DISTANCE ESTIMATES AND PRODUCTS OF TOEPLITZ OPERATORS

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1. Introduction. In this paper we establish some results concerning distance estimates. One of these results produces an equivalent condition to the Axler-Chang-Sarason-Volberg theorem. In order to state our results more precisely, we fix some notations which will be used throughout the paper.

By D we will denote the open unit disc in the complex plane, and by ∂D its boundary. Let L^∞ denote the algebra of bounded measurable functions with respect to the Lebesgue measure on ∂D, and H^∞ denote the subalgebra of L^∞ consisting of all bounded analytic functions in D. We identify L^∞ with $C(X)$, the space of continuous functions on X, where X is the maximal ideal space of L^∞. The algebra $H^\infty+C$ is a closed subalgebra of L^∞; here $C=C(\partial D)$. It is known [14] that $H^\infty+C$ is the smallest closed subalgebra of L^∞ which contains H^∞. A closed subalgebra B of L^∞ which contains H^∞ is usually called a Douglas algebra. The maximal ideal space of B is denoted by $M(B)$. The reader is referred to [16] and [10] for the theory of Douglas algebras and to [9] for uniform algebras. The largest C^*-subalgebra of $H^\infty+C$ will be denoted by QC. Thus $QC=(H^\infty+C)\cap\overline{(H^\infty+C)}$, where bar denotes complex conjugation. The sets in the Shilov decomposition [19] of $M(L^\infty)$ associated with $H^\infty+C$ will be called QC-level sets. For $\phi\in M(H^\infty+C)$ the support of the representing measure for ϕ is called a support set. If A is a closed subspace of a Banach space Y and $x\in Y$, then $\operatorname{dist}(x,A)=\inf\{\|x-y\|: y\in A\}$. The annihilator of A in Y^* will be denoted by $A^\perp$, and $\operatorname{Ext}(A^\perp)$ denotes the set of the extreme points of $\operatorname{ball}(A^\perp)$. If B is a subset of Y, then $\operatorname{co}(B)$ denotes the convex hull of B.

The following results will be established.

THEOREM 1. *If A and B are two Douglas algebras such that $H^\infty+C=A\cap B$, then* $\operatorname{dist}(h,H^\infty+C)=\max\{\operatorname{dist}(h,A),\operatorname{dist}(h,B)\}$ *for all h in L^∞. Conversely, if the above condition is true then $H^\infty+C=A\cap B$.*

This result produces an equivalent condition to the Axler-Chang-Sarason-Volberg theorem. A proof of Theorem 1 appears in Section 2.

THEOREM 2. *Let A and B be two closed subalgebras of $C(X)$, where X is a compact Hausdorff space. Then the following conditions are equivalent:*

(1) $\operatorname{dist}(f,A\cap B)=\max\{\operatorname{dist}(f,A),\operatorname{dist}(f,B)\}$ *for all f in $C(X)$;*
(2) $\operatorname{Co}(\operatorname{ball}A^\perp\cup\operatorname{ball}B^\perp)=\operatorname{ball}(A\cap B)^\perp$;
(3) $\operatorname{Ext}((A\cap B)^\perp)\subset\operatorname{Ext}(A^\perp)\cup\operatorname{Ext}(B^\perp)$.

In Section 3, we show that condition (1) of Theorem 2 is not true in general.

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THEOREM 3. *Let A be a closed subalgebra of $C(X)$ which admits best approximation, where X is a compact Hausdorff space. If E is a weak peak set for A, then* $\text{dist}(f|_E, A|_E)$ *is attainable for any f in $C(X)$.*

THEOREM 4. *If A and B are two Douglas algebras which satisfy any one of the equivalent conditions of Theorem 2, then*

$$\text{dist}(f|_E, A\cap B|_E) = \max\{\text{dist}(f|_E, A|_E), \text{dist}(f|_E, B|_E)\}$$

for any f in L^∞ and for any weak peak set E for $A\cap B$.

A special case of Theorem 3 was obtained by D. Sarason [16, p. 110]. Theorems 3 and 4 appear in Section 4.

2. Products of Toeplitz operators. The relevant facts about products of Toeplitz operators were established in [3] and [20]. General background can be found in [16]. For f in L^∞, the Toeplitz operator on the usual Hardy space of $\partial D, H^2$, with symbol f will be denoted by T_f, and the closed subalgebra of L^∞ generated by H^∞ and f will be denoted by $H^\infty[f]$.

