ON THE ZEROS OF JONQUIÈRE'S FUNCTION WITH A LARGE COMPLEX PARAMETER

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1. Introduction and summary. In this paper we are dealing with Jonquière's function (cf. [10, p. 33], [15, p. 364], [18, p. 280]) defined by its power series

$$f_\kappa(z) := \sum_1^\infty n^\kappa z^n, \quad \kappa = \kappa_0 + i\cdot\kappa_1 \in \mathbf{C} \tag{1}$$

for $|z| < 1$; by analytic continuation it is seen to be holomorphic in the cut plane

$$\mathbf{C}^* := \{z \in \mathbf{C} \mid \text{If } \operatorname{Re} z \geqq 1, \text{ then } \operatorname{Im} z \neq 0\}. \tag{2}$$

If $\kappa = k$ is a positive integer, then f_k is connected with the geometric series by the simple relation

$$f_k(z) = \left(z\frac{d}{dz}\right)^k \frac{1}{1-z}$$

[14, p. 7, problem 46]. Moreover, Jonquière's function is of some significance in various parts of mathematics and physics. For instance, it occurs in analytic number theory [8] as a generalization of Riemann's ζ-function, in summability theory concerning equivalence problems for Césaro and certain discontinuous Riesz means [13, ch. IV, 3], and in research on the structure of polymers [17]. Questions in Riesz summability, especially, require the number and the location of the zeros of f_κ in $\mathbf{C}^*$ when κ is *real*. The first complete result for this case is due to A. Peyerimhoff [12] stating that all zeros in $\mathbf{C}^*$ are real and $\leqq 0$. Moreover, they have order one and their exact number is $k+1$ if $k < \kappa \leqq k+1$, $k \in \mathbf{N}_0$, and 1 if $\kappa \leqq 0$. Different and modified proofs as well as the dependence of the zeros on the real parameter κ were given in a series of papers [2, 3, 4, 5, 6, 7, 11, 12, 16, 19]. In continuation of these investigations we ask the following questions.

(i) In case of *real* κ, how are the zeros distributed on the negative real axis if κ becomes large?

(ii) What can be said about the zeros of f_κ when κ is *complex*?

In view of the close relation of f_κ with Riemann's ζ-function (observe that $f_\kappa(-1) = (2^{\kappa+1}-1)\zeta(-\kappa)$) the second question without any restriction for κ includes the problem of finding all complex zeros of ζ. Fornberg and Kölbig [1] investigated the zeros of $f_\kappa(x)$ in the half-plane $\{\kappa \in \mathbf{C} \mid \operatorname{Re}\kappa < 0\}$, for fixed $x \in (-1, 1)$. Their considerations are restricted to the behaviour of these zeros when $x \to 0$ and $x \to 1^-$. The latter case is used to get a numerical approach to the zeros of the ζ-function. We are interested in the zeros of $f_\kappa(z)$ in the z-plane for fixed $\kappa \in \mathbf{C}$. Treating the first question above, it turns out that most of the arguments used there are also valid for complex κ, and that we can obtain good

Received August 15, 1983. Final revision received May 5, 1984.
Michigan Math. J. 31 (1984).

approximations for the zeros of f_κ in $\mathbf{C}^*$ when $|\kappa|$ becomes large in the angular region

$$W_T := \{\kappa = \kappa_0 + i\kappa_1 \mid \kappa_0 > 0,\ |\kappa_1| \leqq T(\kappa_0+1)\}, \tag{3}$$

$T \geqq 0$ arbitrarily fixed. Thus we restrict κ to values in W_T. We know from the results in [4] that for fixed complex κ the number of zeros of f_κ in $\mathbf{C}^*$ is finite. Now we consider κ on the ray

$$S_\tau := \{\kappa = \kappa_0 + i\kappa_1 \mid \kappa_1 = \tau(\kappa_0+1),\ \kappa_0 > 0\}, \quad \tau \in \mathbf{R} \text{ fixed}, \tag{4}$$

and show that for sufficiently large κ_0 the number of zeros in $\mathbf{C}^*$ is given by $[((\kappa_0+1)\xi_\tau - 1)/2] + 1 + n_0$, where $n_0 \in \{-1, 0, 1\}$ is undetermined and ξ_τ is the solution of a certain transcendental equation satisfying $\frac{3}{2}(1+\tau^2) < \xi_\tau \leqq 2(1+\tau^2)$. The central result says that for large $|\kappa|$ in W_T all zeros in $\mathbf{C}^*$ are approximated by the explicit quantities

$$\tilde{z}_{\kappa,\nu} := -\exp\left(-\pi \operatorname{cotan} \frac{2\nu+1}{\kappa+1}\frac{\pi}{2}\right), \tag{5}$$

$\nu \in \mathbf{N}_0$ suitably, except for "$o(|\kappa|)$ many" which are located in a "small" neighborhood of the origin or (if $0 < \kappa \to \infty$) are sufficiently large in modulus. The error term in this approximation is of the form $O(c^{\kappa_0})$, $0 < c < 1$, where the constants are uniform for zeros in any compact subset in $\mathbf{C}^*$ omitting the origin. If κ is restricted to S_τ in (4) it follows from (5) that the zeros are asymptotically located on the curve (Theorem 2, Lemma 1)

$$z(\xi) = -\exp\left(-\pi \operatorname{cotan} \frac{\pi\xi}{2(1+i\tau)}\right), \quad 0 < \xi < \xi_\tau,\ \tau \geqq 0, \tag{6}$$

which reduces to the negative real axis when $\tau = 0$. For the latter case the asymptotic distribution function of the zeros can also be derived from (5) in explicit form (Theorem 1).

The proofs essentially are based on the so-called Lindelöf–Wirtinger expansion of f_κ giving a representation of the analytic extension onto $\mathbf{C}^*$ (see Section 2).

2. Preliminary results. In this section we use an explicit formula for the analytic extension of f_κ valid in $\mathbf{C}^*$ which is suitable for discussing its zeros. Applying residue calculus [8, 9, 20] or Poisson's sum formula (see also [11]) we obtain from (1) the Lindelöf-Wirtinger expansion

$$f_\kappa(z) = \Gamma(\kappa+1) \sum_{m=-\infty}^{\infty} \frac{1}{(2m\pi i + \log(1/z))^{\kappa+1}}, \quad \operatorname{Re}\kappa > 0, \tag{7}$$

giving the unique analytic extension onto $\mathbf{C}^*$ where for the logarithm the principal branch in $\mathbf{C}^*$ is chosen; that is, $\log 1/z$ is real for real positive z. According to this choice the power in (7) is defined by $(u+iv)^{\kappa+1} = \exp((\kappa+1)\log(u+iv))$, where

$$\log(u+iv) = \tfrac{1}{2}\log(u^2+v^2) + i\arg(u+iv) \tag{8}$$

with

$$\arg(u+iv) = \begin{cases} \pi-\arctan(v/u), & u<0,\ v\geqq 0 \\ \pi/2, & u=0,\ v>0 \\ \arctan(v/u), & u>0 \\ -\pi/2, & u=0,\ v<0 \\ -\pi+\arctan(v/u), & u<0,\ v\leqq 0 \end{cases} \tag{9}$$

arctan being the principal branch, that is, $-\pi/2<\arctan x<\pi/2$ for real x. This exhibits again that (7) is a single valued function in $\mathbf{C}^*$ and f_κ can be extended analytically across the cut from 1 to infinity along the positive real axis with branch points 1 and ∞. For reasons of symmetry we put

$$w:=-z=re^{i\phi},\quad \log(1/w)=\log(1/r)-i\phi,\quad -\pi\leqq\phi\leqq\pi \tag{10}$$

and write

$$\begin{aligned} f_\kappa(z)&=\Gamma(\kappa+1)\sum_{m=-\infty}^{\infty}\frac{1}{((2m+1)\cdot\pi i+\log(1/w))^{\kappa+1}} \\ &=\frac{\Gamma(\kappa+1)}{(\log(1/w)+i\pi)^{\kappa+1}}\{H_\kappa(w)+R_\kappa(w)\} \end{aligned} \tag{11}$$

with

$$H_\kappa(w):=1+\left(\frac{\log(1/w)+i\pi}{\log(1/w)-i\pi}\right)^{\kappa+1} \tag{12}$$

and

$$\begin{aligned} R_\kappa(w):=(\log(1/w)+i\pi)^{\kappa+1}\sum_{m=1}^{\infty}\Bigg\{&\frac{1}{(\log(1/w)+(2m+1)\pi i)^{\kappa+1}} \\ &+\frac{1}{(\log(1/w)-(2m+1)\pi i)^{\kappa+1}}\Bigg\}. \end{aligned} \tag{13}$$

Writing $\kappa+1=(\kappa_0+1)(1+i\tau)$ (see (3) and (4)), by symmetry (see (1)) we may confine our investigations to the case $\tau\geqq 0$. We keep τ fixed throughout; that is, κ tends to infinity on the ray S_τ (see (4)). To recognize H_κ and R_κ as a principal term and a remainder for f_κ in $\mathbf{C}^*$, respectively, we have to consider H_κ and R_κ on the Riemann surface of $H_\kappa(w)$, that is, the surface of $\log([\log(1/w)+i\pi]/[\log(1/w)-i\pi])$ (see (12)). For technical reasons which will become apparent in the proof of Lemma 1 below, we replace the w-plane with a cut from 0 to ∞ along the negative reals (see (10)) by

$$\mathbf{C}_\tau^*:=\left\{w=re^{i\phi}\mid|\phi|<\pi,\ \text{if } r\geqq 1;\ \tau\log\frac{1}{r}<\pi-\phi\leqq\tau\log\frac{1}{r}+2\pi,\ \text{if } r<1\right\},\quad \tau\geqq 0, \tag{14}$$

that is, the cut connecting $w=0$ and $w=-1$ is deformed into the exponential spiral

(15) $$r = r(\phi) = \exp(-(\pi-\phi)/\tau), \quad \phi < \pi, \text{ if } \tau > 0.$$

Now the key to our results is the following approximation of f_κ by H_κ in $\mathbf{C}_\tau^*$.

