ON THE EXTENSION OF SEPARATELY HYPERHARMONIC FUNCTIONS AND H^p-FUNCTIONS

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1. Introduction.

1.1. Grauert and Remmert [7, p. 175] proved that in $\mathbf{C}^n$ analytic sets are removable singularities for plurihyperharmonic functions which are locally bounded below. In fact, their result was extended also for complex spaces [7, Satz 3, p. 181]. Lelong [12, Theorem 2, p. 279] (see also [14, Theorem 4, p. 35] or [8, Theorem 1.2 (b), p. 704]) extended the classical result by showing that in $\mathbf{C}^n$ ($\mathbf{R}^{2n}$-) polar sets are removable in this situation. Using Hausdorff measure, Shiffman [21, Theorem 3, p. 338] gave other extension results for plurihyperharmonic functions.

In Theorem 4.1 below we give a similar result to Lelong's result for functions which are separately hyperharmonic with respect to each complex variable. In fact, we allow our exceptional sets to be slightly larger than polar sets. For this purpose in Section 2 we define n-small sets in $\mathbf{C}^n$. Since polar sets are n-small, also n-polar and sets of finite $(2n-2)$-dimensional Hausdorff measure are n-small. In addition, our sets include n-negligible sets. Note that there are (at least nonmeasurable) n-negligible sets which are not polar (see Remark 2.8 below). For the definition of n-polar and n-negligible sets see [5, Definition 3.10, p. 246], [11, p. 597], [24, Definitions 3.1 and 3.2, p. 32], and [4, p. 284]. In Section 4 we give also some other extension results for functions which are separately hyperharmonic with respect to each complex variable.

In Section 5 we then apply Theorem 4.1 to get extension results for H^p-functions in $\mathbf{C}^n$. Our result, Theorem 5.2 below, includes the results of [11, Theorem 2, p. 597], [4, Remark 3, p. 286], [5, Theorem B, p. 241], and [17, Theorem 3.5, p. 287].

However, before giving the above results we begin in Section 3 with a remark concerning separately hyperharmonic functions. Avanissian [3, Theorem 9, p. 140] (see also [10, Theorem, p. 31]) proved that a separately hyperharmonic function is hyperharmonic if it is locally bounded below. Using a different method, Arsove [2, Theorem 2, p. 622] showed that it is enough that the function has locally an integrable minorant. In Theorem 3.4 below we point out that Arsove's result can also be obtained directly and shortly from Avanissian's result.

For the properties of distributions see [20]. For the properties of hyperharmonic and holomorphic functions see [10], [9], [14] and [19].

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1.2. In addition to the standard notation we use the following, which is partly similar to the notation used in [10].

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In the sequel j, k, l, n, p and q are always positive integers. The empty set is denoted by $\varnothing$. The set of real numbers is denoted by $\mathbf{R}$ and the set of positive real numbers by $\mathbf{R}_+$. In $\mathbf{R}^k$ we use its usual euclidean metric and topology. For each set $A\subset\mathbf{R}^k$ we let $\bar{A}$ and ∂A denote the closure and boundary of A, both taken with respect to $\mathbf{R}^k$. Given two sets A and B in $\mathbf{R}^k$, $A\backslash B$ and $A\Delta B$ are the set-theoretic difference and symmetric difference of A and B, respectively. Moreover, $d(A,B)$ is the distance between the sets A and B. If $C\subset\mathbf{R}^1$, then $A\times C$ is the cartesian product of A and C. If $a\in\mathbf{R}^k$ and $r>0$, we write

$$B^k(a,r)=\{x\in\mathbf{R}^k \mid |x-a|<r\}.$$

The complex plane is denoted by $\mathbf{C}$. The complex space $\mathbf{C}^n$ will be identified with the real space $\mathbf{R}^{2n}$. If $z_0\in\mathbf{C}$ and $r>0$, we write

$$S^1(z_0,r)=\partial B^2(z_0,r), \quad U=B^2(0,1).$$

If $z=(z_1,\dots,z_n)\in\mathbf{C}^n$, we set for each j, $1\leqslant j\leqslant n$, $Z_j=(z_1,\dots,z_{j-1},z_{j+1},\dots,z_n)\in\mathbf{C}^{n-1}$ and $(z_j,Z_j)=z$. Similarly, for each $k\neq j$, $1\leqslant k\leqslant n$, we write $Z_{jk}=Z_{kj}=(z_1,\dots,z_{j-1},z_{j+1},\dots,z_{k-1},z_{k+1},\dots,z_n)\in\mathbf{C}^{n-2}$ and $(z_k,Z_{jk})=(z_k,Z_{kj})=Z_j$. If $G\subset\mathbf{C}^n$ and $z_0=(z_j^0,Z_j^0)\in\mathbf{C}^n$, we write

$$G(z_j^0)=\{Z_j\in\mathbf{C}^{n-1} \mid (z_j^0,Z_j)\in G\}.$$

If $r=(r_1,\dots,r_n)\in\mathbf{R}_+^n$, we write $R_1=(r_2,\dots,r_n)$ and

$$D^n(z_0,r)=B^2(z_1^0,r_1)\times D^{n-1}(Z_1^0,R_1),$$

where

$$D^{n-1}(Z_1^0,R_1)=B^2(z_2^0,r_2)\times\cdots\times B^2(z_n^0,r_n).$$

The α-dimensional Hausdorff outer measure is denoted by H^α. If $A\subset\mathbf{C}^n$, then the notation '$H^\alpha(A)<\sigma_\infty$' means that A is of σ-finite α-dimensional Hausdorff outer measure. For the definition and properties of Hausdorff outer measure see [6, 2.10.2, p. 171] or [9, pp. 220–221]. The Lebesgue measure in $\mathbf{R}^k$ is denoted by m_k. We write $\omega_k=m_k(B^k(0,1))$. The outer logarithmic capacity is denoted by cap*.

