ACTIONS OF SU(2) ON S^7

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1. Introduction. It is well known that SU(2) can act freely on S^7. This paper is concerned with *almost free* actions. An action of a compact Lie group is called *almost free* if every isotropy group is finite, which implies that every orbit has the same dimension as the group. Although SO(3) cannot act freely on S^7, Oliver [5] has shown that SO(3) can act almost freely on S^7. (This shows that a theorem in [2] is false.) The group SO(3) is the quotient by the center C of SU(2), and C consists of 2 elements, namely ± 1 if SU(2) is considered as the group of unit quaternions. Thus Oliver's work shows that there is a non-free but almost free action of SU(2) on S^7. However this action is not effective. The main result of this note is as follows.

THEOREM 1. *Let $G = \mathrm{SU}(2) = S^3$ act smoothly, effectively, and almost freely on S^7. If the fixed point set of the center C is an empty set, that is $F(C) = \varnothing$, then the action is free.*

This theorem may possibly be true in dimension $4k-1$, $k \geqslant 2$.

An example of an *effective* almost free action of SU(2) on S^7 may be constructed at least topologically. In order to do this take the join of SU(2) acting on itself and SU(2) acting on SU(2)/I where I is the doubled icosahedral group. As a space the join is S^7 by the double suspension theorem (see e.g., Cannon, Bull. Am. Math. Soc., 84, 1978, pp. 832–866), and $F(C) = \mathrm{SU}(2)/I$. This may be the only example with $F(C) \neq \varnothing$.

As will be seen the proof amounts essentially to showing that a certain kind of action of N (N = normalizer of the circle group) on a mod p 3-sphere cannot be extended to an action of SU(2) in the way which would be required.

2. General remarks. This section lists two facts about general actions of compact Lie groups which will be useful in this paper. It is assumed that G is a compact Lie group acting on a manifold M with base space M^* and projection $\pi : M \to M^*$.

I. *Let A be a closed connected subset of M which is a cross-section of the orbits it touches. Let all isotropy groups G_x, $x \in A$, be conjugate. Then $G(A)$ is a topological product $G(A) = A \times G/G_a$ for any fixed $a \in A$.*

Proof. For any $x \in A$ there is a homeomorphism $G(a) \to G(x)$ given by $gG_a(a) \to gG_x(x)$, $g \in G$. This determines the desired homeomorphism because G_x, $x \in A$, varies in a continuous way [4]. □

II. *If S^* is a closed path in M^* and if all orbits in S^* are of the same isotropy type, then there is a cross-section S of $\pi^{-1}(S^*)$ and $\pi^{-1}S^* = X \times G/G_a$, for any*

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fixed $a \in S$. The S exists for any fibering which is a local product and has arc connected fibers.

For raising paths see [4].

3. The principal isotropy group. The only finite subgroups of SU(2) which do not contain C are the finite cyclic subgroups of odd order. To see this use the fact that $SU(2)/C = SO(3)$ and that the finite subgroups of SO(3) are known. By the assumption of the theorem these are the only possible isotropy groups. If p is an odd prime, the next lemma gives information on $GF(Z_p)$ that is the orbit under G of the fixed point set $F(Z_p)$; see [2, 3].

LEMMA 1. *Suppose $F(Z_p) \neq \varnothing$. Then $GF(Z_p)$ is a fiber space with base P^2 and fiber $F(Z_p)$. Hence $GF(Z_p)$ is a manifold whose dimension is dimension $F(Z_p)+2$, and which is non-orientable.*

Let T be the circle subgroup which includes Z_p. There is no loss of generality in assuming T includes ± 1, $\pm i$. Let N be the normalizer of T so $N = T \cup jT$. The set $F(Z_p)$ is invariant under N and $g \notin N$ implies $F(Z_p) \cap gF(Z_p) = \varnothing$. This shows that if $g, h \in G$ then $gF(Z_p)$ and $hF(Z_p)$ either coincide or have a null intersection.

Let K be a small smooth 2-cell in G which is a slice of the gN at e. Now suppose $k_1 x = k_2 y$, $k_1, k_2 \in K$, $x, y \in F(Z_p)$. Then

$$k_2^{-1} k_1 x = y \quad \text{and} \quad k_1 = k_2 n \quad (n \in N),$$

and K would not be a slice. This shows that $GF(Z_p)$ is a local product and thus a fiber space with base $G/N = P^2$. It remains to prove that $GF(Z_p)$ is non-orientable. It is known [1] that $F(Z_p)$ is orientable.

