots, l_k)| \leq \|M_k\|_{op}. \tag{2.19}$$

We now consider multilinear convolutions on $\mathbf{R}^n$. In this context we may think of the test functions $C_0^\infty(\mathbf{R}^n)$ as the proper substitute for the trigonometric polynomials $\mathcal{P}(T^n)$. If M_k is a k-multilinear operator on $(C_0^\infty(\mathbf{R}^n))^k$ which commutes with simultaneous translations and takes its values in the space $\mathfrak{M}(\mathbf{R}^n)$ of measurable functions on $\mathbf{R}^n$, then M_k has the form

$$M_k(f_1, \dots, f_k)(x) = \int_{\mathbf{R}^{nk}} e_x(\xi_1 + \dots + \xi_k) \sigma_k(\xi_1, \dots, \xi_k) \prod_{j=1}^{k} \hat{f}_j(\xi_j)\, d\xi_1 \dots d\xi_k. \tag{2.20}$$

Alternatively, we may write

$$\begin{aligned} M_k(f_1, \dots, f_k)(x) &= \int_{\mathbf{R}^{nk}} K_k(u_1, \dots, u_k) \prod_{j=1}^{k} f_j(\tau_{u_j} x)\, du_1 \dots du_k \\ &= K_k * (f_1 \otimes \dots \otimes f_k)(x, \dots, x), \end{aligned} \tag{2.21}$$

where $\hat{K}_k = \sigma_k$. The function $\sigma_k : \mathbf{R}^{nk} \to \mathbf{C}$ is called the *multilinear symbol* of M_k.

3. Transference from $\mathbf{R}^n$. In this section, we use the technique of Coifman and Weiss ([4]) to show that $L^p(\mathbf{R}^n)$ estimates for bounded multilinear convolutions may be transferred to other settings via measure-preserving actions of $\mathbf{R}^n$. We begin with some notation.

Suppose that K_k is a bounded, measurable, compactly supported function on $\mathbf{R}^{nk}$, and let M_k be defined by (2.21). Now let $\{U_t\}$ be a group of measure-preserving transformations of a σ-finite measure space (X, μ) onto itself (i.e., $U_t U_s = U_{t+s}$). We define the "transferred" operator $M_k^\#$ for functions $F_1, \dots, F_k$ on X by setting, for each $x \in X$,

$$M_k^\#(F_1, \dots, F_k)(x) = \int_{\mathbf{R}^{nk}} K_k(u_1, \dots, u_k) \prod_{j=1}^{k} F_j(U_{-u_j} x)\, du_1 \dots du_k. \tag{3.1}$$

Our basic result is the following.

THEOREM 3.1. *Let $1 \leq p < \infty$, and suppose that there is a constant C_p so that whenever $f_1, \dots, f_{k-1} \in L^\infty(\mathbf{R}^n)$, $f_k \in L^p(\mathbf{R}^n)$,*

$$\|M_k(f_1,\dots,f_k)\|_{L^p(\mathbf{R}^n)} \le C_p\left(\prod_{j=1}^{k-1}\|f_j\|_{L^\infty(\mathbf{R}^n)}\right)\|f_k\|_{L^p(\mathbf{R}^n)}. \tag{3.2}$$

Then whenever $F_1,\dots,F_{k-1}\in L^\infty(X)$, $F_k\in L^p(X)$, *we have*

$$\|M_k^\#(F_1,\dots,F_k)\|_{L^p(X)} \le C_p\left(\prod_{j=1}^{k-1}\|F_j\|_{L^\infty(X)}\right)\|F_k\|_{L^p(X)}. \tag{3.3}$$

Proof. For each $w\in\mathbf{R}^n$, U_w is a measure-preserving transformation of X onto itself, so that

$$\int_X |M_k^\#(F_1,\dots,F_k)(x)|^p\,d\mu(x) = \int_X |M_k^\#(F_1,\dots,F_k)(U_wx)|^p\,d\mu(x). \tag{3.4}$$

