TWO COUNTABILITY PROPERTIES OF SETS OF MEASURES

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1. Introduction. Let X be a (Hausdorff) topological space and let $C(X)$ denote the space of bounded continuous real-valued functions on X. The space of (non-negative) bounded σ-additive Baire measures on X is denoted by $M_\sigma(X)$ ($M_\sigma^+(X)$). This paper deals with the following countability properties:

(a) A subset M of $M_\sigma(X)$ is called *countably separated* (c.s.) if there exists a sequence $\{f_n\}$ in $C(X)$ such that for every μ and $\nu \in M$

$$(*) \qquad \int f_n \, d\mu = \int f_n \, d\nu \quad \text{for all } n \Rightarrow \mu = \nu.$$

(b) A subset M of $M_\sigma(X)$ (resp. $M_\sigma^+(X)$) is called *countably determined* (c.d.) in $M_\sigma(X)$ (resp. in $M_\sigma^+(X)$) if there exists a sequence $\{f_n\}$ in $C(X)$ such that for every $\mu \in M_\sigma(X)$ (resp. $\mu \in M_\sigma^+(X)$) and $\nu \in M$

$$\int f_n \, d\mu = \int f_n \, d\nu \quad \text{for all } n \Rightarrow \mu \in M.$$

Countability properties of this kind occur naturally in classical and functional analysis, probability theory and general topology. Here are some examples.

The classical moment problem (see VII.3 in [6]) relates to $\mathbf{R}$ and the particular sequence $f_n(x) = x^n$, $x \in \mathbf{R}$. It is clear that if μ, ν are carried by a bounded closed interval, then $(*)$ holds. If μ, ν are arbitrary Baire measures on $\mathbf{R}$, $(*)$ does not hold, even if all moments are finite (see example on page 227 in [6]). However, a different sequence $\{f_n\}$ exists such that $(*)$ holds for every μ and $\nu \in M_\sigma(\mathbf{R})$, that is, $M_\sigma(\mathbf{R})$ is c.s. In fact this is true in a more general set-up (see §4).

The c.s. property is related to the separability of $C(X)$ as follows: $M_\sigma(X)$ is c.s. if and only if $C(X)$ is separable in the weak topology $\sigma(C(X), M_\sigma(X))$, or equivalently in any locally convex topology which yields $M_\sigma(X)$ as dual space (see §4).

A topological space Y is called *separably submetrizable* [20] if there exists a sequence $\{g_n\}$ in $C(Y)$ which separates points of Y. It is clear that Y is separably submetrizable if and only if Y with its Baire σ-algebra is a countably separated measurable space [5, p. 6] if and only if the set $M = \{\delta_y : y \in Y\}$ of Dirac measures on Y is c.s.

If M is a c.s. subset of $M_\sigma(X)$ and $\{f_n\}$ is as in the definition of the c.s. property, the sequence $g_n : M \to \mathbf{R}$, $n = 1, 2, \ldots$, with

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$$(**) \qquad g_n(\mu) = \int f_n \, d\mu$$

separates the elements of M. Thus M endowed with the relativization of the weak topology $\sigma(M_\sigma(X), C(X))$ is separably submetrizable. The converse is also true when M is a vector subspace of $M_\sigma(X)$, but not in general (see §2).

The c.d. property appears naturally as follows: consider $M_\sigma(X)$ as a subset of $\mathbf{R}^{C(X)}$. Then a set $M \subset M_\sigma(X)$ is c.d. if and only if M is determined by countably many indices in the sense of [15].

A subset Y of a topological space X is called *distinguishable* in X [7, p. 408] if there exists a sequence $\{g_n\}$ in $C(X)$ such that for every $x \in X$ and $y \in Y$

$$g_n(x) = g_n(y) \quad \text{for all } n \Rightarrow x \in Y.$$

Every c.d. set of measures is distinguishable, but not conversely (see §2). The above discussion shows that the c.s. (resp. the c.d.) property of a set of measures M is derived from the notion of separable submetrizability (resp. distinguishability) if we require the sequence $\{g_n\}$ to be of the form $(**)$.