If U is an open set in $\mathbf{R}^k$, then $D(U)$ and $D_+(U)$ mean, as usual, the spaces of testfunctions and non-negative testfunctions in U, respectively. If $\varphi\in D(U)$, then $\Delta\varphi$ means the Laplace of φ. If G is an open set in $\mathbf{C}^n$, $\varphi\in D(G)$ and $1\leqslant j\leqslant n$, then we write in G

$$\Delta_j\varphi(z)=4\frac{\partial^2\varphi}{\partial z_j\,\partial\bar{z}_j}(z).$$

If $p>0$, then the notation '$f\in L^p_{\mathrm{loc}}(G)$' means that f is a complex-valued function defined Lebesgue almost everywhere in G and that $|f|^p$ is Lebesgue integrable in each compact subset of G. The real and imaginary parts of a function f are denoted by $\operatorname{Re} f$ and $\operatorname{Im} f$, respectively; similarly for $\operatorname{Re} z$ and $\operatorname{Im} z$.

If U is a set in $\mathbf{R}^k$ and $u\colon U\to[-\infty,\infty]$, then the lower semicontinuous regularization of u is denoted by u^*, as usual. Thus $u^*\colon U\to[-\infty,\infty]$,

$$u^*(x) = \liminf_{x' \to x} u(x').$$

A *hyperharmonic* function on an open set U of $\mathbf{R}^k$, $k \geq 2$, is a lower semicontinuous function $u: U \to (-\infty, \infty]$ such that for any $a \in U$ there exists arbitrarily small radii $r > 0$ such that

$$u(a) \geq \frac{1}{\omega_k r^k} \int_{B^k(a,r)} u(x)\, dm_k(x).$$

Note that a hyperharmonic function can be identically ∞ on any component of U. A hyperharmonic function $u: U \to (-\infty, \infty]$ is *superharmonic* if $u \not\equiv \infty$ on each component of U. See [10, pp. 7, 9].

If G is an open set in $\mathbf{C}^n$ and $u: G \to (-\infty, \infty]$, then we say that u is *n-hyperharmonic* if u is separately hyperharmonic with respect to each complex variable z_j, $1 \leq j \leq n$, for any fixed values of the other ones. See [10, pp. 35–36]. The function $u: G \to \mathbf{R}$ is *n-harmonic* if u and $-u$ are n-hyperharmonic.

2. Exceptional sets.

2.1. We define a set function for subsets of $\mathbf{C}^n$. The construction is based on the use of the outer logarithmic capacity cap* in $\mathbf{C}$ and on the 2-dimensional Hausdorff outer measure H^2.

2.2. DEFINITION. For each $E \subset \mathbf{C}$ we define $\mathcal{C}^1(E) = \mathrm{cap}^* E$. If $n \geq 2$, $1 \leq j \leq n$ and $\mathcal{C}^{n-1}$ is defined for subsets of $\mathbf{C}^{n-1}$, we define for $E \subset \mathbf{C}^n$

$$\mathcal{C}_j^n(E) = H^2\{z_j \in \mathbf{C} \mid \mathcal{C}^{n-1}\{Z_j \in \mathbf{C}^{n-1} \mid (z_j, Z_j) \in E\} > 0\}.$$

Finally, set $\mathcal{C}^n(E) = \max_{1 \leq j \leq n} \mathcal{C}_j^n(E)$. We say that $E \subset \mathbf{C}^n$ is *n-small* if $\mathcal{C}^n(E) = 0$.

2.3. REMARK. Since H^2 is an outer measure, we see using the subadditivity of the outer logarithmic capacity and induction that $\mathcal{C}^n$ is an outer measure in $\mathbf{C}^n$. Thus $E \subset \mathbf{C}^n$ is n-small if and only if $E \cap U$ is n-small for each $U \subset \mathbf{C}^n$ open. Similarly using induction we see that $E \subset \mathbf{C}^n$ is of zero Lebesgue measure if E is Lebesgue measurable and n-small. There are, however, n-small sets which are Lebesgue nonmeasurable (see Remark 2.8 below).

Next we give two sufficient conditions for n-smallness. In Remarks 2.7 and 2.8 below we show that these conditions are not necessary.

2.4. PROPOSITION. *Let $E \subset \mathbf{C}^n$, $n \geq 2$. Then E is n-small if, for each k, $1 \leq k \leq n$,*

$$H^{2n-2}\{Z_k \in \mathbf{C}^{n-1} \mid \mathrm{cap}^*\{z_k \in \mathbf{C} \mid (z_k, Z_k) \in E\} > 0\} = 0.$$

Proof. We give an induction proof. If $n = 2$, then the assertion is equivalent with $\mathcal{C}^2(E) = 0$.

Suppose then $n \geq 3$. Take j, $1 \leq j \leq n$, arbitrarily. Set for each k, $1 \leq k \leq n$,

$$E_k = \{Z_k \in \mathbf{C}^{n-1} \mid \mathrm{cap}^*\{z_k \in \mathbf{C} \mid (z_k, Z_k) \in E\} > 0\}.$$

By the assumption $H^{2n-2}(E_k) = 0$. Thus by Fubini's theorem there is for each $k \neq j$, $1 \leq k \leq n$, a set $B_{jk} \subset \mathbf{C}$ such that $H^2(B_{jk}) = 0$ and that $H^{2n-4}(E_k(z_j)) = 0$

for each $z_j \notin B_{jk}$, where $E_k(z_j) = \{Z_{jk} \in \mathbf{C}^{n-2} \mid (z_j, Z_{jk}) \in E_k\}$. Set

$$B_j = \bigcup_{\substack{k=1\\k\neq j}}^{n} B_{jk}.$$

Clearly $H^2(B_j)=0$. To show that $\mathcal{C}^{n-1}(E(z_j))=0$ for each $z_j \notin B_j$, where $E(z_j) = \{Z_j \in \mathbf{C}^{n-1} \mid (z_j, Z_j) \in E\}$, we proceed as follows. Choose $k \neq j$, $1 \leqslant k \leqslant n$, arbitrarily. Then

$$\begin{aligned}\{Z_{jk} \in \mathbf{C}^{n-2} \mid \operatorname{cap}^*\{z_k \in \mathbf{C} \mid (z_k, Z_{jk}) \in E(z_j)\} > 0\}\\ = \{Z_{jk} \in \mathbf{C}^{n-2} \mid \operatorname{cap}^*\{z_k \in \mathbf{C} \mid (z_k, Z_k) \in E\} > 0\}\\ = \{Z_{jk} \in \mathbf{C}^{n-2} \mid (z_j, Z_{jk}) \in E_k\} = E_k(z_j).\end{aligned}$$

Since $H^{2n-4}(E_k(z_j))=0$ for each $z_j \notin B_j$, we see by the induction hypothesis that $\mathcal{C}^{n-1}(E(z_j))=0$ whenever $z_j \notin B_j$, concluding the proof. □

2.5. PROPOSITION. *Let $E \subset \mathbf{C}^n$, $n \geqslant 2$. Then E is n-small if, for each k, $1 \leqslant k \leqslant n$, the set*

$$\{Z_k \in \mathbf{C}^{n-1} \mid \operatorname{cap}^*\{z_k \in \mathbf{C} \mid (z_k, Z_k) \in E\} > 0\}$$

is $(n-1)$-small.