LEMMA 1. (i) *Suppose that $\tau, \delta > 0$ are fixed. If $\kappa \to \infty$, $\kappa \in S_\tau$, then*

(16) $$f_\kappa(z) = \frac{\Gamma(\kappa+1)}{(\log(1/w)+i\pi)^{\kappa+1}}\{H_\kappa(w) + O(c^{\kappa_0})\},$$

where $c \in (0,1)$ depends on τ and δ only, when $w \in \mathbf{C}_\tau^* \cap \{w = re^{i\phi} \mid |\log r| \leqq 1/\delta\}$.

(ii) *Suppose that $\phi_0, \delta > 0$ are fixed. If $0 < \kappa \to \infty$ (i.e., $\tau = 0$), then* (16) *holds with $c \in (0,1)$, depending on ϕ_0 and δ only, when*

$$w \in \mathbf{C}_0^* \cap \{w = re^{i\phi} \mid 0 \leqq \pi - \phi \leqq 2\pi - \phi_0, \ |\log r| \leqq 1/\delta\}.$$

(iii) *Suppose that $\phi = 0$ (i.e., $z = -w = -r$) and $0 < \delta \leqq 1/3\pi$. If $0 < \kappa \to \infty$, then for $|\log r| \leqq 1/\delta$,*

(17) $$f_\kappa(-r) = \frac{\Gamma(\kappa+1)}{(\log(1/r)+i\pi)^{\kappa+1}}\left(H_\kappa(-r) + O\left(\frac{1}{\delta}e^{-2\pi^2\kappa\delta^2}\right)\right),$$

the constant involved in the O-term being independent of r and δ.

Proof. According to (11) we have to estimate R_κ in (13). In view of (8), (9), and (10) we have, for $w \in \mathbf{C}_\tau^*$,

$$|R_\kappa(w)|$$
$$\leqq \sum_1^\infty \left\{\frac{\log^2 r + (\pi-\phi)^2}{\log^2 r + ((2m+1)\pi - \phi)^2}\exp[2\tau\cdot(\arg(\log(1/r) + ((2m+1)\pi-\phi)i) - \arg(\log(1/r) + (\pi-\phi)i))]\right\}^{(\kappa_0+1)/2}$$

(18) $$+ \sum_1^\infty \left\{\frac{\log^2 r + (\pi-\phi)^2}{\log^2 r + ((2m+1)\pi + \phi)^2}\exp[2\tau\cdot(\arg(\log(1/r) - ((2m+1)\pi+\phi)i) - \arg(\log(1/r) + (\pi-\phi)i))]\right\}^{(\kappa_0+1)/2}$$

$$=: \sum_1^l {}^{(1)} + \sum_{l+1}^\infty {}^{(1)} + \sum_1^l {}^{(2)} + \sum_{l+1}^\infty {}^{(2)},$$

with obvious notations where l is a positive integer to be chosen suitably below. Next, we consider the function

(19) $$g_\tau(x) := (1+x^2)\exp(-2\tau \arctan x), \quad x \in \mathbf{R}, \ \tau \geqq 0,$$

arctan again being the principal branch as in (9). It is readily verified that g_τ is a convex function with

(20) the minimum at $x = \tau$ and $g_\tau(\tau - h) > g_\tau(\tau + h)$ for all $h > 0$,

where the latter statement holds for fixed $\tau > 0$.

(I) Suppose that $r \geqq 1$. Then, by (14), we have $|\phi| \leqq \pi$. Further, by (8) and (9) we get, for $m \in \mathbf{N}$,

$$\arg(\log(1/r)+((2m+1)\pi-\phi)i)-\arg(\log(1/r)+(\pi-\phi)i) \leqq 0 \tag{21}$$

and

$$\arg(\log(1/r)-((2m+1)\pi+\phi)i)-\arg(\log(1/r)+(\pi-\phi)i) \leqq -\pi/2 \tag{22}$$

which we use to obtain

$$\begin{aligned} \sum_{l+1}^{\infty}{}^{(1)}+\sum_{l+1}^{\infty}{}^{(2)} &\leqq 2\{\log^2 r+(\pi-\phi)^2\}^{(\kappa_0+1)/2} \sum_{l+1}^{\infty} \frac{1}{(2\pi m)^{\kappa_0+1}} \\ &\leqq \frac{2l}{\kappa_0}\left\{\frac{\log^2 r+(\pi-\phi)^2}{4\pi^2 l^2}\right\}^{(\kappa_0+1)/2} \end{aligned} \tag{23}$$

Next, we infer from (18), (21), and (22) that

(24)
$$\sum_{1}^{l}{}^{(1)}+\sum_{1}^{l}{}^{(2)} \leqq l\left\{\frac{\log^2 r+(\pi-\phi)^2}{\log^2 r+(3\pi-\phi)^2}\right\}^{(\kappa_0+1)/2} + l\left\{\frac{\log^2 r+(\pi-\phi)^2}{\log^2 r+(3\pi+\phi)^2}e^{-\tau\pi}\right\}^{(\kappa_0+1)/2}$$

(II) Suppose now that $r<1$. Then, by (14), we have

$$\tau\log(1/r)<\pi-\phi \leqq \tau\log(1/r)+2\pi. \tag{25}$$

Next, we observe that, for $m \in \mathbf{N}$,

$$\arg(\log(1/r)+((2m+1)\pi-\phi)i)-\arg(\log(1/r)+(\pi-\phi)i) \leqq \pi \tag{21$'$}$$

and

$$\arg(\log(1/r)-((2m+1)\pi+\phi)i)-\arg(\log(1/r)+(\pi-\phi)i) \leqq 0, \tag{22$'$}$$

giving (see (18) and (25))

$$\begin{aligned} \sum_{l+1}^{\infty}{}^{(1)}+\sum_{l+1}^{\infty}{}^{(2)} &\leqq \sum_{l+1}^{\infty}\left\{\frac{\log^2 r+(\pi-\phi)^2}{\log^2 r+((2m+1)\pi-\phi)^2}e^{2\pi\tau}\right\}^{(\kappa_0+1)/2} \\ &\quad+\sum_{l+1}^{\infty}\left\{\frac{\log^2 r+(\pi-\phi)^2}{\log^2 r+((2m+1)\pi+\phi)^2}\right\}^{(\kappa_0+1)/2} \\ &\leqq \{(\log^2 r+(\pi-\phi)^2)e^{2\pi\tau}\}^{(\kappa_0+1)/2}\sum_{l+1}^{\infty}\frac{1}{(2m\pi)^{\kappa_0+1}} \\ &\quad+\{\log^2 r+(\pi-\phi)^2\}^{(\kappa_0+1)/2}\sum_{l+1}^{\infty}\frac{1}{(2m\pi-\tau/\delta)^{\kappa_0+1}} \\ &\leqq \frac{2}{\kappa_0}\left(l-\frac{\tau}{2\pi\delta}\right)\left\{\frac{\log^2 r+(\pi-\phi)^2}{(2\pi l-\tau/\delta)^2}e^{2\pi\tau}\right\}^{(\kappa_0+1)/2}, \end{aligned} \tag{26}$$

provided $l>\tau/2\pi\delta$. For estimating $\sum_1^l{}^{(1)}$ and $\sum_1^l{}^{(2)}$, by (9), (note that $\log(1/r)>0$) we have, for $m \in \mathbf{N}$,

$$\arg(\log(1/r)+((2m+1)\pi-\phi)i)=\arctan(((2m+1)\pi-\phi)/\log(1/r))$$
$$\arg(\log(1/r)-((2m+1)\pi+\phi)i)=-\arctan(((2m+1)\pi+\phi)/\log(1/r)).$$