The purpose of this paper is to summarize conditions on X and a set M of measures which satisfies the c.s. and c.d. properties. Our own work on the c.s. property received impetus from the following well-known result.

THEOREM A. *For a compact space X the following are equivalent:*
(i) $M_\sigma^+(X)$ *is* c.s.;
(ii) $M_\sigma^+(X)$ *is metrizable in* $\sigma(M_\sigma(X), C(X))$;
(iii) X *is metrizable;*
(iv) $C(X)$ *is norm-separable.*

A proof of the equivalence of (iv) to (i) and to (iii) follows easily from the Stone–Weierstrass Theorem and the fact that every separably submetrizable compact space is metrizable. The equivalence (ii)$\Leftrightarrow$(iii) is a well-known special case of [21, Part II, Theorem 13].

Thus for a compact space X the c.s. property for $M_\sigma^+(X)$ is fully characterized by the above theorem. In §3 we consider a compact space X and our focus of attention is the set M_X of non-atomic measures in $M_\sigma^+(X)$. We show that the c.s. property for M_X is equivalent to M_X being separable metric; but X need not be metrizable. This property implies the c.d. property for M_X and, when X has no isolated points, the separability of X. Moreover, we show that in general no other implication between these three conditions is valid. Thus the countability properties for M_X describe two classes of compact spaces which properly contain compact metric spaces. We also prove that for compact totally ordered spaces without isolated points, each of the countability properties for M_X is equivalent to the separability of X.

If X is an arbitrary Hausdorff space and $M_\tau(X)$ and $M_t(X)$ denote the spaces of the τ-additive and tight (or Radon) measures on X (see [21]), then the inclusions $M_t(X) \subset M_\tau(X) \subset M_\sigma(X)$ hold. When X is completely regular, there are strict topologies β_σ, β_τ and β_t on $C(X)$, which yield $M_\sigma(X), M_\tau(X)$ and $M_t(X)$ as dual spaces, respectively ([17]; see also [22]). When X is compact, the above strict topologies coincide with supremum-norm topology and, of course, the corresponding spaces of measures coincide as well.

In §4 we generalize Theorem A to non-compact spaces. We show that $M_s(X)$ (for $s=\sigma,\tau$ or t) is c.s. if and only if $(C(X),\beta_s)$ is separable if and only if $M_s(X)$ is separably submetrizable. Summers [20], using a Stone–Weierstrass Theorem for $(C(X),\beta_t)$, has proved that the last two conditions for $s=t$ are equivalent to X being separably submetrizable. Our approach does not make use of Stone–Weierstrass Theorem (in fact, $(C(X),\beta_\sigma)$ and $(C(X),\beta_\tau)$ do not have in general the Stone–Weierstrass property [8]); for $s=\tau$ we have a positive answer to a question of Summers. We also examine the validity of (iii)⇒(i) in Theorem A. In [20] it is proved in an equivalent form that $M_t^+(X)$ is c.s. if X is metrizable with cardinality $\leqslant c$, the cardinal of the continuum. We extend this result to $M_\tau^+(X)$ and show that it cannot be extended to $M_\sigma^+(X)$. Finally, we show that these results remain valid for some non-metrizable spaces.

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NOTATIONS. All topological spaces X are assumed to be at least Hausdorff. A zero set in X is a set of the form $Z=f^{-1}(\{0\})$ for some $f\in C(X)$ and a cozero set in X is a complement of a zero set. The family of Baire (resp. Borel) sets is the σ-algebra $\mathcal{B}(X)$ (resp. $\mathcal{B}_0(X)$) generated by the zero (resp. closed) subsets of X.

If $\mu\in M_\sigma(X)$ and $f\in C(X)$, we write $\mu(f)$ instead of $\int f\,d\mu$ considering μ as a functional on $C(X)$. All topological statements about the spaces of measures will be with respect to the weak topology $\sigma(M_\sigma(X),C(X))$.