The proof is based on induction and is very similar to the proof of Proposition 2.4.

2.6. REMARK. If $E \subset \mathbf{C}^n = \mathbf{R}^{2n}$ is polar, then E is n-small. This follows from [15, Corollary 3.3] and from Proposition 2.4. Thus it follows from [9, Theorem 5.14, p. 228] that E is n-small, if $H^{2n-2}(E) < \sigma_\infty$.

However, for each α, $0 < \alpha < 2$, there are n-small sets $E \subset \mathbf{C}^n$, $n \geqslant 2$, for which $H^{2n-\alpha}(E) > 0$. To get an example take α', $2 - \alpha/n < \alpha' < 2$, and $E' \subset \mathbf{C}$ such that $H^{\alpha'}(E') > 0$ and $H^2(E')=0$. If $E = E' \times \cdots \times E'$, we see by [6, 2.10.25, p. 188] (see also [21, Lemma 1, p. 113]) that $H^{2n-\alpha}(E) > 0$. However, it is easy to see that E is n-small.

2.7. REMARK. The condition in Proposition 2.5 is not necessary for a set to be n-small. This is seen for example from the following. Suppose $n \geqslant 2$ and set $E = \{z \in \mathbf{C}^n \mid z_{n-1},\ z_n \in \mathbf{R}\}$. Then clearly $H^{2n-2}(E) < \sigma_\infty$. Thus E is n-small. However, the condition in Proposition 2.4 does not hold, since the set

$$\begin{aligned}E_n &= \{Z_n \in \mathbf{C}^{n-1} \mid \operatorname{cap}^*\{z_n \in \mathbf{C} \mid (z_n, Z_n) \in E\} > 0\}\\ &= \{Z_n \in \mathbf{C}^{n-1} \mid z_{n-1} \in \mathbf{R}\}\end{aligned}$$

is not $(n-1)$-small.

To see the difference between the above situation with n-negligible sets we recall the following definitions. See also [4, p. 284] and [24, Definitions 3.1 and 3.2, p. 32].

For each $E \subset \mathbf{C}$ we define $\mathcal{E}^1(E) = \operatorname{cap}^* E$. If $n \geqslant 2$, $1 \leqslant j \leqslant n$, and $\mathcal{E}^{n-1}$ is defined for subsets of $\mathbf{C}^{n-1}$, we define for $E \subset \mathbf{C}^n$

$$\mathcal{E}_j^n(E)=\operatorname{cap}^*\{z_j\in\mathbf{C}\mid \mathcal{E}^{n-1}\{Z_j\in\mathbf{C}^{n-1}\mid (z_j,Z_j)\in E\}>0\}.$$

Finally, set $\mathcal{E}^n(E)=\max_{1\leqslant j\leqslant n}\mathcal{E}_j^n(E)$. We say that $E\subset\mathbf{C}^n$ is *n-negligible* if $\mathcal{E}^n(E)=0$.

Using induction one can prove the following characterization of n-negligible sets (observe the difference between n-small sets).

Let $E\subset\mathbf{C}^n$, $n\geqslant 2$. Then E is n-negligible if and only if for each k, $1\leqslant k\leqslant n$, the set

$$\{Z_k\in\mathbf{C}^{n-1}\mid\operatorname{cap}^*\{z_k\in\mathbf{C}\mid (z_k,Z_k)\in E\}>0\}$$

is $(n-1)$-negligible. (Cf. [24, Proposition 3.3, p. 33].)

Using this characterization, induction and Proposition 2.5, we see that n-negligible and thus also n-polar, pluripolar and Borel sets with vanishing Γ-capacity are n-small. See [4, p. 284], [24, Proposition 5.2, p. 41], and [18].

2.8. REMARK. The condition in Proposition 2.4 is not necessary for a set to be n-small. This is seen from the following example.

Let $S\subset\mathbf{R}^2$ be the well-known Sierpinski's set which is Lebesgue nonmeasurable and has exactly one point in common with each line parallel to either of the coordinate axes (see [23, p. 114]). Suppose that $n\geqslant 3$ and set

$$E=\{z\in\mathbf{C}^n\mid (\operatorname{Re} z_1,\operatorname{Im} z_2),\ (\operatorname{Im} z_1,\operatorname{Re} z_2)\in S\}.$$

It is easy to see that E is n-negligible and thus n-small. However, by Fubini's theorem we see that for each k, $3\leqslant k\leqslant n$, the set

$$E_k=\{Z_k\in\mathbf{C}^{n-1}\mid\operatorname{cap}^*\{z_k\in\mathbf{C}\mid (z_k,Z_k)\in E\}>0\}$$

is Lebesgue nonmeasurable. Thus $H^{2n-2}(E_k)>0$ and the condition in Proposition 2.4 does not hold. Note that also E is Lebesgue nonmeasurable and therefore gives an example of an n-negligible set which is not polar. Thus we have answered a question posed by Singman in [24, p. 33].

3. On hyperharmonicity of separately hyperharmonic functions.

3.1. Avanissian [3, Theorem 9, p. 140] (see also [10, Theorem, p. 31]) proved that a separately hyperharmonic function is hyperharmonic if it is locally bounded below. Later Arsove [2, Theorem 1, p. 622] generalized Avanissian's result by showing that it is enough to suppose that the function has locally an integrable minorant. Arsove's proof was based on the use of mean value operators. We show that Arsove's result can be obtained directly from Avanissian's result and from the following, probably well-known lemma. See also [2, Lemma 1, p. 624].