Then from (18) and (19) we conclude, by (20) and (25), that

$$\begin{aligned}\sum_1^l{}^{(1)}+\sum_1^l{}^{(2)}&=\sum_1^l\left\{g_\tau\left(\frac{\pi-\phi}{\log(1/r)}\right)\Big/g_\tau\left(\frac{(2m+1)\pi-\phi}{\log(1/r)}\right)\right\}^{(\kappa_0+1)/2}\\&\quad+\sum_1^l\left\{g_\tau\left(\frac{\pi-\phi}{\log(1/r)}\right)\Big/g_\tau\left(-\frac{(2m+1)\pi+\phi}{\log(1/r)}\right)\right\}^{(\kappa_0+1)/2}\\&\leqq l\left\{g_\tau\left(\frac{\pi-\phi}{\log(1/r)}\right)\Big/g_\tau\left(\frac{\pi-\phi}{\log(1/r)}+\frac{2\pi}{\log(1/r)}\right)\right\}^{(\kappa_0+1)/2}\\&\quad+l\left\{g_\tau\left(\frac{\pi-\phi}{\log(1/r)}\right)\Big/g_\tau\left(\frac{\pi-\phi}{\log(1/r)}-\frac{4\pi}{\log(1/r)}\right)\right\}^{(\kappa_0+1)/2}\end{aligned}\tag{27}$$

Now, combining (23), (24), (26), and (27), we obtain, by (18),

$$\begin{aligned}|R_\kappa(w)|&\leqq\frac{2l}{\kappa_0}\left\{\frac{\log^2 r+(\pi-\phi)^2}{4\pi^2l^2}\right\}^{(\kappa_0+1)/2}+l\left\{\frac{\log^2 r+(\pi-\phi)^2}{\log^2 r+(3\pi-\phi)^2}\right\}^{(\kappa_0+1)/2}\\&\quad+l\left\{\frac{\log^2 r+(\pi-\phi)^2}{\log^2 r+(3\pi+\phi)^2}e^{-\tau\pi}\right\}^{(\kappa_0+1)/2}\end{aligned}\tag{28}$$

if $r\geqq 1$, $|\phi|\leqq\pi$ and

$$\begin{aligned}|R_\kappa(w)|&\leqq\frac{2}{\kappa_0}\left(l-\frac{\tau}{2\pi\delta}\right)\left\{\frac{\log^2 r+(\pi-\phi)^2}{(2\pi l-\tau/\delta)^2}e^{2\pi\tau}\right\}^{(\kappa_0+1)/2}\\&\quad+l\left\{g_\tau\left(\frac{\pi-\phi}{\log(1/r)}\right)\Big/g_\tau\left(\frac{\pi-\phi}{\log(1/r)}+\frac{2\pi}{\log(1/r)}\right)\right\}^{(\kappa_0+1)/2}\\&\quad+l\left\{g_\tau\left(\frac{\pi-\phi}{\log(1/r)}\right)\Big/g_\tau\left(\frac{\pi-\phi}{\log(1/r)}-\frac{4\pi}{\log(1/r)}\right)\right\}^{(\kappa_0+1)/2}\end{aligned}\tag{29}$$

if $r<1$ and $\tau\log(1/r)<\pi-\phi\leqq\tau\log(1/r)+2\pi$.

(i) Suppose that $\tau>0$ and $|\log r|\leqq 1/\delta$. Choosing l in (28) such that $l=[(\log^2 r+(3\pi-\phi)^2)^{1/2}/2\pi]+1$, a straightforward estimation leads to

$$|R_\kappa(w)|\leqq M(\delta)c(\tau,\delta)^{\kappa_0},\quad \kappa_0\geqq 1\tag{28$'$}$$

for some $c(\tau,\delta)\in(0,1)$. Next, in (29) we obtain from (19), (20), and (25)

$$\begin{aligned}\frac{g_\tau\left(\frac{\pi-\phi}{\log(1/r)}\right)}{g_\tau\left(\frac{\pi-\phi}{\log(1/r)}+\frac{2\pi}{\log(1/r)}\right)}&\leqq\max\left\{\frac{g_\tau(\tau)}{g_\tau\left(\tau+\frac{2\pi}{\log(1/r)}\right)},\frac{g_\tau\left(\tau+\frac{2\pi}{\log(1/r)}\right)}{g_\tau\left(\tau+\frac{4\pi}{\log(1/r)}\right)}\right\}\\&\leqq\max\left\{\frac{g_\tau(\tau)}{g_\tau(\tau+2\pi\delta)},\frac{1}{4}\right\}\end{aligned}\tag{30}$$

and

$$\frac{g_\tau\left(\dfrac{\pi-\phi}{\log(1/r)}\right)}{g_\tau\left(\dfrac{\pi-\phi}{\log(1/r)}-\dfrac{4\pi}{\log(1/r)}\right)} \leqq \frac{g_\tau\left(\tau+\dfrac{2\pi}{\log(1/r)}\right)}{g_\tau\left(\tau-\dfrac{2\pi}{\log(1/r)}\right)} \leqq \frac{g_\tau(\tau+2\pi\delta)}{g_\tau(\tau-2\pi\delta)}, \tag{31}$$

where in addition we have used in (30) that $(h>0)$

$$\frac{d}{dx}(g_\tau(x)/g_\tau(x+h)) \begin{cases} <0 & \tau \leqq x < x_0 \\ >0 & x_0 < x \end{cases}$$

for some $x_0 > \tau$ and in (31) that $g_\tau(\tau+h)/g_\tau(\tau-h)$ is a decreasing function with respect to $h>0$. Now, using (20), a very similar estimate as above leads to

$$|R_\kappa(w)| \leqq M(\tau,\delta)c(\tau,\delta)^{\kappa_0}, \quad \kappa_0 \geqq 1, \tag{29'}$$

completing the proof for part (i).

(ii) The estimate for $r \geqq 1$ is an analogue to (28) in the preceding part. For (29) we only mention that (30) becomes

$$\frac{g_0\left(\dfrac{\pi-\phi}{\log(1/r)}\right)}{g_0\left(\dfrac{\pi-\phi}{\log(1/r)}+\dfrac{2\pi}{\log(1/r)}\right)} \leqq \max\left\{\frac{1}{1+(2\pi\delta)^2}, \frac{1}{4}\right\} \tag{30'}$$

and (31) has to be replaced by

$$\frac{g_0\left(\dfrac{\pi-\phi}{\log(1/r)}\right)}{g_0\left(-\dfrac{3\pi+\phi}{\log(1/r)}\right)} = \frac{\log^2 r+(\pi-\phi)^2}{\log^2 r+(3\pi+\phi)^2}, \tag{31'}$$

implying part (ii).

(iii) If $\tau=\phi=0$ and $|\log r| \leqq 1/\delta$, then inequalities (28) and (29) coincide and become

$$|R_\kappa(-r)| \leqq \frac{2l}{\kappa_0}\left\{\frac{\log^2 r+\pi^2}{(2\pi l)^2}\right\}^{(\kappa+1)/2} + 2l\left\{\frac{\log^2 r+\pi^2}{\log^2 r+(3\pi)^2}\right\}^{(\kappa+1)/2}$$

Further, the choice $l := [(\log^2 r+(3\pi)^2)^{1/2}/2\pi]+1$ leads to

$$R_\kappa(-r) = O\left(\frac{1}{\delta}\left\{\frac{1+\pi^2\delta^2}{1+9\pi^2\delta^2}\right\}^{(\kappa+1)/2}\right) = O\left(\frac{1}{\delta}e^{-2\pi^2\kappa\delta^2}\right)$$

as $\kappa \to \infty$ (observe that $\delta \leq 1/3\pi$). Now the proof of Lemma 1 is complete. □

REMARKS. (i) It should be emphasized that the estimates in the proof of Lemma 1 are good enough to give explicit bounds for R_κ (κ finite) required for numerical calculations. But for the purpose of this paper O-estimates are sufficient.

(ii) The restriction $-\pi+\phi_0 \leqq \phi \leqq \pi$ in part (ii) of the Lemma only is caused by extracting the factor $(\log(1/w)+i\pi)^{-\kappa-1}$ in (11). Extracting $(\log(1/w)-i\pi)^{-\kappa-1}$ would result in $-\pi \leqq \phi \leqq \pi-\phi_0$.

Next, we discuss the zeros of H_κ (see (12)), which turn out to be "good approximations" for those of f_κ in $\mathbf{C}^*$ (see Theorems 1 and 2).