If $f:X\to Y$ is a continuous function, we define $f_*:M_\sigma(X)\to M_\sigma(Y)$ by $f_*(\mu)(g)=\mu(g\circ f)$ for all $g\in C(Y)$. It is well known that f_* is continuous, $f_*(M_s(X))\subset M_s(Y)$ for $s=\sigma$, τ or t and $f_*(\mu)(B)=\mu(f^{-1}(B))$ for $B\in\mathcal{B}(Y)$.

For a Baire (resp. Borel) measure μ on X and a measurable set A, we denote by μ_A the Baire (resp. Borel) measure on X with $\mu_A(B)=\mu(A\cap B)$. The cardinal of a set A is denoted by $\operatorname{card}(A)$.

2. Preliminary results. This section contains some useful characterizations of the c.s. and c.d. properties for sets of measures on an arbitrary topological space X and connections to some known countability properties.

PROPOSITION 2.1. *For a subset M of $M_\sigma(X)$ the following are equivalent:*

(i) *M is* c.s.;

(ii) *there is a countable family H of bounded Baire measurable functions such that if $\mu,\nu\in M$, then*

$$\int f\,d\mu=\int f\,d\nu \quad \text{for all } f\in H\Rightarrow \mu=\nu;$$

(iii) *there is a countably generated σ-algebra $\mathcal{B}$ of Baire sets such that if $\mu,\nu\in M$, then*

$$\mu(B)=\nu(B) \quad \text{for all } B\in\mathcal{B}\Rightarrow\mu=\nu;$$

(iv) *there is a continuous function* f *from* X *onto a separable metric space such that if* $\mu, \nu \in M$, *then*

$$f_*(\mu) = f_*(\nu) \Rightarrow \mu = \nu.$$

PROPOSITION 2.2. *For a subset* M *of* $M_\sigma(X)$ (*resp.* $M_\sigma^+(X)$) *the following are equivalent:*

(i) M *is* c.d. *in* $M_\sigma(X)$ (*resp. in* $M_\sigma^+(X)$);

(ii) *there is a countable family* H *of bounded Baire measurable functions such that if* $\mu \in M_\sigma(X)$ (*resp.* $\mu \in M_\sigma^+(X)$) *and* $\nu \in M$, *then*

$$\int f\,d\mu = \int f\,d\nu \quad \text{for all } f \in H \Rightarrow \mu \in M;$$

(iii) *there is a countably generated* σ*-algebra* $\mathcal{B}$ *of Baire sets such that if* $\mu \in M_\sigma(X)$ (*resp.* $\mu \in M_\sigma^+(X)$) *and* $\nu \in M$, *then*

$$\mu(B) = \nu(B) \quad \text{for all } B \in \mathcal{B} \Rightarrow \mu \in M;$$

(iv) *there is a continuous function* f *from* X *onto a separable metric space such that if* $\mu \in M_\sigma(X)$ (*resp.* $\mu \in M_\sigma^+(X)$) *and* $\nu \in M$, *then*

$$f_*(\mu) = f_*(\nu) \Rightarrow \mu \in M.$$

The proofs of the above propositions are based on the method used in [11, 3.2 and 3.3] and are omitted.

The next two propositions contain some cases where the c.s. and c.d. properties coincide with the separable submetrizability and distinguishability, respectively.

PROPOSITION 2.3. (i) *For a compact subset* M *of* $M_\sigma(X)$ *we have:* M *is* c.s. *if and only if* M *is separably submetrizable if and only if* M *is metrizable.*

(ii) *For a compact subset* M *of* $M_\sigma(X)$ (*resp.* $M_\sigma^+(X)$) *we have:* M *is* c.d. *if and only if* M *is distinguishable if and only if* M *is* G_δ *in* $M_\sigma(X)$ (*resp. in* $M_\sigma^+(X)$).