THEOREM A (Axler-Chang-Sarason-Volberg Theorem). *For f and g in L^∞, the following conditions are equivalent:*

(i) $T_{\bar f}T_g - T_{\bar f g}$ *is compact;*

(ii) $H^\infty[f]\cap H^\infty[g] \subset H^\infty + C$;

(iii) *For each support set S, either $f|_S$ or $g|_S$ is in $H^\infty|_S$.*

The implication from condition (i) to condition (ii) is due to A. L. Volberg [20]; the other implications are due to S. Axler, A. Chang and D. Sarason and can be found in [3].

We will show that the following condition can be added to the equivalent conditions (i)–(iii) of Theorem A: (iv) Either

$$\text{dist}(h, H^\infty + C) = \max\{\text{dist}(h, H^\infty[f]), \text{dist}(h, H^\infty[g])\}$$

for all h in L^∞, or

$$\text{dist}(h, H^\infty) = \max\{\text{dist}(h, H^\infty[f]), \text{dist}(h, H^\infty[g])\}$$

for all h in L^∞.

First we establish the following result.

THEOREM 1. *Let A and B be two Douglas algebras such that $H^\infty + C = A\cap B$. Then* $\text{dist}(h, H^\infty + C) = \max\{\text{dist}(h, A), \text{dist}(h, B)\}$ *for all h in L^∞. Conversely, if the above condition is satisfied then $H^\infty + C = A\cap B$.*

The following results will be needed in our proof of Theorem 1.

THEOREM B ([18], [21, p. 52]). *Let f and g be functions in L^∞ such that, for each support set S, either $f|_S$ or $g|_S$ is in $H^\infty|_S$. Then, for each QC-level set E, either $f|_E$ or $g|_E$ is in $H^\infty|_E$.*

THEOREM C ([12, Theorem 3.4]). *Let A and B be two Douglas algebras; then $M(A\cap B) = M(A)\cup M(B)$.*

Proof of Theorem 1. Assume that $H^\infty + C = A \cap B$. Let $h \in L^\infty$. Then

$$\operatorname{dist}(h, H^\infty + C) = \left|\int h\, d\mu\right|,$$

for some $\mu \in \operatorname{Ext}(H^\infty + C)^\perp$, [11, p. 419]. Since $\operatorname{supp}\mu$ (=support of μ) is an antisymmetric set for $H^\infty + C$, then $\operatorname{supp}\mu \subset E_0$, for some QC-level set E_0. We claim that $\mu \in A^\perp \cup B^\perp$. Suppose not. Then there exist $f \in A$ and $g \in B$ such that $\int f\, d\mu \neq 0$ and $\int g\, d\mu \neq 0$. By Theorem C, $M(H^\infty + C) = M(A) \cup M(B)$. Thus for each support set S, either $f|_S$ or $g|_S$ is in $H^\infty|_S$ (recall that $A|_S = H^\infty|_S$). By Theorem B we get either $f|_E$ or $g|_E$ is in $H^\infty|_E$ for each QC-level set E. In particular, $f|_{E_0}$ or $g|_{E_0}$ is in $H^\infty|_{E_0}$. This shows that $\int f\, d\mu = 0$ or $\int g\, d\mu = 0$. This contradiction establishes our claim that $\mu \in A^\perp \cup B^\perp$. So let us say, for example, that $\mu \in A^\perp$. Then $\operatorname{dist}(h, A) \geqslant |\int h\, d\mu| = \operatorname{dist}(h, H^\infty + C)$. Since the reverse inequality is clear ($H^\infty + C \subseteq A$), it follows that $\operatorname{dist}(h, H^\infty + C) = \operatorname{dist}(h, A)$. Consequently, $\operatorname{dist}(h, H^\infty + C) = \max\{\operatorname{dist}(h, A), \operatorname{dist}(h, B)\}$.

Conversely, assume that $\operatorname{dist}(h, H^\infty + C) = \max\{\operatorname{dist}(h, A), \operatorname{dist}(h, B)\}$ for all h in L^∞. By taking $h \in H^\infty + C$, the equality shows that $H^\infty + C \subset A \cap B$. By taking $h \in A \cap B$, the equality shows that $A \cap B \subset H^\infty + C$, and that ends the proof of Theorem 1. □

In virtue of what has just been proved, condition (iv) can be added to the equivalent conditions (i)–(iii) of Theorem A.

3. A general result on distance estimates. This section contains a proof of Theorem 2 and several examples. One of these examples shows that condition (1) of Theorem 2 is not true in general.