LEMMA 2. *Suppose that $\tau \geqq 0$ is fixed. Then all zeros of H_κ in $\mathbf{C}_\tau^*$ are located on the curve K_τ with representation*

$$w(\xi) := r(\xi)e^{i\phi(\xi)} = \exp\left(-\pi \operatorname{cotan} \frac{\pi\xi}{2(1+i\tau)}\right), \quad 0<\xi\leqq\xi_\tau, \tag{32}$$

with

$$r(\xi) := \exp\left(-\pi \cdot \frac{\sin(\pi\xi/(1+\tau^2))}{\cosh(\pi\xi\tau/(1+\tau^2)) - \cos(\pi\xi/(1+\tau^2))}\right), \tag{33}$$

$$\phi(\xi) := -\pi \cdot \frac{\sinh(\pi\xi\tau/(1+\tau^2))}{\cosh(\pi\xi\tau/(1+\tau^2)) - \cos(\pi\xi/(1+\tau^2))}, \tag{34}$$

and

$$\xi_\tau := \max\left\{\xi \in (0, 2(1+\tau^2)] \mid \exp\left(\frac{\pi\xi\tau}{1+\tau^2}\right)\cos\left(\frac{\pi\xi}{1+\tau^2}\right) = 1\right\}. \tag{35}$$

For fixed $\kappa+1 = (\kappa_0+1)(1+i\tau)$ on S_τ, all zeros of H_κ in $\mathbf{C}_\tau^* \cup \{w = re^{-i\pi} \mid r \geqq 1\}$ are given by

$$w_{\kappa,\nu} := w\left(\frac{2\nu+1}{\kappa_0+1}\right) = \exp\left(-\pi \operatorname{cotan} \frac{2\nu+1}{\kappa+1}\frac{\pi}{2}\right), \quad \nu = 0, \ldots, N, \tag{36}$$

where

$$N = N(\kappa_0, \tau) = [((\kappa_0+1)\xi_\tau - 1)/2]. \tag{37}$$

REMARK. (i) As an immediate consequence of (35) we have

$$\tfrac{3}{2}(1+\tau^2) < \xi_\tau \leqq 2(1+\tau^2) \quad \text{and} \quad \xi_\tau \sim 2(1+\tau^2) \text{ as } \tau \to 0, \infty.$$

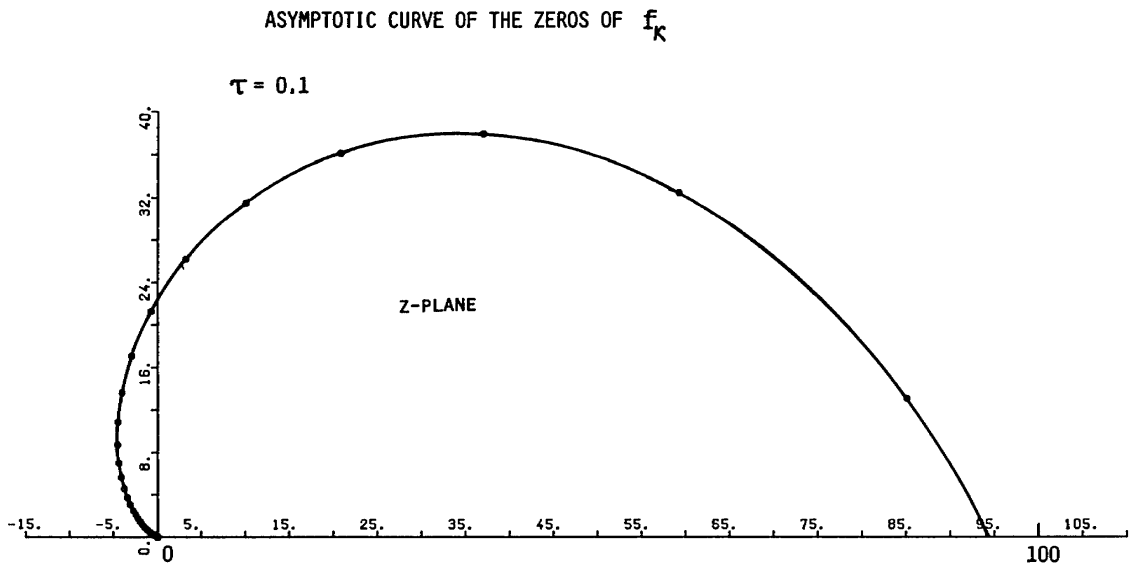

Figure 1

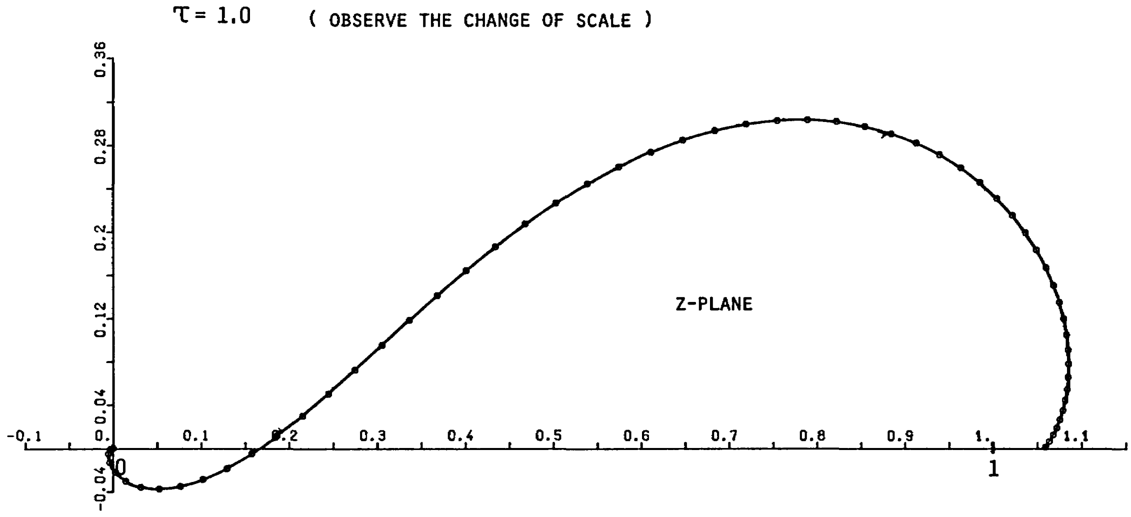

Figure 2

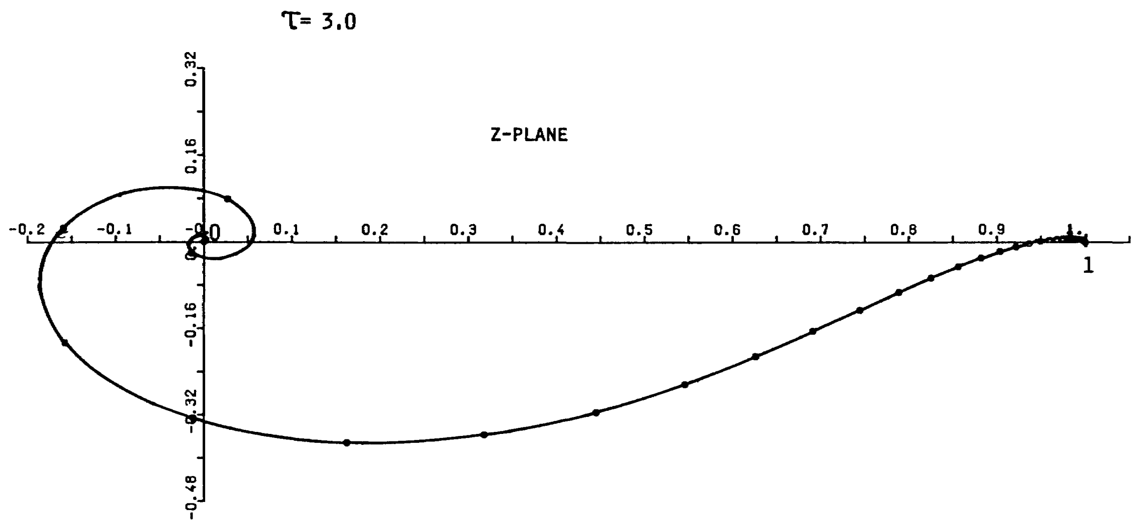

Figure 3

(ii) It follows from (32), (33), and (34) that K_0 is the positive real axis (in the w-plane!) and for $\tau > 0$, K_τ is a spiral with the origin as an asymptotic point ($\xi \to 0$). Moreover we have $r(\xi) < 1$ if $\xi < 1+\tau^2$, $r(\xi) > 1$ if $\xi > 1+\tau^2$, and K_τ traverses the unit circle in $w(1+\tau^2) = \exp(-i\pi \tanh(\pi\tau/2))$. K_τ "leaves" $\mathbf{C}_\tau^*$ at $w(\xi_\tau) \in (-\infty, -1)$ and $w(\xi_\tau) \to -1$ as $\tau \to \infty$ and $w(\xi_\tau) \to -\infty$ as $\tau \to 0$. For various values of τ we have drawn $-K_\tau$ (the curve of the zeros in the z-plane!) in Figures 1, 2, and 3.