Let $\alpha^*(t)$, $0 \leqslant t \leqslant 1$ with $\alpha^*(0) = \alpha^*(1) =$ point of P^2 corresponding to N, and let this path reverse the local orientation in P^2. If π is the projection from G to G/N let $\alpha(t)$ be a path above $\alpha^*(t)$, that is, $\pi\alpha(t) = \alpha^*(t)$. Such a path is known to exist. The end $\alpha(1)$ is in N, that is, in $T \cup jT$.

As was mentioned, $F(Z_p)$ is an orientable manifold. If $\alpha(1) \in T$, then $\alpha(1): F(Z_p) \to F(Z_p)$ cannot reverse orientation. If $\alpha(1) \in jT$, assume $\alpha(1) = j$. The element j is of period 4. If it reverses orientation it must have a fixed point, since $F(Z_p)$ is a 3-sphere modulo the rationals. If j has a fixed point so does $j^2 = -1$, which is impossible by assumption. Hence $\alpha(1)$ does not reverse the orientation of $F(G)$. Thus $\alpha(t)$ does not reverse the fiber orientation. On the other hand it does reverse the orientation of a transverse 2-cell, because $\alpha^*(t)$ reverses the local orientation in $GF(Z_p)$. This proves that $GF(Z_p)$ is not orientable. □

REMARK. The facts above also show that the inverse of a simple closed curve in P^2 has the mod p homology of $S^1 \times F(Z_p)$. This is because a circuit of the curve preserves the orientation.

LEMMA 2. *The principal isotropy group is e.*

The principal isotropy group must be odd cyclic since every isotropy group is odd cyclic and it remains to prove that it is trivial. Suppose not and that the principal isotropy group is a non-trivial odd cyclic group A. The set $F(A)$ must touch every orbit. Let T be the circle group containing A, and N the normalizer of T, which is also the normalizer for any subgroup of T. Then N leaves $F(A)$ invariant. Let $a_1 \in A$ generate A_1 of prime order p. Then $F(A) \subset F(A_1)$; $F(A_1)$ is a sphere mod p which cannot be S^7. By Lemma 1, $GF(A_1)$ is a non-orientable manifold and it must be S^7. This contradiction proves the Lemma. □

4. First part of proof. It has been shown that the principal isotropy group is e. Suppose now for some prime p that $F(Z_p) \neq \varnothing$, so that $F(Z_p)$ is a mod p sphere of dimension 1, 3, 5, 7. If it had dimension 7, G would not be effective and if it had dimension 5, $GF(Z_p)$ would be all of S^7 and the principal isotropy group would contain Z_p. If $F(Z_p)$ is 1-dimensional, $GF(Z_p)$ is an orbit $G(x)$, $x \in F(Z_p)$. But $G(x)$ contains $N(x)$ and $N(x) \subset F(Z_p)$. On the other hand $N(x)$ contains 2 components. This shows that $F(Z_p)$ is 3-dimensional.

The proof of the theorem consists in showing that $F(Z_p)$ cannot be 3-dimensional because if it is we are led to a contradiction. The contradiction is found by computing the mod p cohomology of $GF(Z_p)$ in two ways which lead to conflicting results.

For the first calculation notice that $GF(Z_p)$ is fibered by the sets $gF(Z_p)$, $g \in G$, and that the base space $B^* = P^2$, $\pi: GF(Z_p) \to B^*$, $gF(Z_p) \to pt$. Let S^* be a simple closed curve in B^* which cannot be shrunk to a point. Then in $\pi^{-1}S^*$ there is a cross-section S.

This situation has been discussed in Lemma 1, and it was shown that traversing the closed path S does not reverse the orientation of the fiber $gF(Z_p)$. Now $F(Z_p)$ is a mod p 3-sphere and this argument shows that the mod p mapping sheaf for $\pi: GF(Z_p) \to P^2 = B^*$ is constant. The mod p homology of P^2 is trivial and it follows that the mod p homology of $GF(Z_p)$ is the same as that of S^3.

5. Conclusion of proof. To obtain the proof we consider $GF(Z_p)$ from another point of view. Computing the homology of $GF(Z_p)$ from this point of view leads to a contradictory result and this contradiction proves that $F(Z_p) = \varnothing$ which proves the theorem.

As in the previous section, assume that $F(Z_p) \neq \varnothing$ so that $F(Z_p)$ is a mod p 3-sphere. Again T is the circle group containing Z_p and N the normalizer. Let A be the principal isotropy group for the action of T on $F(Z_p)$. Then A is a cyclic group of order $p^r m$, $r \geqslant 1$, and m not divisible by p. There may be exceptional orbits for the action of T on $F(Z_p)$ and the union of these will be denoted by E.