Proof of Theorem 2. Suppose that condition (1) is valid. By [8, p. 415] and the Kreĭn–Milman theorem we have

$$\overline{\operatorname{co}}(\operatorname{Ext}(A^\perp) \cup \operatorname{Ext}(B^\perp)) = \operatorname{co}(\operatorname{ball}(A^\perp) \cup \operatorname{ball}(B^\perp)).$$

Thus if condition (2) is not valid, then there exists $\mu_0 \in \operatorname{ball}((A \cap B)^\perp)$, and $\mu_0 \notin \overline{\operatorname{co}}(\operatorname{Ext}(A^\perp) \cup \operatorname{Ext}(B^\perp))$. Hence by [22, p. 108] there exists a continuous linear functional f such that $f(\mu_0) > 1$ and $|f(\mu)| \leqslant 1$ for all $\mu \in \overline{\operatorname{co}}(\operatorname{Ext}(A^\perp) \cup \operatorname{Ext}(B^\perp))$. Note that the set $\overline{\operatorname{co}}(\operatorname{Ext}(A^\perp) \cup \operatorname{Ext}(B^\perp))$ is a balanced convex subset of $(C(X))^*$ and that $(C(X))^*$ with its w^*-topology is a locally convex linear topological vector space. Thus we can consider f to be an element of $C(X)$. Consequently, $\operatorname{dist}(f, A \cap B) \geqslant \int f\, d\mu_0 > 1$, while $\operatorname{dist}(f, A) \leqslant 1$ and $\operatorname{dist}(f, B) \leqslant 1$. This contradiction shows that condition (1) implies condition (2).

Now, suppose that condition (2) is valid. Let $\mu \in \operatorname{Ext}((A \cap B)^\perp)$ be written as $\mu = \sum \alpha_i \mu_i + \sum \beta_i v_i$, where $\alpha_i \geqslant 0$, $\beta_i \geqslant 0$, $\sum \alpha_i + \sum \beta_i = 1$, $\mu_i \in \operatorname{ball}(A^\perp)$, $v_i \in \operatorname{ball}(B^\perp)$, for each i. Since μ is extreme, $\mu = \mu_i$ for some i, or $\mu = v_i$ for some i. Thus $\mu \in A^\perp \cup B^\perp$ and hence $\mu \in \operatorname{Ext}(A^\perp) \cup \operatorname{Ext}(B^\perp)$. This shows that condition (2) implies condition (3).

Finally, assume that condition (3) is valid. Let $f \in C(X)$. Then $\operatorname{dist}(f, A \cap B) = |\int f\, d\mu|$, for some $\mu \in \operatorname{Ext}((A \cap B)^\perp)$. By condition (3), $\mu \in \operatorname{Ext}(A^\perp) \cup \operatorname{Ext}(B^\perp)$. So assume that $\mu \in \operatorname{Ext}(A^\perp)$. Thus $\operatorname{dist}(f, A) \geqslant |\int f\, d\mu| = \operatorname{dist}(f, A \cap B)$. Hence

$\operatorname{dist}(f, A\cap B)=\max\{\operatorname{dist}(f,A), \operatorname{dist}(f,B)\}$. This shows that condition (3) implies condition (1). This ends the proof of Theorem 2. □

REMARK. Theorem 2 can be extended to arbitrary finite intersection of closed subalgebras of $C(X)$.

EXAMPLES. (1) Let H be a fixed closed subalgebra of $C(X)$. If A and B are closed subalgebras of $C(X)$ containing H such that A/H and B/H are M-ideals of $C(X)/H$, then $\operatorname{dist}(f, A\cap B)=\max\{\operatorname{dist}(f,A), \operatorname{dist}(f,B)\}$ for every f in $C(X)$. (See Section 4 for the definition of an M-ideal.) This is due to the fact that $\operatorname{co}(\operatorname{ball}(A^\perp)\cup \operatorname{ball}(B^\perp))=\operatorname{ball}(A\cap B)^\perp$ [4, p. 37].