Proof of Lemma 2. First we observe that all zeros of H_κ (on its Riemann surface) are the solutions of the equation (see (12))

$$\left(\frac{\log(1/w)+i\pi}{\log(1/w)-i\pi}\right)^{\kappa+1} = -1$$

which clearly are given by $w_{\kappa,\nu}$, $\nu \in \mathbf{Z}$, in (36), and they are located on the curve with representation $w(\xi)$, $\xi \in \mathbf{R}$, in (32). All we have to show is that $w(\xi) \in \mathbf{C}_\tau^*$ precisely for $\xi \in (0, \xi_\tau)$ (see (35)). To this end we verify that

$$\tau \log \frac{1}{r(\xi)} < \pi - \phi(\xi) < \tau \log \frac{1}{r(\xi)} + 2\pi, \quad 0 < \xi < 1 + \tau^2 \tag{38}$$

$$r(\xi) \geqq 1, \ -\pi \leqq \phi(\xi) \leqq -\pi \tanh(\pi\tau/2), \quad 1 + \tau^2 \leqq \xi < \xi_\tau \tag{39}$$

$$w(\xi) \notin \mathbf{C}_\tau^*, \quad \xi \geqq \xi_\tau \text{ or } \xi < 0. \tag{40}$$

We consider the map $w = \exp(-\pi \operatorname{cotan} \pi\zeta)$ and determine the inverse image of $\mathbf{C}_\tau^*$ in the ζ-plane (see (32)). Writing

$$\zeta = \frac{1}{2\pi i} \log \left\{ \frac{(1/\pi i) \log(1/w) + 1}{(1/\pi i) \log(1/w) - 1} \right\} \tag{41}$$

where the logarithms are defined in (8), (9), and (10), we decompose (41) by

$$w' := \frac{1}{\pi i} \log(1/w), \quad \log(1/w) = \log(1/r) - i\phi,$$

$$\zeta' := \frac{w' + 1}{w' - 1},$$

$$\zeta := \frac{1}{2\pi i} \log \zeta', \quad \log \zeta' = \log|\zeta'| + i \arg \zeta', \quad 0 \leqq \arg \zeta' < 2\pi.$$

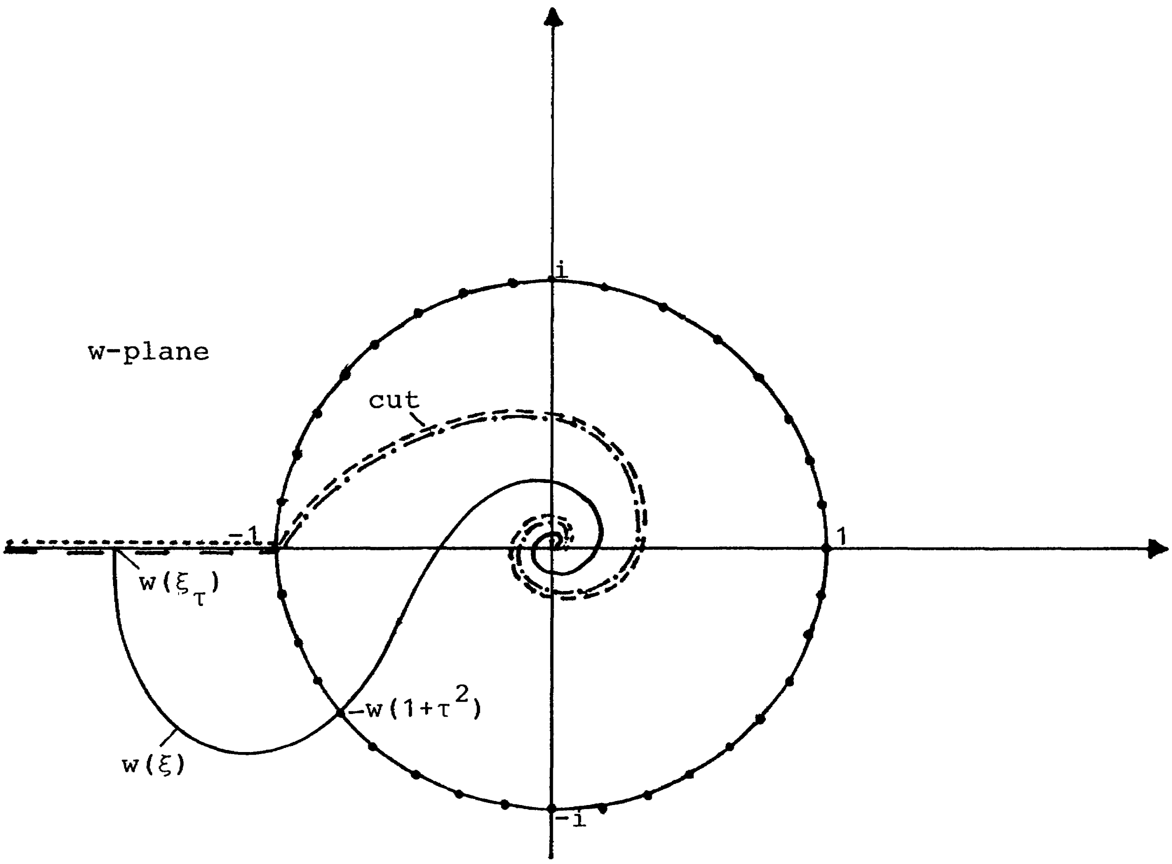

Figure 4

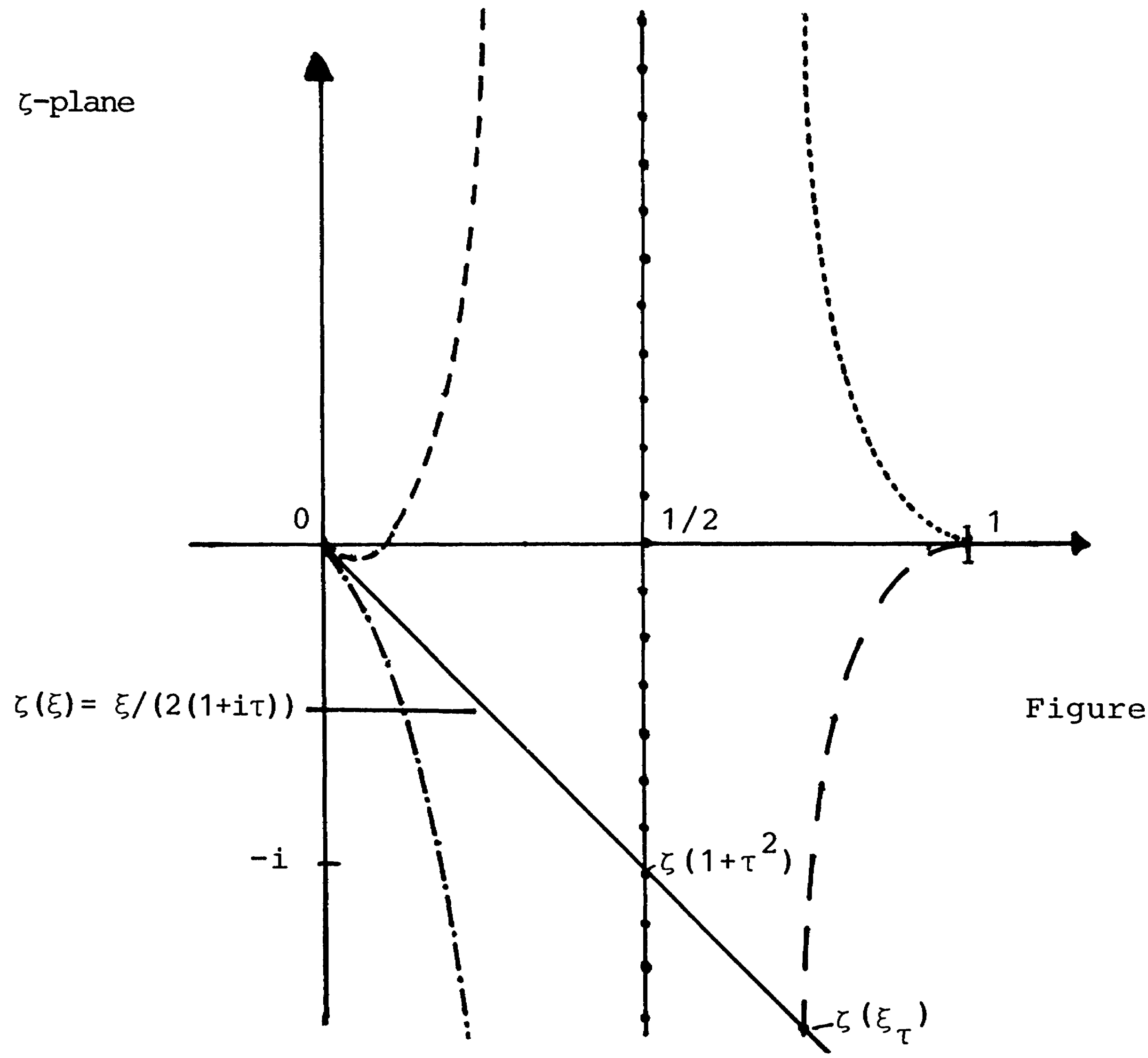

Figure 5

Now a straightforward computation gives that $\mathbf{C}_\tau^*$ is mapped into the strip $\{\zeta \mid 0<\operatorname{Re}\zeta<1\}$ not containing the half-line $\{\zeta=\xi/2(1+i\tau) \mid \xi<0\}$, which proves $w(\xi)\notin \mathbf{C}_\tau^*$ when $\xi<0$, that is, part of (40).