Proof. (i) Every compact separably submetrizable space is metrizable, so it is enough to show that the metrizability of M implies the c.s. property. Since M is also compact, it is second countable. Thus we can find $\mu_n \in M$, $f_n \in C(X)$, $\epsilon_n > 0$, for $n = 1, 2, \ldots$, such that all finite intersections of the sets

$$U_n = \{\mu \in M : |(\mu - \mu_n)(f_n)| < \epsilon_n\}, \quad n = 1, 2, \ldots$$

form a base for the open sets in M. Then $\{f_n\}$ separates measures in M.

(ii) Every compact distinguishable set is G_δ, so it is enough to show that if M is G_δ then M is c.d. Using the compactness of M, we can find $\mu_n \in M$, $f_n \in C(X)$, $\epsilon_n > 0$, for $n = 1, 2, \ldots$, such that M is the intersection of some finite unions of the sets

$$U_n = \{\mu \in M_\sigma(X) : |(\mu - \mu_n)(f_n)| < \epsilon_n\}, \quad n = 1, 2, \ldots.$$

Then $\{f_n\}$ satisfies the definition of the c.d. property for M. □

PROPOSITION 2.4. (i) *If M is a vector subspace of $M_\sigma(X)$, then M is* c.s. *if and only if M is separably submetrizable.* (ii) *If M is an arbitrary subset of the vector space $M_\sigma(X)$, then M is* c.d. *if and only if M is distinguishable in $M_\sigma(X)$.*

The proof follows directly from [11, Lemma 2.8 (ii)].

We now show by means of examples that the assumptions on M in Propositions 2.3 and 2.4(i) are necessary and that Proposition 2.4(ii) is not true if we replace $M_\sigma(X)$ by its positive cone $M_\sigma^+(X)$.

EXAMPLES 2.5. (a) Let S be the Sorgenfrey line (i.e. the real line with the topology of the right half-open intervals) and M the set of all Dirac measures on S. Then M is c.s. (and separably submetrizable) but not metrizable since M is homeomorphic to S.

(b) The set of all non-negative measures on $\mathbf{R}$ with compact support is c.d. (and distinguishable) in $M_\sigma^+(\mathbf{R})$ but not G_δ (cf. [11, Proposition 2.7]).

(c) Let $X=Y\cup\{\infty\}$ be the one-point compactification of a discrete space Y with cardinality $\aleph_1$ and

$$M=\{\mu\in M_\sigma^+(X):\mu^*(\{\infty\})=0\}.$$

Then M is homeomorphic to $M_\tau^+(Y)$, hence metrizable (see [21, Part II, Theorem 13]) and separably submetrizable since $\operatorname{card}(M)\leqslant c$ (see [20, Corollary 3.3]). M is also G_δ in $M_\sigma^+(X)$ (see [21, Part II, Theorem 17]). However, M is neither c.s. nor c.d. in $M_\sigma^+(X)$. This follows from the fact that every continuous function on X is constant on the complement of a countable subset.

(d) Assume that there are no real-valued measurable cardinals. Let X be a discrete space with $\operatorname{card}(X)>c$ and M the set of Dirac measures on X. Then $M_\sigma^+(X)=M_\tau^+(X)$ is metrizable [21, Part II, Theorem 13] and M is closed in $M_\sigma^+(X)$. Thus M is metrizable, distinguishable and G_δ in $M_\sigma^+(X)$. Since $\operatorname{card}(X)>c$, it follows from the next proposition that M is neither c.s. nor c.d.

PROPOSITION 2.6. *For the set M of Dirac measures on a completely regular space X the following are equivalent:*

(i) *M is* c.s. (*that is, X is separably submetrizable*);

(ii) *M is* c.d.

Proof. (i)⇒(ii). We have that there is a continuous one-to-one function $f:X\to\mathbf{R}^{\mathbf{N}}$. The result now follows easily from Proposition 2.2(iv).