3.2. LEMMA. *Let $U\subset\mathbf{R}^p$, $p\geqslant 1$, and $V\subset\mathbf{R}^q$, $q\geqslant 1$, be open sets. Let $f\colon U\times V\to(-\infty,\infty]$ be a function which satisfies the following conditions.*

(1) *For each $y\in V$ the function $f_y\colon U\to(-\infty,\infty]$, $f_y(x)=f(x,y)$, is lower semicontinuous.*

(2) *For each $x\in U$ the function $f_x\colon V\to(-\infty,\infty]$, $f_x(y)=f(x,y)$, is lower semicontinuous. Moreover, for each $x\in U$*

$$\lim_{r\to 0} \frac{1}{\omega_q r^q} \int_{B^q(b,r)} f_x(y)\, dm_q(y) = f_x(b)$$

for all $b \in V$.
Then f *is measurable.*

Proof. Take V_1 open such that $\bar{V}_1 \subset V$ is compact. Take $M>0$ arbitrarily. It is sufficient to show that the function $g\colon U\times V_1 \to \mathbf{R}$, $g(x,y)=\inf\{f(x,y),M\}$, is measurable.

For each r, $0<r<r_0=d(V_1, \mathbf{R}^q\setminus V)$, define $h_r\colon U\times V_1\to\mathbf{R}$,

$$h_r(x,y) = \frac{1}{\omega_q r^q}\int_{B^q(y,r)} g(x,z)\, dm_q(z).$$

Note that h_r is well-defined. Using the lower semicontinuity of the functions $g_{ky}\colon U\to\mathbf{R}$, $g_{ky}(x)=\sup\{g(x,y),-k\}$, $y\in V_1$, $k=1,2,\ldots$, Fatou's lemma and Monotone convergence, we see that for each $y\in V_1$ the function $h_{ry}\colon U\to\mathbf{R}$, $h_{ry}(x)=h_r(x,y)$, is measurable. On the other hand, for each $x\in U$ the function $h_{rx}\colon V_1\to\mathbf{R}$, $h_{rx}(y)=h_r(x,y)$, is continuous, since

$$\begin{aligned}|h_{rx}(y)-h_{rx}(y_0)| &= \frac{1}{\omega_q r^q}\left|\int_{B^q(y,r)} g(x,z)\, dm_q(z) - \int_{B^q(y_0,r)} g(x,z)\, dm_q(z)\right| \\ &\leqslant \frac{1}{\omega_q r^q}\int_{B^q(y,r)\Delta B^q(y_0,r)} |g(x,z)|\, dm_q(z)\end{aligned}$$

for all $y, y_0\in V_1$. It is straightforward to see that h_r is measurable. (In fact, it is a classical result, originally due to Lebesgue, that a function is measurable if it is measurable with respect to the first variable and continuous with respect to the other variable.)

To show that $h_r\to g$ as $r\to 0$, we proceed as follows. Take $(x,y)\in U\times V_1$ arbitrarily. For each r, $0<r<r_0$, we have

$$h_r(x,y) = \frac{1}{\omega_q r^q}\int_{B^q(y,r)} g(x,z)\, dm_q(z) \leqslant \frac{1}{\omega_q r^q}\int_{B^q(y,r)} f(x,z)\, dm_q(z).$$

Thus using the condition (2) we get

$$\limsup_{r\to 0} h_r(x,y)\leqslant g(x,y).$$

On the other hand, using the lower semicontinuity of g_x we see that

$$\liminf_{r\to 0} h_r(x,y)\geqslant g(x,y).$$

Therefore $h_r\to g$ as $r\to 0$. Hence g and also f is measurable. □

3.3. REMARK. It is well known that a function which is separately lower (upper) semicontinuous need not be measurable. As an example serves the characteristic function χ_S of Sierpinski's set S explained in Remark 2.8.

Note that by [10, Proposition 2 b), p. 10] the condition (2) is satisfied if the functions f_x, $x\in U$, are hyperharmonic.

3.4. THEOREM [2, Theorem 1, p. 622]. *Let G be an open set in $\mathbf{R}^{p+q}$, $p, q \geq 1$. Let $v: G \to (-\infty, \infty]$ be separately hyperharmonic, i.e. let v satisfy the following conditions.*

(1) *For each $y \in \mathbf{R}^q$ the function*

$$\{x \in \mathbf{R}^p \mid (x, y) \in G\} \ni x \mapsto v(x, y) \in (-\infty, \infty]$$

is hyperharmonic.

(2) *For each $x \in \mathbf{R}^p$ the function*

$$\{y \in \mathbf{R}^q \mid (x, y) \in G\} \ni y \mapsto v(x, y) \in (-\infty, \infty]$$

is hyperharmonic.

If v has locally an integrable minorant, then v is hyperharmonic.

Proof. For each $k = 1, 2, \ldots$ define $v_k: G \to \mathbf{R}$, $v_k(x, y) = \inf\{v(x, y), k\}$. By [10, b), p. 8] v_k is separately hyperharmonic. By [10, a), p. 8] it is sufficient to show that each v_k, $k = 1, 2, \ldots$, is hyperharmonic. Observe first that by [10, Proposition 2 b), p. 10] and by Lemma 3.2, v_k is measurable and thus locally integrable. To show that for each $(a, b) \in G$ and $r > 0$ such that $\bar{B}^{p+q}((a, b), r) \subset G$ we have

$$v_k(a, b) \geq \frac{1}{\omega_{p+q} r^{p+q}} \int_{B^{p+q}((a, b), r)} v_k(x, y) \, dm_{p+q}(x, y),$$

just proceed as in the proof of Avanissian's theorem [10, pp. 32–33]. Hence v_k is nearly superharmonic, and thus locally bounded below. Therefore the lower semicontinuity of v_k follows as in the proof of Avanissian's theorem [10, p. 32]. □

3.5. COROLLARY. *Let G be an open set in $\mathbf{C}^n$. Let $u: G \to (-\infty, \infty]$ be n-hyperharmonic. If u has locally an integrable minorant, then u is hyperharmonic.*

4. Extension of n-hyperharmonic functions.

4.1. THEOREM. *Let G be an open set in $\mathbf{C}^n$. Let $E \subset G$ be closed in G and n-small. Let $u: G \setminus E \to (-\infty, \infty]$ be n-hyperharmonic. If u is locally bounded below in G, then u has a unique n-hyperharmonic extension $u^*: G \to (-\infty, \infty]$.*

Proof. We give an induction proof. For each $k = 1, 2, \ldots$ define $u_k: G \setminus E \to \mathbf{R}$, $u_k(z) = \inf\{u(z), k\}$. By [10, b), p. 8] u_k is n-hyperharmonic. Since u_k is locally bounded below in G, u_k is superharmonic by Corollary 3.5. Hence u_k is locally integrable in G and thus defines a distribution in G. By [10, a), p. 8] it is sufficient to show that u_k has a unique n-hyperharmonic extension to G.