Next, it follows from (33) and (34) that K_τ, starting with $\xi=0$, leaves the unit circle in $w(1+\tau^2)=\exp(-i\pi\tanh(\pi\tau/2))$, and then, staying outside the unit circle, passes the real axis at $w(\xi_\tau)>1$, ξ_τ being the unique solution of $\phi(\xi)=-\pi$, subject to $1+\tau^2<\xi\leqq 2(1+\tau^2)$, which is equivalent to (35). For $\xi\geqq\xi_\tau$ the geometrical considerations above (see also Figure 5) imply that $w(\xi)\notin \mathbf{C}_\tau^*$, and hence (39) and (40) are true. Finally, by (33) and (34), we rewrite (38) as

$$\tau\sin\frac{\pi\xi}{1+\tau^2}<\cosh\frac{\pi\xi\tau}{1+\tau^2}-\cos\frac{\pi\xi}{1+\tau^2}+\sinh\frac{\pi\xi\tau}{1+\tau^2}$$

$$<\tau\sin\frac{\pi\xi}{1+\tau^2}+2\cosh\frac{\pi\xi\tau}{1+\tau^2}-2\cos\frac{\pi\xi}{1+\tau^2}.$$

The first inequality is true, since $\cos(\pi\xi/(1+\tau^2)) < \cosh(\pi\xi\tau/(1+\tau^2))$, $\xi > 0$, and the second is equivalent to $(x = \pi\xi/(1+\tau^2))$

$$e^{\tau x}\cos x < 1 + \tau e^{\tau x}\sin x, \quad 0 < x < \pi$$

which follows by considering derivatives. Now the proof of Lemma 2 is complete. □

3. Main results. In this section we translate the results on the zeros of H_κ into those of f_κ via the approximation of Lemma 1. First we deal with the case of real $\kappa > 0$ (i.e., $\tau = 0$). A. Peyerimhoff [12, Theorem 4, p. 204] proved that the zeros of f_κ in $\mathbf{C}^*$ are all $\leqq 0$ and simple. Writing $k < \kappa \leqq k+1$, $k \in \mathbf{N}_0$, they are exactly $k+1$ in number; if $k = 2m$ or $k = 2m+1$, $m \in \mathbf{N}_0$, then exactly m of them are located in the interval $(-1, 0)$. Moreover $z = -1$ is a zero if and only if κ is an even positive integer.

According to this result we assume the zeros $z_{\kappa,\nu}$ of f_κ, $\nu = 0, \ldots, k$, to be numbered such that

$$z_{\kappa,k} < z_{\kappa,k-1} < \cdots < z_{\kappa,1} < z_{\kappa,0} = 0. \tag{42}$$

Further, for $-x_2 < -x_1 < 0$ we denote by

$$n(\kappa; x_1, x_2) \text{ the number of } z_{\kappa,\nu}\text{'s in } [-x_2, -x_1]. \tag{43}$$

Then we obtain more precise information about the zeros, when κ becomes large, from the following.

THEOREM 1. *Suppose that $\kappa > 0$ and $\delta \in (0, 1/3\pi]$ is fixed.*

(i) *If κ is sufficiently large, then*

$$z_{\kappa,\nu} = -\exp\left(-\pi \operatorname{cotan} \frac{2\nu+1}{\kappa+1}\frac{\pi}{2}\right) + O(c^\kappa), \tag{44}$$

where

$$\frac{\kappa+1}{\pi} \operatorname{arc\,cotan} \frac{1}{\pi\delta} < \nu < (\kappa+1)\left(1 - \frac{1}{\pi} \operatorname{arc\,cotan} \frac{1}{\pi\delta}\right) \tag{45}$$

and $c \in (0,1)$ depends only on δ.

(ii) *The asymptotic distribution of the zeros is given by the limit* (see (43))

$$\lim_{\kappa\to\infty} \frac{1}{k+1} n(\kappa; x_1, x_2) = \frac{1}{\pi} \operatorname{arc\,cotan}\left(\frac{1}{\pi}\log\frac{1}{x_2}\right) - \frac{1}{\pi} \operatorname{arc\,cotan}\left(\frac{1}{\pi}\log\frac{1}{x_1}\right). \tag{46}$$

Proof. We use the approximation of f_κ by H_κ in Lemma 1 (iii), and in view of Lemma 2 we put

$$z_{\kappa,\nu} = -\exp(-\pi \operatorname{cotan} x_{\kappa,\nu}), \quad x_{\kappa,\nu} := \frac{2\nu+1}{\kappa+1}\frac{\pi}{2} - \epsilon_{\kappa,\nu}, \tag{47}$$

where $\epsilon_{\kappa,\nu} \in \mathbf{R}$ and $\nu \in \mathbf{N}$ have to be chosen suitably.

(i) In order to apply Lemma 1 (iii), we restrict $\nu = \nu(\kappa)$ and $\epsilon_{\kappa,\nu}$ subject to

$$\pi|\operatorname{cotan} x_{\kappa,\nu}| \leqq 1/\delta \tag{48}$$

which gives, as $\kappa \to \infty$ (observe $x_{\kappa,\nu} \in (0, \pi)$),

$$
\begin{aligned}
f_\kappa(z_{\kappa,\nu}) &= \frac{\Gamma(\kappa+1)}{(\pi \operatorname{cotan} x_{\kappa,\nu}+i\pi)^{\kappa+1}}\left\{1+\left(\frac{\pi\operatorname{cotan} x_{\kappa,\nu}+i\pi}{\pi\operatorname{cotan} x_{\kappa,\nu}-i\pi}\right)^{\kappa+1}+O\left(\frac{1}{\delta}e^{-2\pi^2\delta^2\kappa}\right)\right\} \\
&= \frac{2\Gamma(\kappa+1)\sin^{\kappa+1} x_{\kappa,\nu}}{\pi^{\kappa+1}}\left\{(-1)^\nu \sin((\kappa+1)\epsilon_{\kappa,\nu})+O\left(\frac{1}{\delta}e^{-2\pi^2\delta^2\kappa}\right)\right\}.
\end{aligned} \tag{49}
$$

To ensure that f_κ changes sign at $z_{\kappa,\nu}$ and that $-w_{\kappa,\nu}$ (see (36)) is an approximation for $z_{\kappa,\nu}$, we require in addition to (48) that

$$
\begin{aligned}
(\kappa+1)\epsilon_{\kappa,\nu} &= o(1) \\
e^{-2\pi^2\delta^2\kappa} &= o((\kappa+1)\epsilon_{\kappa,\nu})
\end{aligned} \tag{50}
$$

as $\kappa\to\infty$. Thus we choose

$$\epsilon_{\kappa,\nu} := e^{-2\pi^2\delta^2\kappa} =: c^\kappa$$

in accordance with (50). Now, by (48), a straightforward computation leads from (47) to (44) and (45) is implied by (48).

(ii) Given $-x_2<-x_1<0$ we fix $\delta\in(0,1/3\pi]$ such that $|\log x_i|\leqq 1/\delta$, that is, $e^{-1/\delta}\leqq x_1<x_2\leqq e^{1/\delta}$. Thus we may apply (44) or (47) to those ν with $-x_2\leqq z_{\kappa,\nu}\leqq -x_1$ and obtain, as $k\to\infty$,

$$
\begin{aligned}
\frac{1}{k+1}n(\kappa;x_1,x_2) &= \frac{1}{k+1}|\{\nu\in\mathbf{N}\mid -x_2\leq z_{\kappa,\nu}\leq -x_1\}| \\
&= \frac{1}{k+1}\left|\left\{\nu\in\mathbf{N}\mid \frac{\kappa+1}{\pi}\left(\operatorname{arc\,cotan}\left(\frac{1}{\pi}\log\frac{1}{x_1}\right)+\epsilon_{\kappa,\nu}\right)-\frac{1}{2}\leqq\nu\right.\right. \\
&\qquad\left.\left.\leqq\frac{\kappa+1}{\pi}\left(\operatorname{arc\,cotan}\left(\frac{1}{\pi}\log\frac{1}{x_2}\right)+\epsilon_{\kappa,\nu}\right)-\frac{1}{2}\right\}\right| \\
&\to \frac{1}{\pi}\operatorname{arc\,cotan}\left(\frac{1}{\pi}\log\frac{1}{x_2}\right)-\frac{1}{\pi}\operatorname{arc\,cotan}\left(\frac{1}{\pi}\log\frac{1}{x_1}\right).
\end{aligned}
$$

REMARK. It should be mentioned that it is possible to enlarge the interval (45) for ν by letting $\delta=\delta_\kappa$ tend to zero at a certain rate. Essentially this is due to approximation (17) in Lemma 1 containing the explicit dependence of the O-term on δ. However, then the geometric rate of approximation in (44) is lost.

Further investigations of f_κ gave rise to A. Peyerimhoff's conjecture that the moduli of the relative extrema are increasing and decreasing in the intervals $(-\infty,-1)$ and $(-1,0)$, respectively, which was supported by some numerical calculations as well. Observing the simple recurrence relation $f_{\kappa+1}(z)=zf_\kappa'(z)$ (see (1)) we show that this is true asymptotically by the following.