(ii)⇒(i). Let $\{f_n\}$ be as in the definition of the c.d. property for M and $x,y\in X$ such that $\delta_x(f_n)=\delta_y(f_n)$ for all n. Then $(\frac12\delta_x+\frac12\delta_y)(f_n)=\delta_y(f_n)$ for all n. It follows that $\frac12\delta_x+\frac12\delta_y\in M$ and, since X is completely regular, $\delta_x=\delta_y$. □

We recall that a subset M of $M_\sigma^+(X)$ is *uniformly regular* (C^*-uniformly regular in [2]) if there is a separable continuous pseudometric d on X such that $\mu(Z)=\mu(\bar{Z}^d)$, for every $\mu\in M$ and every zero set Z in X. (Here $\bar{Z}^d$ denotes the closure of Z with respect to the topology of d.) In the case where $M=\{\mu\}$, the measure μ is called uniformly regular [1].

PROPOSITION 2.7. *If $M\subset M_\sigma^+(X)$ is uniformly regular then M is both* c.s. *and* c.d.

Proof. Let d be as in the definition of uniform regularity for M and let $\mathcal{B}$ be the σ-algebra of all subsets of X which are Borel with respect to the topology of d. Then $\mathcal{B}$ is a countably generated sub-σ-algebra of $\mathcal{B}(X)$. Moreover, if $\mu \in M_\sigma^+(X)$, $\nu \in M$ and $\mu(B)=\nu(B)$ for all $B \in \mathcal{B}$, then $\nu(Z)=\nu(\bar{Z}^d)=\mu(\bar{Z}^d) \geqslant \mu(Z)$ for every zero set Z in X. By the property of regularity, we have $\mu \leqslant \nu$ and, since $\mu(X)=\nu(X)$, we deduce that $\mu=\nu$. It follows from Propositions 2.1 and 2.2 that M is c.s. and c.d. in $M_\sigma^+(X)$. □

The converse of the above proposition is not correct in general. Indeed, let X be any completely regular space which admits a G_δ point x that has no countable base (for example, take X to be a countable dense subspace of $[0,1]^c$ with the product topology). Then the set $M=\{\delta_x\}$ is c.s. and c.d. in $M_\sigma^+(X)$. This follows from Proposition 2.3 since for any $f \in C(X)$, $f \geqslant 0$ with $\{x\}=f^{-1}(\{0\})$ we have

$$M=\bigcap_{n=1}^{\infty}\left\{\mu \in M_\sigma^+(X): |\mu(f)|<\frac{1}{n} \text{ and } |\mu(1)-\delta_x(1)|<\frac{1}{n}\right\},$$

so M is G_δ in $M_\sigma^+(X)$. However, M is not uniformly regular since x has no countable base.

We finish this section with the following characterization of uniform regularity.

PROPOSITION 2.8. *A subset M of $M_\sigma^+(X)$ is uniformly regular if and only if there is a sequence $\{U_n\}$ of cozero sets in X such that if U is any cozero set then*

$$\mu(U \setminus \bigcup \{U_n : U_n \subset U\})=0$$

for all $\mu \in M$.

Proof. Assume that $\{U_n\}$ is as in the statement of the proposition and, for each n, let $f_n : X \to [0,1]$ be a continuous function with $U_n=\{x \in X : f_n(x)>0\}$. Define $d(x,y)=\sum_{n=1}^{\infty}(1/2^n)|f_n(x)-f_n(y)|$. Then d is a separable continuous pseudometric on X and each U_n is d-open. If U is any cozero set in X, then

$$\bigcup\{U_n : U_n \subset U\} \subset \operatorname{int}_d(U) \subset U.$$

Therefore $\mu(U)=\mu(\operatorname{int}_d(U))$ for all $\mu \in M$.

Conversely, assume that M is uniformly regular. Let d be as in the definition of uniform regularity and $\{U_n\}$ be a countable base for the topology of d. Then each U_n is cozero in X. Moreover, if U is any cozero set in X, then $\operatorname{int}_d(U)= \bigcup\{U_n : U_n \subset U\}$. Therefore $\mu(U \setminus \bigcup\{U_n : U_n \subset U\})=0$ for all $\mu \in M$. □

3. Compact spaces. All measures in this section are assumed to be non-negative; in particular, the c.d. property is always referred to $M_\sigma^+(X)$. Here we study the c.s. and c.d. properties for the set of non-atomic measures on a compact space X. We note that every $\mu \in M_\sigma^+(X)$ has a unique regular Borel extension, so we assume that μ is defined on the Borel sets of X.