In view of [10, Theorem 2, p. 25] we may suppose that $n \geq 2$. Take j, $1 \leq j \leq n$, arbitrarily. Since E is n-small, there is a set $B_j \subset \mathbf{C}$ such that $H^2(B_j) = 0$ and that for each $z_j \notin B_j$ the set

$$E(z_j) = \{Z_j \in \mathbf{C}^{n-1} \mid (z_j, Z_j) \in E\}$$

is $(n-1)$-small. Thus for each $z_j \notin B_j$ the function $u_{kz_j}: (G \setminus E)(z_j) \to \mathbf{R}$,

$$u_{kz_j}(Z_j) = u_k(z_j, Z_j),$$

has—in the case $n=2$ by [10, Theorem 2, p. 25] and in the case $n\geqslant 3$ by the induction hypothesis—a unique $(n-1)$-hyperharmonic extension $u^*_{kz_j}\colon G(z_j)\to\mathbf{R}$. Since by Corollary 3.5 $u^*_{kz_j}$ is also superharmonic, we see by [10, Theorem 1, p. 11] that

$$\text{(A)}\qquad \int u_{kz_j}(Z_j)\Delta\varphi_{z_j}(Z_j)\,dm_{2n-2}(Z_j)\leqslant 0$$

for each testfunction $\varphi_{z_j}\in D_+(G(z_j))$.

Take now a testfunction $\varphi\in D_+(G)$ arbitrarily. Using Fubini's theorem we get

$$\begin{aligned}(n-1)&\int u_k(z)\Delta\varphi(z)\,dm_{2n}(z)\\ &=\sum_{j=1}^{n}\int u_k(z)\Big(\sum_{\substack{l=1\\ l\neq j}}^{n}\Delta_l\varphi(z)\Big)dm_{2n}(z)\\ &=\sum_{j=1}^{n}\int_{\mathbf{C}\setminus B_j}\Big(\int u_k(z_j,Z_j)\Big(\sum_{\substack{l=1\\ l\neq j}}^{n}\Delta_l\varphi(z_j,Z_j)\Big)dm_{2n-2}(Z_j)\Big)dm_2(z_j)\\ &=\sum_{j=1}^{n}\int_{\mathbf{C}\setminus B_j}\Big(\int u_{kz_j}(Z_j)\Delta\varphi_{z_j}(Z_j)\,dm_{2n-2}(Z_j)\Big)dm_2(z_j),\end{aligned}$$

where $\varphi_{z_j}\colon G(z_j)\to\mathbf{R}$, $\varphi_{z_j}(Z_j)=\varphi(z_j,Z_j)$, when $z_j\in\mathbf{C}$. Since $\varphi_{z_j}\in D_+(G(z_j))$ for each $z_j\notin B_j$, we see by (A) that

$$\int u_k(z)\Delta\varphi(z)\,dm_{2n}(z)\leqslant 0.$$

Thus by [10, Corollary 1, p. 13] there is a superharmonic function $v_k\colon G\to(-\infty,\infty]$ such that $v_k=u_k$ almost everywhere in $G\setminus E$. From the superharmonicity of u_k in $G\setminus E$ it follows by [10, Proposition 3, p. 11] that v_k is the unique superharmonic extension of u_k to G. To see that v_k is actually n-hyperharmonic, we proceed as follows.

Define $w_k\colon G\to(-\infty,\infty]$,

$$w_k(z)=\begin{cases} v_k(z)=u_k(z), & \text{when } z\in G\setminus E,\\ \infty, & \text{when } z\in E.\end{cases}$$

Since $m_{2n}(E)=0$, w_k is locally integrable in G. Since v_k is superharmonic and $w_k=v_k$ almost everywhere in G, w_k is nearly superharmonic. Thus the function $w^*_k\colon G\to(-\infty,\infty]$,

$$w^*_k(z)=\liminf_{z'\to z} w_k(z'),$$

is a superharmonic extension of u_k to G, and in fact $w^*_k=v_k$ by [10, Proposition 3, p. 11].

Take j, $1\leqslant j\leqslant n$, arbitrarily. For each $z_j\notin B_j$ the function $w_{kz_j}\colon G(z_j)\to(-\infty,\infty]$, $w_{kz_j}(Z_j)=w_k(z_j,Z_j)$, is nearly superharmonic, since the set $E(z_j)$ is

$(n-1)$-small. Thus $m_{2n-2}(E(z_j))=0$, and the function u_{kz_j} has a unique $(n-1)$-hyperharmonic and thus a superharmonic extension to $G(z_j)$. Thus by [10, Proposition 1, p. 33] for each $z_j \in \mathbf{C}$ the function $w^*_{kz_j}: G(z_j) \to (-\infty, \infty]$, $w^*_{kz_j}(Z_j) = w^*_k(z_j, Z_j)$, is hyperharmonic. However, if $z_j \notin B_j$ then the function $w^*_{kz_j}$ is $(n-1)$-hyperharmonic. This follows from the facts that then $w^*_{kz_j}$ is superharmonic, $u^*_{kz_j}$ is $(n-1)$-hyperharmonic and superharmonic, $w^*_{kz_j} = u^*_{kz_j}$ in $G(z_j) \setminus E(z_j)$, $m_{2n-2}(E(z_j))=0$ and [10, Proposition 3, p. 11].