COROLLARY TO THEOREM 1. *Suppose that $\kappa>0$ and $\delta\in(0,1/3\pi]$ is fixed. Then for sufficiently large κ we have*

$$
\begin{aligned}
f_\kappa(z_{\kappa+1,\nu}) &= 2\frac{\Gamma(\kappa+1)}{\pi^{\kappa+1}}\sin^{\kappa+1}\left(\frac{2\nu+1}{\kappa+2}\frac{\pi}{2}+O(\kappa\cdot c^\kappa)\right) \\
&\quad\times\left\{(-1)^\nu\sin\left(\frac{2\nu+1}{\kappa+2}\cdot\frac{\pi}{2}+O(\kappa\cdot c^\kappa)\right)+O(c^\kappa)\right\},
\end{aligned}
$$

where c and ν are specified as in Theorem 1.

Proof. As in the proof of Theorem 1, for ν satisfying (45) we obtain from Lemma 1 (iii) and (47), as $\kappa \to \infty$,

$$f_\kappa(z_{\kappa+1,\nu}) = \frac{\Gamma(\kappa+1)}{(\pi \operatorname{cotan} x_{\kappa+1,\nu} + i\pi)^{\kappa+1}} \left\{ 1 + \left(\frac{\pi \operatorname{cotan} x_{\kappa+1,\nu} + i\pi}{\pi \operatorname{cotan} x_{\kappa+1,\nu} - i\pi} \right)^{\kappa+1} + O(c^\kappa) \right\},$$

and thus the corollary. (Observe also formula (49).) □

Next, we turn to complex κ. In [4] it was shown that for every fixed $\kappa \in \mathbf{C}$, f_κ has a finite number of zeros in $\mathbf{C}^*$ only. For large κ on S_τ, $\tau > 0$ (see (41)), we extend this result considerably by the following.

THEOREM 2. *Suppose that $\tau, \delta > 0$ are fixed such that*

$$\sup_{z \in -K_\tau \cap \mathbf{C}^*} |z| < \frac{1}{2} e^{1/\delta}, \tag{51}$$

$\kappa + 1 = (\kappa_0 + 1)(1 + i\tau)$ *and* $n(\kappa_0, \tau)$ *is the number of zeros of* f_κ *in* $\mathbf{C}^*$.

(i) *If κ_0 is sufficiently large, then*

$$n(\kappa_0, \tau) = [((\kappa_0 + 1)\xi_\tau - 1)/2] + 1 + n_0, \tag{52}$$

where $n_0 \in \{-1, 0, 1\}$ *is undetermined and* ξ_τ *is defined in* (35).

(ii) *If* $z_{\kappa,\nu}$ *denote the zeros of* f_κ *in* $\mathbf{C}^* \cap \{z \mid \log(1/|z|) \leqq 1/\delta\}$, *then we have*

$$z_{\kappa,\nu} = -\exp\left(-\pi \operatorname{cotan} \frac{2\nu + 1}{\kappa + 1} \frac{\pi}{2} \right) + O(c^{\kappa_0}) \tag{53}$$

as $\kappa_0 \to \infty$, *where* $\nu = \nu_0(\kappa_0, \delta), \dots, n(\kappa_0, \tau)$ (ν_0 *a suitable positive integer*) *and* $c \in (0, 1)$ *depends on* δ *only.*

REMARK. (i) Part (ii) states that except for a small neighborhood of the origin the zeros of f_κ can be approximated by those of H_κ; in particular the $z_{\kappa,\nu}$ are located on the curve $-K_\tau$ (in the z-plane, see Lemma 2 and the following remark) asymptotically.

(ii) For sufficiently small δ, from (33) we may determine $\nu_0(\kappa_0, \delta) \approx [\kappa_0 \cdot \delta]$, approximately.

Proof of Theorem 2. First we prove part (ii). Since for fixed κ [4, p. 375]

$$f_\kappa(z) \sim \frac{-1}{\Gamma(1-\kappa)} \frac{1}{(\log(-z))^\kappa} \quad \text{as } z \to \infty,\ z \in \mathbf{C}^*,$$

we may assume without loss of generality that δ in (51) is already chosen such that f_κ has no zero for $|z| > e^{1/\delta}$. Condition (51) then guarantees that $-K_\tau \subset \mathbf{C}^* \cap \{z \mid |z| \leqq e^{1/\delta}\}$ and all zeros of f_κ in $\mathbf{C}^*$ are already contained in $\{z \mid |z| \leqq e^{1/\delta}\}$. Then for $|\log|z|| \leqq 1/\delta$ we may use the approximation (16) in Lemma 1

$$f_\kappa(z) = \frac{\Gamma(\kappa+1)}{(\log(-1/z) + i\pi)^{\kappa+1}} (H_\kappa(-z) + O(c^{\kappa_0})). \tag{16'}$$

The zeros of H_κ in $\{z \mid |\log|z|| \leqq 1/\delta\}$ are given by

(5) $$\tilde{z}_{\kappa,\nu} = -\exp\left(-\pi \operatorname{cotan} \frac{2\nu+1}{\kappa+1}\frac{\pi}{2}\right), \quad \nu_0 \leqq \nu \leqq N,$$

where $\nu_0 = \nu_0(\kappa, \delta)$ is chosen suitably from $\mathbf{N}$ (Lemma 2). Next, we consider a fixed z with $|\log|z|| \leqq 1/\delta$ and write

(54) $$z = -\exp\left(-\pi \operatorname{cotan}\left(\frac{2\nu+1}{\kappa+1}\frac{\pi}{2} - \epsilon\right)\right)$$

where $\epsilon = \epsilon(\nu, z)$, and suppose that

(55) $$|z - \tilde{z}_{\kappa,\mu}| \geqq c^{\kappa_0} \quad \text{for all } \tilde{z}_{\kappa,\mu}.$$

From (16′) we obtain (compare (49))

(56) $$\begin{aligned} f_\kappa(z) = {} & \frac{\Gamma(\kappa+1)}{\pi^{\kappa+1}\left(\operatorname{cotan}\left(\frac{2\nu+1}{\kappa+1}\frac{\pi}{2} - \epsilon\right) + i\right)^{\kappa+1}} \\ & \times \left\{ H_\kappa\left(\exp\left(-\pi \operatorname{cotan}\left(\frac{2\nu+1}{\kappa+1}\cdot\frac{\pi}{2} - \epsilon\right)\right)\right) + O(c^{\kappa_0})\right\} \\ = {} & \frac{\Gamma(\kappa+1)}{\pi^{\kappa+1}} \sin^{\kappa+1}\left(\frac{2\nu+1}{\kappa+1}\frac{\pi}{2} - \epsilon\right) i(-1)^\nu \cdot e^{i\epsilon(\kappa+1)} \cdot \{1 - e^{-2i\epsilon(\kappa+1)} + O(c^{\kappa_0})\}, \end{aligned}$$

and from (5), by Cauchy's formula (see Figure 6),

$$\begin{aligned} c^{\kappa_0} & \leqq |z - z_{\kappa,\mu}| \\ & = \left| \frac{1}{2\pi i} \int_\gamma e^{-\pi \operatorname{cotan} \pi\zeta} \left\{ \frac{1}{\zeta - \left(\frac{1}{2}\cdot\frac{2\nu+1}{\kappa+1} - \frac{\epsilon}{\pi}\right)} - \frac{1}{\zeta - \frac{1}{2}\cdot\frac{2\mu+1}{\kappa+1}} \right\} d\zeta \right| \\ & \leqq M(\delta, z) \left| \frac{\nu-\mu}{\kappa+1} - \frac{\epsilon}{\pi} \right|. \end{aligned}$$

Thus we have

(57) $$|\epsilon(\kappa+1) - r\pi| \geqq M_1(\delta, z)|\kappa+1|c^{\kappa_0} \quad \text{for all } r \in \mathbf{Z}.$$

Finally we conclude (observe the periodicity of the exponential) from (57) (choose $r = 0$) for $-\pi/2 \leqq \operatorname{Re} v \leqq \pi/2$, $|v| \geqq |\kappa+1|c^{\kappa_0}M_1$,

$$|1 - e^{-2iv}| = \left|\frac{1 - e^{-2iv}}{v}\right| |v| \geqq M_2(\delta, z)|\kappa+1|c^{\kappa_0}$$

if $\operatorname{Im} v \geqq -M_1|\kappa+1|c^{\kappa_0}$, and

$$|1 - e^{-2iv}| \geqq 1 - \exp(-2M_1|\kappa+1|c^{\kappa_0}) \geqq M_3(\delta, z)|\kappa+1|c^{\kappa_0}$$

if $\operatorname{Im} v < -M_1|\kappa+1|c^{\kappa_0}$. Combining these estimates together with (56) we have shown that $f_\kappa(z) \neq 0$, for z satisfying (55), provided κ_0 is sufficiently large. Finally, applying the argument principle to (56) with $\epsilon = c^{\kappa_0}e^{i\theta}$ $(0 \leqq \theta \leqq 2\pi)$, in (54) we