Our first result is to show that the converse of Proposition 2.7 is true for compact spaces.

THEOREM 3.1. *Let X be a compact space and $M \subset M_\sigma^+(X)$. The following are equivalent:*

(i) *M is* c.s. *and* c.d.;

(ii) *M is uniformly regular.*

Proof. We prove only (i) $\Rightarrow$ (ii) since, by Proposition 2.7, the converse is true in general.

By Proposition 2.1 and 2.2, there is a compact metric space T and a continuous function f from X onto T such that $f_*: M_\sigma^+(X) \to M_\sigma^+(T)$ is one-to-one on M and $M = f_*^{-1}(f_*(M))$. Let $\{V_n\}$ be a countable base for the topology of T and set $U_n = f^{-1}(V_n)$. We show that $\{U_n\}$ satisfies the condition of Proposition 2.8.

Let W be any cozero subset of X and set $W_0 = \{x \in W : f^{-1}(\{f(x)\}) \subset W\}$. Clearly, W_0 is open in X and we claim that $\mu(W_0) = \mu(W)$ for all $\mu \in M$. Suppose that this is not the case. Then there is a compact set $K \subset W \setminus W_0$ and $\mu \in M$ such that $\mu(K) > 0$. Let $L = (X \setminus W) \cap f^{-1}(f(K))$ and observe that $f(K) = f(L)$. It follows that there is $\nu \in M_\sigma^+(X)$ such that $\nu(L) = \nu(X)$ and $f_*(\nu) = f_*(\mu_K)$ (see e.g. [11, Lemma 2.2]). If we set $\lambda = \nu + \mu_{(X \setminus K)}$ then $f_*(\lambda) = f_*(\mu_K) + f_*(\mu_{(X \setminus K)}) = f_*(\mu)$ and $\mu \neq \lambda$ since $\lambda(K) = 0$. This is a contradiction since $\mu \in M = f_*^{-1}(f_*(M))$ and f_* is one-to-one on M.

Now let $V = T \setminus f(X \setminus W)$. It is easy to see that $W_0 \subset f^{-1}(V) \subset W$, hence $\mu(W \setminus f^{-1}(V)) = 0$ for all $\mu \in M$. Since V is open in T, $f^{-1}(V)$ is the union of some U_n's and the proof is complete. □

If X is a compact space we denote by M_X the set of all non-atomic measures in $M_\sigma^+(X)$. The space X is called scattered if it has no nonempty perfect subsets, which is equivalent to $M_X = \{0\}$ (see [10]).

PROPOSITION 3.2. *For any compact space X, M_X is* c.d. *if and only if there is a compact metric space T and $f: X \to T$ continuous, onto such that $f^{-1}(\{t\})$ is scattered for all $t \in T$.*

Proof. Let $f: X \to T$ be continuous, onto, where T is a compact metric space. By Propositions 2.1 and 2.2, it is enough to show that $M_X = f_*^{-1}(f_*(M_X))$ if and only if $f^{-1}(\{t\})$ is scattered for all $t \in T$.

If $f^{-1}(\{t\})$ is not scattered, there is $\mu \in M_X$ such that $\mu(f^{-1}(\{t\})) = \mu(X) = 1$. Let $x \in f^{-1}(\{t\})$ and $\nu = \delta_x$. Then we have $f_*(\mu) = f_*(\nu) = \delta_t$, so $\nu \in f_*^{-1}(f_*(M_X)) \setminus M_X$.