Now let N be a positive integer so that supp K_k is contained in $\{u\in\mathbf{R}^n\colon |u|\le N\}^k$; let M be any positive integer; let χ denote the characteristic function of the set $\{y\in\mathbf{R}^n\colon |y|\le M\}+\{y\in\mathbf{R}^n\colon |y|\le N\}$; and let Λ_n denote the volume of the solid unit sphere in $\mathbf{R}^n$. Then we may write

$$\begin{aligned}
&M^n\Omega_n\|M_k^\#(F_1,\dots,F_k)\|_{L^p(X)}^p\\
&\quad= \int_{|w|\le M}\int_X |M_k^\#(F_1,\dots,F_k)(U_wx)|^p\,d\mu(x)\,dw\\
&\quad= \int_X\int_{|w|\le M}\left|\int_{\mathbf{R}^{nk}} K_k(u_1,\dots,u_k)\prod_{j=1}^{k} F_j(U_{w-u_j}x)\,du_1\dots du_k\right|^p dw\,d\mu(x)\\
&\quad\le \int_X\int_{\mathbf{R}^n}\left|\int_{\mathbf{R}^{nk}} K_k(u_1,\dots,u_k)\chi(w-u_k)\prod_{j=1}^{k} F_j(U_{w-u_j}x)\,du_1\dots du_k\right|^p dw\,d\mu(x)\\
&\quad\le \int_X C_p^p\left(\int_{\mathbf{R}^n}|F_k(U_wx)\chi(w)|^p\,dw\right)\left(\prod_{j=1}^{k-1}\operatorname*{ess\,sup}_{w\in\mathbf{R}^n}|F_j(U_wx)|\right)^p d\mu(x)\\
&\quad\le C_p^p\left(\prod_{j=1}^{k-1}\|F_j\|_{L^\infty(X)}^p\right)\int_{\mathbf{R}^n}\chi(w)\int_X|F_k(U_wx)|^p\,d\mu(x)\,dw\\
&\quad\le C_p^p\left(\prod_{j=1}^{k-1}\|F_j\|_{L^\infty(X)}^p\right)\Omega_n(M+N)^n\|F_k\|_{L^p(X)}^p.
\end{aligned}$$

Thus

$$\|M_k^\#(F_1,\dots,F_k)\|_{L^p(X)}^p \le \left(\frac{M+N}{M}\right)^n C_p^p\left(\prod_{j=1}^{k-1}\|F_j\|_{L^\infty(X)}^p\right)\|F_k\|_{L^p(X)}^p. \tag{3.5}$$

Now, M was an arbitrary positive integer, and so (3.5) evidently holds in the limit as $M\to\infty$; taking pth roots gives (3.3). □

We now consider a number of applications of Theorem 3.1. Let us recall a few facts regarding tensor products of multilinear operators. If m is a positive integer and $f,g\colon\mathbf{R}^m\to\mathbf{C}$, then the tensor product $f\otimes g\colon\mathbf{R}^{2m}\to\mathbf{C}$ is defined by

$$(3.6)\qquad f\otimes g((u_1, v_1), \ldots, (u_m, v_m)) = f(u_1, \ldots, u_m)g(v_1, \ldots, v_m).$$

Similarly, suppose M_k and N_k are two (not necessarily bounded) multilinear convolutions, each of which acts on $(L^\infty(\mathbf{R}^n)^{k-1} \times L^p(\mathbf{R}^n))$. The tensor product $M_k \otimes N_k$ is defined on $(L^\infty \otimes L^\infty)^{k-1} \times (L^p \otimes L^p)$ by setting

$$(3.7)\qquad M_k\otimes N_k(f_1\otimes g_1, \ldots, f_k\otimes g_k)(x, y) = M_k(f_1, \ldots, f_k)(x) N_k(g_1, \ldots, g_k)(y).$$

B. C. Krikeles has shown (see [5]) that if M_k and N_k are defined in terms of the multilinear symbols σ_k and μ_k, respectively, then $M_k \otimes N_k$ may be extended to all of $(L^\infty(\mathbf{R}^{2n}))^{k-1} \times L^p(\mathbf{R}^{2n})$ by means of the symbol $\sigma_k \otimes \mu_k$. Moreover, he has shown that if M_k, N_k are bounded multilinear convolutions, their tensor product is not necessarily bounded. We can, by virtue of Theorem 3.1, give a necessary condition for the boundedness of $M_k \otimes N_k$.