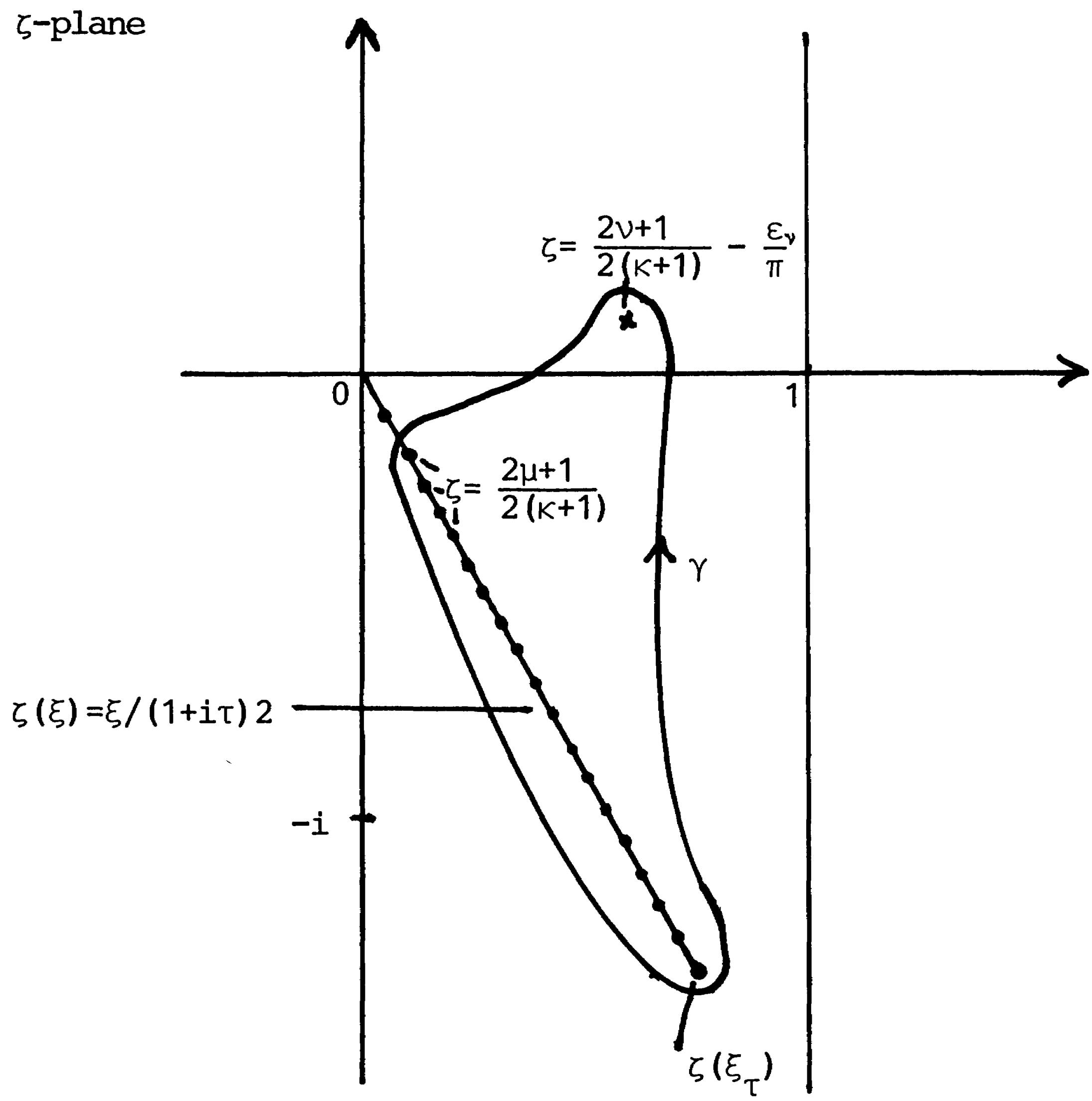

Figure 6

see that z surrounds a zero $z_{\kappa,\nu}$ of f_κ. In a way similar to the deduction of (44) from (47) we obtain (53).

It remains to show the formula for $n(\kappa_0, \tau_0)$ in part (i) for every $\tau_0 > 0$. In the following we choose κ_0 large enough but fixed and start with the case $\tau = 0$. Then we know that for $k < \kappa_0 \leqq k+1$, f_κ has exactly $k+1$ nonpositive zeros $z_{\kappa,\nu}$, $\nu = 0, \ldots, k = n(\kappa_0, 0) - 1$, ordered as in (42). To $z_{\kappa,\nu}$ we attach the corresponding zero of H_κ, $\tilde{z}_{\kappa,\nu}$, for $\nu = 0, \ldots, k$. In the region $R_\delta = \{z \mid |\log|z|| \leqq 1/\delta\}$ the $\tilde{z}_{\kappa,\nu}$ are good approximations for $z_{\kappa,\nu}$ (see (53)) for all $\tau \in [0, \tau_0]$. Observing that

(i) $z_{\kappa,\nu}$ and $\tilde{z}_{\kappa,\nu}$ are continuous in $\tau \geqq 0$,

(ii) in R_δ a zero of f_κ is always accompanied by a zero of H_κ and conversely,

(iii) the zeros of f_κ remain separated in R_δ because of (53),

(iv) $|\tilde{z}_{\kappa,0}| \leqq \cdots \leqq |\tilde{z}_{\kappa,\nu_\tau}| \leqq 1 \leqq |\tilde{z}_{\kappa,\nu_\tau+1}|$, with $\nu_\tau := [(\kappa_0+1)(1+\tau^2)]$,

a zero $z_{\kappa,\nu}$ of f_κ in $R_\delta \cap \{z \mid |z| \leqq 1\}$ has again number ν if the zeros of f_κ are ordered with respect to their moduli, for all $\tau \in [0, \tau_0]$. The zeros outside of the unit circle can be controlled by (53) for $\tau=\tau_0$. Hence in order to count the number of zeros of f_κ we count the zeros of H_κ, but we have to be careful with the last one located near the cut from 1 to ∞. Thus

$$n(\kappa_0,\tau_0)=N(\kappa_0,\tau_0)+1+\begin{cases} 1 & \text{if } z_{\kappa,N+1}\in\mathbf{C}^* \ (\tilde z_{\kappa,N+1}\notin\mathbf{C}^* \text{ by definition}) \\ 0 & z_{\kappa,N+1}\notin\mathbf{C}^*,\ z_{\kappa,N}\in\mathbf{C}^* \\ -1 & z_{\kappa,N}\notin\mathbf{C}^* \end{cases}$$

provided κ_0 is large enough. Note that the approximation in Lemma 1 also holds in a neighborhood of $z=-w(\xi_\tau)$ on the surface of f_κ. Now the proof of Theorem 2 is complete. □

REMARK. (i) Actually the proof of Theorem 2 shows that the zeros $z_{\kappa,\nu}$ in (53) are simple.

(ii) A rough estimation for the "small" zeros of f_κ (for large κ) is given by the first-step approximation of Graeffe's algorithm for the zeros of $\sum_{n=1}^{m} n^\kappa z^n$, $m\in\mathbf{N}$. We obtain, for small m,

$$z_\nu=-\exp[-\kappa_0\log(\nu+1/\nu)-i\kappa_1\log(\nu+1/\nu)] \quad \nu=1,\dots,m-1.$$

Compare these with the approximations in [1] for the case $x\to 0$.

(iii) As a numerical example we consider the case $\kappa_0=5$, $\tau=0$, 0.5. Here we obtain, up to an error $<10^{-4}$,

	$\tilde z_{\kappa,\nu}=(r,\phi)$	$z_{\kappa,\nu}=(r,\phi)$	$z_\nu=(r,\phi)$
$\tau=0$	0.0432, π	0.0431, π	0.0313, π
	0.4309, π	0.4306, π	0.1316, π
	2.3225, π	2.3225, π	
	23.2039, π	23.2039, π .	

Note that

$$f_5(z)=\left(z\frac{d}{dz}\right)^5\frac{1}{1-z} \quad \text{and}$$

$$z_{5,1(4)}=-\frac{13+\sqrt{105}}{2}\overset{+}{(-)}\sqrt{\frac{(13+\sqrt{105})^2}{4}-1}$$

$$z_{5,2(3)}=-\frac{13-\sqrt{105}}{2}\overset{+}{(-)}\sqrt{\frac{(13-\sqrt{105})^2}{4}-1}\ .$$

$\tau=0.5$	0.0355, 0.7880	0.0357, 0.8110	0.0313, 1.062
	0.2738, 1.2728	0.2738, 1.2758	0.1316, 1.925
	0.8184, 1.1652	0.8182, 1.1664	
	1.5558, 0.7225	1.5556, 0.7236	
	1.8320, 0.1244	1.8330, 0.1225	

We see that even for small κ_0 the approximations are rather good. The quality of approximation decreases with increasing τ. In case $\kappa_0=10$, at least 4 relevant digits are correct in all cases above. Furthermore $N(\kappa_0,\tau)+1=n(\kappa_0,\tau)$ in our example. Especially the points $\tilde{z}_{\kappa,\nu}$ can be used as starting points for the Newton algorithm to compute the zeros of f_κ.

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