(2) If A and B are closed subalgebras of $C(X)$ such that every measure in $A^\perp$ is singular to every measure in $B^\perp$, then $(A\cap B)^\perp=A^\perp\oplus B^\perp$ [6, Theorem 11.3]. Let $\mu\in \operatorname{Ext}((A\cap B)^\perp)$ and suppose that $\mu=\mu_1+\mu_2$, $\mu_1\neq 0$, $\mu_2\neq 0$, $\mu_1\in A^\perp$ and $\mu_2\in B^\perp$. Then $\mu=\|\mu_1\|(\mu_1/\|\mu_1\|)+\|\mu_2\|(\mu_2/\|\mu_2\|)$. Since $\|\mu_1\|+\|\mu_2\|=1$, we get $\mu=\mu_1/\|\mu_1\|=\mu_2/\|\mu_2\|$. This shows that $\mu\in\operatorname{Ext}(A^\perp)\cup\operatorname{Ext}(B^\perp)$. By Theorem 2 we get $\operatorname{dist}(f, A\cap B)=\max\{\operatorname{dist}(f,A), \operatorname{dist}(f,B)\}$ for all f in $C(X)$.

(3) Let A and B be two closed subalgebras of $C(X)$ such that $A+B$ is not norm closed; then condition (1) of Theorem 2 is not valid in this case. Indeed, if A and B are any two closed subalgebras of $C(X)$ which satisfy Theorem 2, then condition (2) implies easily that $(A\cap B)^\perp=A^\perp+B^\perp$, which implies by [5, Lemma 2.7.7] that $A+B$ is norm closed in $C(X)$.

(4) In [1], Adamjan, Arov and Kreĭn gave an example of a function $v\in C$ which has no nearest point in $A_0=H^\infty\cap C$. A modification of the example was used by the authors of [7, p. 57]. It is shown in [7, Theorem 10.5] that there is an $f\in H^\infty$ with $\operatorname{dist}(f,C)=1$ and $\operatorname{dist}(f,H^\infty\cap C)=2$. Thus condition (1) of Theorem 2 is not true in general. Note that in this example, $A+B=H^\infty+C$ is norm closed in L^∞, in contrast with Example 3.

4. Local best approximation. A subspace $\mathcal{J}$ of a Banach space Y is called an M-ideal if there exists an L-projection $P\colon Y^*\to\mathcal{J}^\perp$ such that $\|\mu\|=\|P\mu\|+\|\mu-P\mu\|$ for all $\mu\in Y^*$; here P is onto. If $\mathcal{J}$ is an M-ideal of Y and $y\in Y$, then there exists $x\in\mathcal{J}$ such that $\operatorname{dist}(y,\mathcal{J})=\|y-X\|$, [4, p. 126].

In this section we give proofs of Theorems 3 and 4. A special case of Theorem 3 was obtained by D. Sarason. He showed, in the case $A=H^\infty$ and $E=X_\alpha=\{\phi\in M(L^\infty)\colon \phi(z)=\alpha\}$, that $\operatorname{dist}(f|_{X_\alpha}, H^\infty|_{X_\alpha})$ is attainable [16, p. 110], where he used the fact that $H^\infty+C$ admits best approximation ([2], [23]).

To prove Theorem 3, we use an M-ideal approach. We need the following lemma.

LEMMA 1. *If A is a closed subalgebra of $C(X)$ and E is a weak peak set for A then* $\operatorname{dist}(f|_E, A|_E)=\operatorname{dist}(f,A_E)$ *for any f in $C(X)$.*

By definition, $A_E=\{f\in C(X)\colon f|_E\in A|_E\}$. It is a closed subalgebra of $C(X)$.

Proof of Lemma 1. Let $f\in C(X)$; then $\operatorname{dist}(f,A_E)=|\int f\,d\mu|$ for some $\mu\in \operatorname{Ext}(A_E^\perp)$. Thus $\operatorname{dist}(f,A_E)=|\int (f-g)\,d\mu|$ for all $g\in A$. Hence $\operatorname{dist}(f,A_E)\leqslant \|f|_E-g|_E\|$ for all $g\in A$. Consequently, $\operatorname{dist}(f,A_E)\leqslant\|f|_E-A|_E\|$. On the other hand, $\|f|_E-A|_E\|=\|f|_E-A_E|_E\|\leqslant \operatorname{dist}(f,A_E)$. Thus $\operatorname{dist}(f,A_E)=\operatorname{dist}(f|_E,A|_E)$. This ends the proof of Lemma 1. □