To show that w^*_k is n-hyperharmonic it is by [10, Proposition 1, p. 33] sufficient to show that for each l, $1 \leqslant l \leqslant n$,

$$\text{(B)} \qquad \int w^*_k(z)\, \Delta_l \varphi(z)\, dm_{2n}(z) \leqslant 0$$

for all $\varphi \in D_+(G)$. For this purpose take $j \neq l$, $1 \leqslant j \leqslant n$, arbitrarily. Using Fubini's theorem we get

$$\int w^*_k(z)\, \Delta_l \varphi(z)\, dm_{2n}(z) = \int_{\mathbf{C}\setminus B_j} \left(\int w^*_{kz_j}(Z_j)\, \Delta_l \varphi_{z_j}(Z_j)\, dm_{2n-2}(Z_j) \right) dm_2(z_j).$$

Since we have above seen that the functions $w^*_{kz_j}$, $z_j \notin B_j$, are $(n-1)$-hyperharmonic and superharmonic, it follows from [10, Proposition 1, p.33] that

$$\int w^*_{kz_j}(Z_j)\, \Delta_l \varphi_{z_j}(Z_j)\, dm_{2n-2}(Z_j) \leqslant 0$$

for each $z_j \notin B_j$. Thus (B) holds, and we have shown that w^*_k is n-hyperharmonic, concluding the proof. □

4.2. COROLLARY. *Let G be an open set in $\mathbf{C}^n$. Let $E \subset G$ be closed in G and polar. Let $u: G \setminus E \to (-\infty, \infty]$ be n-hyperharmonic. If u is locally bounded below in G, then u has a unique n-hyperharmonic extension $u^*: G \to (-\infty, \infty]$.*

4.3. COROLLARY. *Let G be an open set in $\mathbf{C}^n$. Let $E \subset G$ be closed in G and let $H^{2n-2}(E) < \sigma_\infty$. Let $u: G \setminus E \to (-\infty, \infty]$ be n-hyperharmonic. If u is locally bounded below in G, then u has a unique n-hyperharmonic extension $u^*: G \to (-\infty, \infty]$.*

4.4. COROLLARY (cf. [5, Theorem 3.11, p. 246] and [24, Theorems 4.8 and 4.9, p. 38]). *Let G be an open set in $\mathbf{C}^n$. Let $E \subset G$ be closed in G and n-polar or, more generally, n-negligible. Let $u: G \setminus E \to (-\infty, \infty]$ be n-hyperharmonic. If u is locally bounded below in G, then u has a unique n-hyperharmonic extension $u^*: G \to (-\infty, \infty]$.*

4.5. REMARK. Lelong [12, Theorem 2, p. 279] (see also [14, Theorem 4, p. 35] or [8, Theorem 1.2 (b), p. 704]) proved a similar result to Corollary 4.2 for plurihyperharmonic functions. Lelong's method of proof was to reduce the situation to the case of hyperharmonic functions by a convenient characterization of plurihyperharmonic functions [12, Theorem 1, p. 273] (see also [14, Theorem 1, p. 18] and [8, p. 707]). Hyvönen has proved Corollary 4.2 with a method similar to Lelong's method.

4.6. REMARK. Shiffman [22, Theorem 3 (i), p. 338] proved a similar result to Corollary 4.3 for plurihyperharmonic functions. In fact, our method in proving that the function u_k has a superharmonic extension to G seems to be slightly similar to the idea of Shiffman in [22, p. 338]. Shiffman [22, Theorem 3 (ii), p. 338] (see also [8, Theorem 1.2 (c), p. 704]) proved also that a plurihyperharmonic function extends plurihyperharmonically across a set E, if $H^{2n-2}(E)=0$. Proceeding as in the proof of Theorem 4.1 we get a similar result for n-hyperharmonic functions, although under the rather heavy assumption that the function has locally in G an integrable minorant.

4.7. THEOREM. *Let G be an open set in $\mathbf{C}^n$. Let $E\subset G$ be closed in G and let $H^{2n-2}(E)=0$. Let $u: G\setminus E\to(-\infty,\infty]$ be n-hyperharmonic. If u has locally in G an integrable minorant, then u has a unique n-hyperharmonic extension $u^*: G\to(-\infty,\infty]$.*

4.8. REMARK. We do not know whether the assumption of the existence of an integrable minorant is really necessary. Anyway, if we demand that $H^n(E)=0$ and $n\geqslant 2$ then this assumption can be dropped. The key to the proof of this fact is the following lemma which, with its proof, has its origin in [4, Lemma, p. 284]. In fact, Cegrell gave a similar result for n-negligible sets.

4.9. LEMMA. *Let $E\subset\mathbf{C}^n$ be such that $H^n(E)=0$. Then there is a point $z_0\in E$ and j, $1\leqslant j\leqslant n$, such that*

$$H^1\{z_j\in\mathbf{C}\mid(z_j,Z_j^0)\in E\}=0.$$

Proof. We give an induction proof. If $n=1$, the assertion follows from [21, Corollary 2, p. 114].

Suppose then that $n\geqslant 2$. Suppose, on the contrary, that no point of E satisfies the assertion of the Lemma. Set

$$E_1=\{z_1\in\mathbf{C}\mid H^{n-1}\{Z_1\in\mathbf{C}^{n-1}\mid(z_1,Z_1)\in E\}>0\}.$$

By [6, 2.10.25, p. 188] (see also [21, Lemma 1, p. 113]), $H^1(E_1)=0$. If $z_1\notin E_1$, it follows from the induction hypothesis and the antithese that $(z_1,Z_1)\notin E$ for each $Z_1\in\mathbf{C}^{n-1}$. Thus $E\subset E_1\times\mathbf{C}^{n-1}$ and so every point of E satisfies the assertion of the Lemma, a contradiction. □

4.10. THEOREM. *Let G be an open set in $\mathbf{C}^n$, $n\geqslant 2$. Let $E\subset G$ be closed in G and let $H^n(E)=0$. Let $u: G\setminus E\to(-\infty,\infty]$ be n-hyperharmonic and locally bounded below in $G\setminus E$. Then u has a unique n-hyperharmonic extension $u^*: G\to(-\infty,\infty]$, which is locally bounded below in G.*

Proof. Let G' be the set of those points $z\in G$ which have a neighborhood U in G such that u is bounded below in U. Since $H^n(E)=0$, it follows from Corollary 4.3 that u has a unique n-hyperharmonic extension $u^*: G'\to(-\infty,\infty]$. To show that $G'=G$ we proceed as follows.