We wish to find the mod p cohomology of $GF(Z_p)$. Let

$$GF(Z_p)/G = B^* = P^2 = F(Z_p)/N.$$

Since $F(Z_p)$ is a 3-sphere mod p, $F(Z_p)/T$ *is* S^2 and $B^* = P^2$.

The base $B^* = GF(Z_p)/G$ will be taken to be the union

$$B^* = M^* \cup D^*, \quad M^* \cap D^* = S^1,$$

where M^* is a Möbius band and D^* is a 2-disc. It may be assumed that M^* contains no points corresponding to exceptional orbits. Assume that $b_1^*, \ldots, b_k^*$ are points in the interior of D^* corresponding to all the exceptional orbits. To include the case where there are no exceptional orbits let b_0^* be an additional point corresponding to a regular orbit. Let D_0^*, $D_1^*, \ldots, D_k^*$ be 2-discs in the interior of D^* with D_i^* having center b_i^*. Let π be the projection $\pi: GF(Z_p) \to B^*$. The orbits in $\pi^{-1}(M^*)$ are all principal orbits. The following sketches a method of constructing a cross-section of $\pi^{-1}M^*$. We take M^* to be the union

$$M^* = \bigcup_{i=1}^{t} R_i^*,$$

where R_i^* is a 2-cell and $R_i^* \cap R_{i+1}^*$ is a 1-cell, $i=1, \ldots, t-1$ with orientation as usual and with R_t^* being attached to R_1^* with a reversal of orientation. Above each R_i^* there is a cross-section R_i. Of course R_1 and R_2 may not coincide along the relevant edge. But we may slide R_2 using elements of G so that R_2 becomes properly attached to R_1. Proceed this way and finally attach R_t to R_{t-1}. We then attach the second edge of R_t to R_1 being careful to keep the other edge of R_t attached as it was. This completes the construction of a Möbius band M which is a cross-section of $\pi^{-1}(M^*)$. In all cases it is the fact that attachments are along an arc that makes them possible.

We may continue to add 2-cells to M^* and to attach corresponding cross-sections to M provided each new 2-cell intersects the part already constructed in an arc (or 2 disjoined arcs). In this way we may form an M^* which is no longer a Möbius band (if $k>0$) but includes all of B^* except for the interior of the D_i^*. The notation is now changed so that M^* is a Möbius band with k holes so that

$$B^* = M^* \bigcup_{i=0}^{k} D_i^*$$

and M is now a cross-section above this new M^*.

To compute $H^*(GF(Z_p)$ use will be made of a cohomology sequence with compact supports and with coefficients mod p. Let $x_0, x_1, \ldots, x_k$ be points of S^7 with x_0 on a regular orbit and $x_1, \ldots, x_k$ on the exceptional orbits. Let

$$U = GF(Z_p) - \bigcup_{0}^{k} \pi^{-1}D_i^*.$$

Notice that $H^*\pi^{-1}D_i^* = H^*G(x_k)$, and that U is homeomorphic to $M_1 \times G/G_A$ ($M_1 = \operatorname{Int} M$). Since $M_1 = P^2 - [(k+1)$ discs$]$,

$$H^0(M_1) = 0, \quad H^1(M_1) = kZ_p, \quad H^2(M_1) = 0 \bmod p,$$

where kZ_p means the direct sum of k copies of Z_p. The cohomology sequence to be used is as follows, where the first and third columns are computed from the data above and the center column (the one being sought) is filled in the only possible way.

$$
\begin{array}{ccccc}
H^0U & \to H^0GF(Z_p) & \to H^0 \bigcup_0^k G(x_i) \to \\
0 & \to \quad Z_p & \to \quad (k+1)Z_p \\
H^1U & \to H^1GF(Z_p) & \to H^1 \bigcup_0^k G(x_i) \\
kZ_p & \to \quad Z_p & \to \quad (k+1)Z_p \\
H^2U & \to H^2GF(Z_p) & \to H^2 \bigcup_0^k G(x_i) \\
kZ_p & \to \quad Z_p & \to \quad (k+1)Z_p \\
H^3U & \to H^3GF(Z_p) & \to H^3 \bigcup_0^k G(x_i) \\
kZ_p & \to \quad Z_p & \to \quad (k+1)Z_p \\
H^4U & \to H^4GF(Z_p) & \to H^4 \bigcup_0^k G(x_i) \\
kZ_p & \to \quad 0 & \to \quad 0 \\
H^5U & \to H^5GF(Z_p) & \to H^5 \bigcup_0^k G(x_i) \\
0 & \to \quad 0 & \to \quad 0.
\end{array}
$$

Thus from this point of view $H^*GF(Z_p)$ is different from that of the previous calculation and this contradiction proves the $F(Z_p)=\varnothing$, which proves the theorem. □

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