COROLLARY 3.1.1. *Let M_k, N_k be as above, and suppose that each of the symbols σ_k, μ_k is the Fourier transform of a bounded, measurable, compactly supported function on $\mathbf{R}^{nk}$. Suppose, moreover, that $M_k \otimes N_k$ is a bounded multilinear convolution from $(L^\infty(\mathbf{R}^{2n}))^{k-1} \times L^p(\mathbf{R}^{2n})$ to $L^p(\mathbf{R}^{2n})$. Then the operator T_k, defined in terms of the symbol $\sigma_k \mu_k$, is a bounded multilinear convolution from $(L^\infty(\mathbf{R}^n))^{k-1} \times L^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$.*

Proof. Let $\hat{K} = \sigma_k$, and $\hat{L} = \mu_k$. Then if $f_1, \ldots, f_{k-1} \in L^\infty(\mathbf{R}^{2n})$ and $f_k \in L^p(\mathbf{R}^{2n})$, we have

$$(3.8)\qquad \begin{aligned} &M_k\otimes N_k(f_1, \ldots, f_k)(x, y) \\ &= \int_{\mathbf{R}^{2nk}} K\otimes L(x_1, y_1, \ldots, x_n, y_n) \prod_{j=1}^{k} f_j(x - x_j, y - y_j)\, dx_1\, dy_1 \ldots dx_n\, dy_n. \end{aligned}$$

Moreover, it is not difficult to see that for $F_1, \ldots, F_{k-1} \in L^\infty(\mathbf{R}^n)$ and $F_k \in L^p(\mathbf{R}^n)$, we have

$$(3.9)\qquad \begin{aligned} &T_k(F_1, \ldots, F_k)(z) \\ &= \int_{\mathbf{R}^{2nk}} K\otimes L(x_1, y_1, \ldots, x_n, y_n) \prod_{j=1}^{k} F_j(z - (x_j + y_j))\, dx_1\, dy_1 \ldots dx_n\, dy_n. \end{aligned}$$

Now define, for every $w = (x, y) \in \mathbf{R}^{2n}$ and for every $z \in \mathbf{R}^n$, $U_w z = z + (x + y)$. Clearly $\{U_w\}$ is a group of measure preserving transformations indexed by $\mathbf{R}^{2n}$, and

$$(3.10)\qquad \begin{aligned} &T_k(F_1, \ldots, F_k)(z) \\ &= \int_{\mathbf{R}^{2nk}} K\otimes L(x_1, y_1, \ldots, x_n, y_n) \prod_{j=1}^{k} F_j(U_{-w_j} z)\, dw_1 \ldots dw_n, \end{aligned}$$

where $w_j = (x_j, y_j)$. The result is now immediate from Theorem 3.1. □

Theorem 3.1 may also be used to transfer results on the analyticity of operator-valued functions from $\mathbf{R}^n$ to other measure spaces. The following Corollary is immediate.

COROLLARY 3.1.2. *Suppose that* $T: L^\infty(\mathbf{R}^n) \to \mathcal{L}(L^p(\mathbf{R}^n))$ *is analytic at the origin in* $L^\infty(\mathbf{R}^n)$, *and suppose that* T *has the series expansion*

$$T(b)f = \sum_{k=1}^{\infty} M_k(b, \dots, b, f), \tag{3.11}$$

where

$$\begin{aligned} &M_k(b, \dots, b, f)(x) \\ &\quad = \int_{\mathbf{R}^{nk}} K_k(u_1, \dots, u_k) \left(\prod_{j=1}^{k-1} b(x-u_j) \right) f(x-u_k)\, du_1 \dots du_k. \end{aligned} \tag{3.12}$$

Then if we define $T^\#: L^\infty(X) \to \mathcal{L}(L^p(X))$ *by*

$$T^\#(B)F = \sum_{k=1}^{\infty} M_k^\#(B, \dots, B, F) \tag{3.13}$$

with notation as in Theorem 3.1, *then* $T^\#$ *is analytic at the origin in* $L^\infty(X)$, *and the operators* $M_k^\#$ *satisfy the same estimates as do the operators* M_k.