Set $E'=G\setminus G'$ and suppose, on the contrary, that $E'\neq\varnothing$. Since $E'\subset E$, there is by Lemma 4.9 a point $z_0\in E'$ and j, $1\leqslant j\leqslant n$, such that

$$H^1\{z_j\in\mathbf{C}\mid(z_j,Z_j^0)\in E'\}=0.$$

For the simplicity of notation we may suppose that $j=1$. Thus there is $r_1>0$ such that $\bar{B}^2(z_1^0,r_1)\times\{Z_1^0\}\subset G$ and that $S^1(z_1^0,r_1)\times\{Z_1^0\}\subset G\backslash E'$. Since E' is closed in G, there is $R_1=(r_2,\ldots,r_n)\in\mathbf{R}_+^{n-1}$ such that $\bar{B}^2(z_1^0,r_1)\times\bar{D}^{n-1}(Z_1^0,R_1)\subset G$ and $S^1(z_1^0,r_1)\times\bar{D}^{n-1}(Z_1^0,R_1)\subset G\backslash E'$. Since u^* is lower semicontinuous, there is $m\in\mathbf{R}\cup\{\infty\}$ such that $u^*(z_1,Z_1)\geqslant m$ for each $(z_1,Z_1)\in S^1(z_1^0,r_1)\times D^{n-1}(Z_1^0,R_1)$.

Set $E_1=\{Z_1\in\mathbf{C}^{n-1}\mid(\mathbf{C}\times\{Z_1\})\cap E'=\varnothing\}$. By the minimum principle [10, Proposition, pp. 6–7] we see that $u^*(z_1,Z_1)\geqslant m$ for each

$$(z_1,Z_1)\in B^2(z_1^0,r_1)\times(D^{n-1}(Z_1^0,R_1)\cap E_1).$$

Since $H^{2n-2}(\mathbf{C}^{n-1}\backslash E_1)=0$, it follows (e.g. by [10, Proposition 2 b′), p. 10]) that $u(z_1,Z_1)\geqslant m$ for each $(z_1,Z_1)\in(B^2(z_1^0,r_1)\times D^{n-1}(Z_1^0,R_1))\backslash E$. Hence $z_0\in G'$, a contradiction. Thus the theorem is proved. □

4.11. COROLLARY. *Let G be an open set in $\mathbf{C}^n$, $n\geqslant 2$. Let $E\subset G$ closed in G and let $H^n(E)=0$. Let $u\colon G\backslash E\to\mathbf{R}$ be n-harmonic. Then u has a unique n-harmonic extension $u^*\colon G\to\mathbf{R}$.*

Proof. By [13, p. 561] (see also [10, Theorem, p. 54]) u is harmonic. Thus by Theorem 4.10 u has a unique n-harmonic extension $u^*\colon G\to\mathbf{R}$. □

5. Extension of H^p-functions.

5.1. THEOREM. *Let G be an open set in $\mathbf{C}^n$. Let $E\subset G$ be closed in G and n-small. Let $f\colon G\backslash E\to\mathbf{C}$ be a holomorphic function such that for some $p>0$, $|f|^p$ has an n-hyperharmonic majorant u in $G\backslash E$. If u is superharmonic, then f has a unique holomorphic extension $f^*\colon G\to\mathbf{C}$.*

Proof. We give an induction proof. By Theorem 4.1, u has a unique n-hyperharmonic extension $u^*\colon G\to(-\infty,\infty]$. Since u is superharmonic, we see by Corollary 3.5 that u^* is superharmonic. Hence $|f|^p$ is locally integrable in G and thus defines a distribution in G. We show first that $|f|^p$ has a unique subharmonic extension to G.

In view of [16, Théorème 20, p. 182] (see also [11, Theorem 1, p. 597]) we may suppose that $n\geqslant 2$. Take j, $1\leqslant j\leqslant n$, arbitrarily. Let $B_j\subset\mathbf{C}$ and $E(z_j)$, $z_j\notin B_j$, be as in the proof of Theorem 4.1, however, with the additional property that for each $z_j\notin B_j$ the function $u_{z_j}\colon(G\backslash E)(z_j)\to(-\infty,\infty]$, $u_{z_j}(Z_j)=u(z_j,Z_j)$, is superharmonic. Actually, B_j can be furnished with this additional property using Fubini's theorem. If $z_j\notin B_j$ and $f_{z_j}\colon(G\backslash E)(z_j)\to\mathbf{C}$, $f_{z_j}(Z_j)=f(z_j,Z_j)$, then $|f_{z_j}(Z_j)|^p\leqslant u_{z_j}(Z_j)$ for all $Z_j\in(G\backslash E)(z_j)$. By [10, Corollary 1, p. 10] $|f_{z_j}|^p$ has a harmonic majorant in $(G\backslash E)(z_j)$. If $n=2$ then by [16, Théorème 20, p. 182] (see also [11, Theorem 1, p. 597]) and if $n\geqslant 3$ then by the induction hypothesis f_{z_j} has a unique holomorphic extension $f_{z_j}^*\colon G(z_j)\to\mathbf{C}$. Thus by [10, Theorem 1, p. 11]

$$\text{(C)}\qquad \int|f_{z_j}(Z_j)|^p\Delta\varphi_{z_j}(Z_j)\,dm_{2n-2}(Z_j)\geqslant 0$$

for each testfunction $\varphi_{z_j} \in D_+(G(z_j))$.

Take now a testfunction $\varphi \in D_+(G)$ arbitrarily. Using Fubini's theorem and proceeding as in the proof of Theorem 4.1 we see with the aid of (C) that

$$\int |f(z)|^p \Delta\varphi(z)\, dm_{2n}(z) \geq 0.$$

Thus by [10, Corollary 1, p. 13] there is a subharmonic function $v: G \to [-\infty, \infty)$ such that $v = |f|^p$ almost everywhere in $G \backslash E$. From [10, Proposition 3, p. 11] it follows that $v = |f|^p$ in $G \backslash E$.