Proof of Theorem 3. First, we note that A_E/A is an M-ideal of $C(X)/A$. To see this, we identify $(C(X)/A)^*=A^\perp$ and $(A_E/A)^\perp=(A_E)^\perp$. Define the L-projection $P: A^\perp \to (A_E)^\perp$ by $P\mu=\chi_E\mu$, $\mu\in A^\perp$. It is easy to check that it is an L-projection. Thus, we get $\operatorname{dist}(f,A_E)=\operatorname{dist}(f-g,A)$, for some $g\in A_E$. Since A admits best approximation, then we have $\operatorname{dist}(f,A_E)=\|f-g-h\|$, for some $h\in A$. By Lemma 1, $\operatorname{dist}(f|_E,A|_E)=\|f-G\|$, $G\in A_E$. Take $k\in A$ such that $k|_E=G|_E$. Observe that $\operatorname{dist}(f|_E,A|_E)\leqslant\|f|_E-k|_E\|\leqslant\|f-G\|=\operatorname{dist}(f|_E,A|_E)$. This completes the proof of Theorem 3. □

Proof of Theorem 4. Define $(A\cap B)_E=\{f\in L^\infty: f|_E\in(A\cap B)|_E\}$, $A_E=\{f\in L^\infty: f|_E\in A|_E\}$ and $B_E=\{f\in L^\infty: f|_E\in B|_E\}$. All these algebras are Douglas algebras. We claim that $(A\cap B)_E=A_E\cap B_E$. It is clear that $(A\cap B)_E\subset A_E\cap B_E$. Let $m\in M((A\cap B)_E)$. Then $m\in M(A\cap B)$ with supp $\mu_m\subset E$; here μ_m is the representing measure for m. By Theorem C, $m\in M(A)$ or $m\in M(B)$. So assume that $m\in M(A)$. Then $m\in M(A_E)$. Thus we have $M((A\cap B)_E)=M(A_E)\cup M(B_E)$, which is equal to $M(A_E\cap B_E)$ by Theorem C. By the Chang-Marshall theorem [15] we get $(A\cap B)_E=A_E\cap B_E$. This proves the claim.

Now, let $f\in L^\infty$. Then $\operatorname{dist}(f, (A\cap B)_E)=|\int f\,d\mu_0|$, for some

$$\mu_0\in \operatorname{Ext}((A\cap B)_E^\perp).$$

We claim that μ_0 is an extreme point of $\operatorname{ball}((A\cap B)^\perp)$. To see this, suppose that $\mu_0=\frac{1}{2}\mu_1+\frac{1}{2}\mu_2$, μ_1 and $\mu_2\in\operatorname{ball}((A\cap B)^\perp)$. Thus $\mu_0=\frac{1}{2}\chi_E\mu_1+\frac{1}{2}\chi_E\mu_2$; here χ_E is the characteristic function of E. The measures $\chi_E\mu_1$ and $\chi_E\mu_2$ both belong to $((A\cap B)_E^\perp)$. Since $\mu_0\in\operatorname{Ext}((A\cap B)_E^\perp)$, we get $\mu_0=\chi_E\mu_1=\chi_E\mu_2$. Since $\|\mu_1\|=\|\chi_E\mu_1\|+\|\mu_1-\chi_E\mu_1\|$, $\|\chi_E\mu_1\|=1$ and $\|\mu_1\|\leqslant 1$, we get $\|\mu_1-\chi_E\mu_1\|=0$. This shows that $\mu_1=\chi_E\mu_1$. Hence $\mu_1\in((A\cap B)_E^\perp)$. Similarly, $\mu_2\in((A\cap B)_E^\perp)$. Since μ_0 is an extreme point of $((A\cap B)_E^\perp)$, we get $\mu_0=\mu_1=\mu_2$. This proves our claim that $\mu_0\in\operatorname{Ext}((A\cap B)^\perp)$.

By Theorem 2, $\mu_0\in\operatorname{Ext}(A^\perp)\cup\operatorname{Ext}(B^\perp)$. So let us say, for example, that $\mu_0\in A^\perp$. Since support $\mu_0\subset E$, we get $\mu_0\in(A_E)^\perp$. Thus $\operatorname{dist}(f,A_E)\geqslant|\int f\,d\mu_0|=\operatorname{dist}(f, (A\cap B)_E)$. This shows that

$$\operatorname{dist}(f, (A\cap B)_E)=\max\{\operatorname{dist}(f,A_E), \operatorname{dist}(f,B_E)\}.$$

By Lemma 1, we get $\operatorname{dist}(f|_E, (A\cap B|_E)=\max\{\operatorname{dist}(f|_E,A|_E), \operatorname{dist}(f|_E,B|_E)\}$. This ends the proof of Theorem 4. □

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