Finally, we can use Theorem 3.1 to extend a result of deLeeuw (see [6], Chapter 7, Theorem 3.8) to the multilinear case. Consider the action of $\mathbf{R}^n$ on T^n by simple translation (modulo the lattice points). We define, for $x \in \mathbf{R}^n$, $U_x = \tau_{-x}$; that is, for every $\theta \in T^n$, $U_x\theta = \theta + x \pmod{\Lambda_n}$. As in Theorem 3.1, we suppose that K_k is bounded, measurable, and compactly supported on $\mathbf{R}^{nk}$, and we define M_k according to (2.21). Recalling that M_k also has the representation (2.20), where $\sigma_k = \hat{K}_k$, we define the *periodization* of M_k to be the operator defined for functions $F_1, \dots, F_k$ on T^n by

$$M_k^\circ(F_1, \dots, F_k)(\theta) = \sum_{m \in \Lambda_n} \left(\sum_{|l| = m} \sigma_k(l_1, \dots, l_k) \prod_{j=1}^{k} \hat{F}_j(l_j) \right) e_m(\theta). \tag{3.14}$$

We claim that M_k° is precisely $M_k^\#$; for $M_k^\#$ is evidently a multilinear convolution commuting with simultaneous translations, so that, by Proposition 2.1,

$$M_k^\#(F_1, \dots, F_k)(\theta) = \sum_{m \in \Lambda_n} \left(\sum_{|l| = m} \mu_k(l_1, \dots, l_k) \prod_{j=1}^{k} \hat{F}_j(l_j) \right) e_m(\theta), \tag{3.15}$$

where $\mu_k(l_1, \dots, l_k) = M_k^\#(e_{l_1}, \dots, e_{l_k})(0)$. But

$$\begin{aligned} \mu_k(l_1, \dots, l_k) &= \int_{\mathbf{R}^{nk}} K_k(u_1, \dots, u_k) \left(\prod_{j=1}^{k} e_{-l_j}(u_j) \right) du_1 \dots du_k \\ &= \int_{\mathbf{R}^{nk}} \exp(-2\pi i (l_1, \dots, l_k) \cdot (u_1, \dots, u_k)) K_k(u_1, \dots, u_k)\, du_1 \dots du_k \\ &= \sigma_k(l_1, \dots, l_k). \end{aligned} \tag{3.16}$$

Thus $M_k^\circ = M_k^\#$, and we have the following.

COROLLARY 3.1.3. *Let* K_k *be a bounded, measurable, compactly supported function on* $\mathbf{R}^{nk}$, *and define the multilinear operator* M_k *according to* (2.20) *and* (2.21). *Let* $1 \le p < \infty$ *and suppose that there is a constant* C_p *such that if* $f_1, \dots, f_{k-1} \in L^\infty(\mathbf{R}^n)$ *and* $f_k \in L^p(\mathbf{R}^n)$ *we have*

$$\|M_k(f_1, \dots, f_k)\|_{L^p(\mathbf{R}^n)} \le C_p \left(\prod_{j=1}^{k-1} \|f_j\|_{L^\infty(\mathbf{R}^n)} \right) \|f_k\|_{L^p(\mathbf{R}^n)}. \tag{3.17}$$

Then, if M_k° *is the periodization of* M_k, *defined by* (3.14) *for functions* $F_1, \dots, F_{k-1} \in L^\infty(T^n)$, $F_k \in L^p(T^n)$, *we have*

$$\|M_k^\circ(F_1, \dots, F_k)\|_{L^p(T^n)} \le C_p \left(\prod_{j=1}^{k-1} \|F_j\|_{L^\infty(T^n)} \right) \|F_k\|_{L^p(T^n)}. \tag{3.18}$$

We can, indeed, say a bit more.

COROLLARY 3.1.4. *Under the hypotheses of Corollary* 3.1.3, *we have* $\|\sigma_k\|_{L^\infty(\mathbf{R}^{kn})} \le C_p$, *where* $\sigma_k = \hat{K}_k$.