To show that f has a unique holomorphic extension to G it is sufficient to show that $\operatorname{Re} f$ and $\operatorname{Im} f$ have harmonic extensions to G. Since v and thus $|f|^p$ are locally bounded above in G, f and also $\operatorname{Re} f$ and $\operatorname{Im} f$ are locally bounded in G. Since $\operatorname{Re} f$ and $\operatorname{Im} f$ are n-harmonic, it follows from Theorem 4.1 that they have n-harmonic and thus harmonic extensions to G, concluding the proof. □

5.2. THEOREM. *Let $E \subset U^n$ be closed in U^n and n-small. Let $f: U^n \backslash E \to \mathbf{C}$ be a holomorphic function such that for some $p > 0$, $|f|^p$ has an n-harmonic majorant u in $U^n \backslash E$. Then f has a unique holomorphic extension $f^*: U^n \to \mathbf{C}$ such that $|f^*|^p$ has an n-harmonic majorant in U^n.*

Proof. By Theorem 4.1, u has a unique n-hyperharmonic extension $u^*: U^n \to (-\infty, \infty]$, which by Corollary 3.5 is also superharmonic. By Theorem 5.1, f has a unique holomorphic extension $f^*: U^n \to \mathbf{C}$. To show that $|f^*|^p$ has an n-harmonic majorant in U^n we proceed as follows.

Take a nondecreasing sequence $r_k \to 1$ as $k \to \infty$. For each $k = 1, 2, \ldots$ set $B_k = B^2(0, r_k)$ and define $u_k^1: U^n \to \mathbf{R}$,

$$u_k^1(z_1, Z_1) = \begin{cases} \int |f^*(\eta, Z_1)|^p \, d\mu_{z_1}^{B_k}(\eta), & \text{when } z_1 \in B_k, \\ |f^*(z_1, Z_1)|^p, & \text{when } z_1 \in U \backslash B_k. \end{cases}$$

Here $\mu_{z_1}^{B_k}$ is the harmonic measure at $z_1 \in B_k$ over ∂B_k (see [10, pp. 4–5]). Using [10: Theorem, p. 5; Proposition 1, pp. 9–10; and c), p. 8] we see that u_k^1, $k = 1, 2, \ldots$, is a nondecreasing sequence of continuous, n-subharmonic (i.e. subharmonic in each complex variable separately) functions such that for each $Z_1 \in U^{n-1}$ the subharmonic function $u_{kZ_1}^1: U \to \mathbf{R}$, $u_{kZ_1}^1(z_1) = u_k^1(z_1, Z_1)$, is harmonic in B_k. Since $|f^*|^p$ is n-subharmonic and u^* is n-hyperharmonic, we see by the minimum principle [10, Proposition, pp. 6–7] that $|f^*|^p \leq u_k^1 \leq u^*$ for each $k = 1, 2, \ldots$.

Taking then similarly successive Poisson modifications with respect to the variables $z_2, \ldots, z_n$ we get a nondecreasing sequence of continuous, n-subharmonic functions u_k^n, $k = 1, 2, \ldots$, which are n-harmonic in $D^n(0, r_k^*)$ such that $|f^*|^p \leq u_k^n \leq u^*$. Here $r_k^* = (r_k, \ldots, r_k)$ (n copies). Using then Harnack's theorem [10, Corollary, p. 6] we get the desired n-harmonic majorant to $|f^*|^p$ in U^n. □

5.3. COROLLARY. *Let $E \subset U^n$ be closed in U^n and polar. Let $f: U^n \backslash E \to \mathbf{C}$ be a holomorphic function such that for some $p > 0$, $|f|^p$ has an n-harmonic majorant u in $U^n \backslash E$. Then f has a unique holomorphic extension $f^*: U^n \to \mathbf{C}$ such that $|f^*|^p$ has an n-harmonic majorant in U^n.*

5.4. COROLLARY ([17, Theorem 3.5, p. 286]). *Let $E \subset U^n$ be closed in U^n and let $H^{2n-2}(E) < \sigma_\infty$. Let $f: U^n \setminus E \to \mathbf{C}$ be a holomorphic function such that for some $p > 0$, $|f|^p$ has an n-harmonic majorant u in $U^n \setminus E$. Then f has a unique holomorphic extension $f^*: U^n \to \mathbf{C}$ such that $|f^*|^p$ has an n-harmonic majorant in U^n.*

5.5. COROLLARY (cf. [11, Theorem 2, p. 597] and [4, Remark 3, p. 286]). *Let $E \subset U^n$ be n-polar or, more generally, n-negligible. Let $f: U^n \setminus E \to \mathbf{C}$ be a holomorphic function such that for some $p > 0$, $|f|^p$ has an n-harmonic majorant in $U^n \setminus E$. Then f has a unique holomorphic extension $f^*: U^n \to \mathbf{C}$ such that $|f^*|^p$ has an n-harmonic majorant in U^n.*

5.6. THEOREM (cf. [8, Theorem 1.1 (d), p. 703]). *Let G be an open set in $\mathbf{C}^n$. Let $E \subset G$ be closed in G and n-small. Let $f: G \setminus E \to \mathbf{C}$ be a holomorphic function such that $f \in L^2_{\mathrm{loc}}(G)$. Then f has a unique holomorphic extension $f^*: G \to \mathbf{C}$.*

Proof. Proceed essentially as in the proof of Theorem 5.1. Actually, the only difference is that use [1, Theorem B (a), p. 530] (see also [17, Lemma 2.4, p. 284]) instead of [16, Théorème 20, p. 182] or [11, Theorem 1, p. 597] when inferring that the functions f_{z_j}, $z_j \notin B_j$, have holomorphic extensions $f^*_{z_j}$ to $G(z_j)$.

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One can show that though $\mathcal{C}^n$ (see Definition 2.2) is not a capacity it is a precapacity. For the definition of capacity and precapacity see, for example, U. Cegrell, *On product capacities with application to complex analysis,* Séminaire Pierre Lelong–Henri Skoda (Analyse), Années 1978/79 (French), pp. 33–45, Lecture Notes in Math., 822, Springer, Berlin, 1980.

In Remark 2.6 it was stated that polar sets are n-small. This follows also from Lemma 6, p. 115 in A. Sadullaev, *Rational approximation and pluripolar sets* (Russian), Math. Sb. 119, No 1 (1982), 96–118.

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