Proof. For $\lambda > 0$, define M_k^λ to be the multilinear operator obtained from M_k by replacing K_k in (2.21) by its dilate $K_{k,\lambda} = \lambda^{-nk} K_k(\bullet \lambda^{-1})$. It is not difficult to show that M_k^λ is a bounded multilinear operator, and indeed, for $f_1, \dots, f_{k-1} \in L^\infty(\mathbf{R}^n)$ and $f_k \in L^p(\mathbf{R}^n)$,

$$\|M_k^\lambda(f_1, \dots, f_k)\|_{L^p(\mathbf{R}^n)} \le C_p \left(\prod_{j=1}^{k-1} \|f_j\|_{L^\infty(\mathbf{R}^n)} \right) \|f_k\|_{L^p(\mathbf{R}^n)}. \tag{3.19}$$

Therefore, by Corollary 3.1.3, each periodization $(M_k^\lambda)^\circ$ satisfies the analogous estimate in $L^p(T^n)$. By Corollary 2.1.1, it follows that, for all $l_1, \dots, l_k \in \Lambda_n$, and each $\lambda > 0$,

$$|\sigma_k(\lambda l_1, \dots, \lambda l_k)| \le C_p. \tag{3.20}$$

Now since σ_k is the Fourier transform of an L^1 function, it is continuous. Therefore, since $|\sigma_k|$ is bounded by C_p on a dense subset of $\mathbf{R}^{nk}$, we must have $\|\sigma_k\|_\infty \le C_p$. □

4. "Deperiodization" from T^n. In this section we obtain a partial converse to Corollary 3.1.3, which generalizes a known result for multiplier transformations (see [6], Chapter 7, Theorem 3.18).

THEOREM 4.1. *Let* $1 \le p < \infty$, *and suppose that* σ_k *is the Fourier transform of a bounded, measurable, compactly supported function on* $\mathbf{R}^{nk}$. *Define the multilinear operator* M_k *according to* (2.20) *and for each* $\lambda > 0$, *define* M_k^λ, $(M_k^\lambda)^\circ$ *as in the proof of Corollary* 3.1.4. *Suppose, moreover, that there is a constant* C_p *such that for all* $\lambda > 0$ *and for all* $F_1, \dots, F_{k-1} \in L^\infty(T^n)$, $F_k \in L^p(T^n)$,

$$\|(M_k^\lambda)^\circ(F_1, \dots, F_k)\|_{L^p(T^n)} \le C_p \left(\prod_{j=1}^{k-1} \|F_j\|_{L^\infty(T^n)} \right) \|F_k\|_{L^p(T^n)}. \tag{4.1}$$

Then, for all $f_1, \dots, f_{k-1} \in L^\infty(\mathbf{R}^n)$, $f_k \in L^p(\mathbf{R}^n)$, *we have*

$$\|M_k(f_1, \dots, f_k)\|_{L^p(\mathbf{R}^n)} \le C_p \left(\prod_{j=1}^{k-1} \|f_j\|_{L^\infty(\mathbf{R}^n)} \right) \|f_k\|_{L^p(\mathbf{R}^n)}. \tag{4.2}$$

Proof. It clearly suffices to assume that $f_1, \dots, f_k \in C_0^\infty(\mathbf{R}_n)$. For $\lambda > 0$, define $\tilde{f}_{j,\lambda}$, the periodized dilate of f_j, by setting

$$\tilde{f}_{j,\lambda} = \lambda^{-n} \sum_{m \in \Lambda_n} f_j\left(\frac{x+m}{\lambda}\right) \tag{4.3}$$

(see [6]). By the Poisson summation formula, we have

$$\tilde{f}_{j,\lambda}(x) = \sum_{m \in \Lambda_n} \hat{f}_j(\lambda m) e_m(x). \tag{4.4}$$

We claim that, for every $x \in \mathbf{R}^n$,

$$M_k(f_1, \dots, f_k)(x) = \lim_{\lambda \to 0} \lambda^{kn} (M_k^\lambda)^\circ (\tilde{f}_{1,\lambda}, \dots, \tilde{f}_{k,\lambda})(\lambda x), \tag{4.5}$$

where $(M_k^\lambda)^\circ (\tilde{f}_{1,\lambda}, \dots, \tilde{f}_{k,\lambda})$ is defined on all of $\mathbf{R}^n$ by periodic extension from T^n. To see this, observe that

$$\begin{aligned} &\lambda^{kn} (M_k^\lambda)^\circ (\tilde{f}_{1,\lambda}, \dots, \tilde{f}_{k,\lambda})(\lambda x) \\ &= \lambda^{kn} \sum_{m \in \Lambda_n} \left(\sum_{|l| = m} \sigma_k(\lambda l_1, \dots, \lambda l_k) \prod_{j=1}^{k} \hat{f}_j(\lambda l_j) \right) e_m(\lambda x) \end{aligned} \tag{4.6}$$

is a Riemann sum for the integral

$$\begin{aligned} &\int_{\mathbf{R}^{nk}} \sigma_k(\xi_1, \dots, \xi_k) \left(\prod_{j=1}^{k} \hat{f}_j(\xi_k) \right) e_x(\xi_1 + \cdots + \xi_k)\, d\xi_1 \dots d\xi_k \\ &= M_k(f_1, \dots, f_k)(x), \end{aligned} \tag{4.7}$$

and this establishes the claim. Now, if we choose a nonnegative function $\eta \in C_0^\infty(\mathbf{R}^n)$ satisfying $\eta(0) = 1$ and

$$\sum_{m \in \Lambda_n} (\eta(x+m))^p = 1 \quad \text{for all } x \in \mathbf{R}^n \tag{4.8}$$

(see [6], Chapter 7, Lemma 3.2.1), then we have

$$\lim_{\lambda \to 0} \lambda^{kn} (M_k^\lambda)^\circ (\tilde{f}_{1,\lambda}, \dots, \tilde{f}_{k,\lambda})(\lambda x) \eta(\lambda x) = M_k(f_1, \dots, f_k)(x) \tag{4.9}$$

for every x, since η is continuous and $\eta(0) = 1$. Then we have

$$\begin{aligned} &\int_{\mathbf{R}^n} |\lambda^{kn} (M_k^\lambda)^\circ (\tilde{f}_{1,\lambda}, \dots, \tilde{f}_{k,\lambda})(\lambda x) \eta(\lambda x)|^p\, dx \\ &= \lambda^{n(kp-1)} \int_{\mathbf{R}^n} |(M_k^\lambda)^\circ (\tilde{f}_{1,\lambda}, \dots, \tilde{f}_{k,\lambda})(y) \eta(y)|^p\, dy \\ &= \lambda^{n(kp-1)} \sum_{m \in \Lambda_n} \int_{T^n} |(M_k^\lambda)^\circ (\tilde{f}_{1,\lambda}, \dots, \tilde{f}_{k,\lambda})(y)|^p (\eta(y+m))^p\, dy \\ &= \lambda^{n(kp-1)} \int_{T^n} |(M_k^\lambda)^\circ (\tilde{f}_{1,\lambda}, \dots, \tilde{f}_{k,\lambda})(y)|^p\, dy \\ &\le \lambda^{n(kp-1)} C_p^p \int_{T^n} |\tilde{f}_{k,\lambda}(y)|^p\, dy \left(\prod_{j=1}^{k-1} \operatorname*{ess\,sup}_{y \in T^n} |\tilde{f}_j(y)| \right)^p. \end{aligned} \tag{4.10}$$

For λ sufficiently small, the supports of $\lambda^{-n} f_j(\bullet\lambda^{-1})$ lie entirely within T^n, and so this last expression equals

$$\begin{aligned}\lambda^{n(kp-1)} C_p^p \int_{\mathbf{R}^n} |\lambda^{-n} f_k(y\lambda^{-1})|^p\, dy \left(\prod_{j=1}^{k-1} \operatorname*{ess\,sup}_{y \in \mathbf{R}^n} |\lambda^{-n} f_j(y\lambda^{-1})| \right)^p \\ = C_p^p \|f\|_{L^p(\mathbf{R}^n)}^p \left(\prod_{j=1}^{k-1} \|f_j\|_{L^\infty(\mathbf{R}^n)} \right)^p.\end{aligned} \tag{4.11}$$

Therefore

$$\begin{aligned}\liminf_{\lambda \to 0} \int_{\mathbf{R}^n} |\lambda^{kn} (M_k^\lambda)^\circ (\tilde{f}_{1,\lambda}, \ldots, \tilde{f}_{k,\lambda})(\lambda x) \eta(\lambda x)|^p\, dx \\ \leq C_p^p \|f_k\|_{L^p(\mathbf{R}^n)}^p \left(\prod_{j=1}^{k-1} \|f_j\|_{L^\infty(\mathbf{R}^n)} \right)^p.\end{aligned} \tag{4.12}$$

In view of (4.9) and (4.12), the result follows by Fatou's Lemma. □

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