Now assume that $f^{-1}(\{t\})$ is scattered for every $t \in T$. If $\mu \in M_X$, then $f_*(\mu) \in M_T$. Indeed, if $\{t\}$ were an atom for $f_*(\mu)$ then $\mu(f^{-1}\{t\}) > 0$, so $f^{-1}(\{t\})$ would not be scattered. Moreover, if $\mu \in M_\sigma^+(X)$ has an atom, then $f_*(\mu)$ has an atom. Therefore $f_*^{-1}(f_*(M_X)) = M_X$. □

COROLLARY 3.3. *Let X_1 and X_2 be compact spaces such that M_{X_1} and M_{X_2} are* c.d. *Then $M_{X_1 \times X_2}$ is* c.d.

Proof. Let T_1 and T_2 be compact metric spaces, $f_1: X_1 \to T_1$ and $f_2: X_2 \to T_2$ continuous, onto, satisfying the condition of Proposition 3.2 for X_1 and X_2, respectively. Then the function $f: X_1 \times X_2 \to T_1 \times T_2$, defined by $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$, satisfies the same condition for $X_1 \times X_2$. □

THEOREM 3.4. *For any compact space X the following are equivalent:*
(i) M_X *is* c.s.;
(ii) M_X *is* c.s. *and* c.d.;
(iii) M_X *is separable metrizable.*

Proof. (i)$\Rightarrow$(ii). Let T be a compact metric space and $f: X \to T$ continuous, onto such that f_* is one-to-one on M_X (Proposition 2.1). By Proposition 2.2, it is enough to show that $M_X = f_*^{-1}(f_*(M_X))$.

Let $\mu \in M_X$ and assume (if possible) that there is an atom $\{t\}$ for $f_*(\mu)$. Then $\mu(f^{-1}(\{t\})) > 0$ and we can find two disjoint subsets A and B of $f^{-1}(\{t\})$ with $\mu(A) > 0$ and $\mu(B) > 0$. We set

$$\mu_1 = \frac{1}{\mu(A)} \cdot \mu_A \quad \text{and} \quad \mu_2 = \frac{1}{\mu(B)} \cdot \mu_B .$$

Then $\mu_1 \in M_X$, $\mu_2 \in M_X$ and $f_*(\mu_1) = f_*(\mu_2) = \delta_t$. This contradiction shows that $f_*(M_X) \subset M_T$. Moreover, if $\mu \in M_\sigma^+(X)$ has an atom, then $f_*(\mu)$ has an atom. Therefore $M_X = f_*^{-1}(f_*(M_X))$.

(ii)$\Rightarrow$(iii). As above there is a continuous function f from X onto a compact metric space T such that $f_*: M_X \to M_T$ is continuous, one-to-one and onto. Moreover if we set $M_\sigma^r(X) = \{\mu \in M_\sigma^+(X) : \mu(X) \leqslant r\}$ and $M_X^r = \{\mu \in M_X : \mu(X) \leqslant r\}$ for any $r > 0$, then we have

$$f_*(M_\sigma^r(X) \backslash M_X^r) \subset M_\sigma^r(T) \backslash M_T^r .$$

Thus, if F is a closed subset of M_X^r, then

$$f_*(F) = f_*(\mathrm{Cl}_{M_\sigma^r(X)} F) \cap M_T^r$$

is closed in M_T^r (since $M_\sigma^r(X)$ is compact). It follows that $f_*: M_X^r \to M_T^r$ is a homeomorphism. Since the sets of the form $\{\mu \in M_X^r : \mu(X) < r\}$, $r > 0$, form an open covering of M_X, $f_*: M_X \to M_T$ is a homeomorphism, so M_X is separable metrizable.

(iii)$\Rightarrow$(i). This is, in essence, proved in Proposition 2.3 (i). □

A family of open subsets of a space X is called a pseudobase if every nonempty open subset of X contains a nonempty member of the family.

The next corollary follows from Theorems 3.1, 3.4 and Proposition 2.8. When the space X is totally disconnected, the result was first proved by S. Argyros using different methods.

COROLLARY 3.5. *If X is a compact space without isolated points and M_X is* c.s., *then X has a countable pseudobase and, in particular, X is separable.*

Proof. By Theorems 3.1 and 3.4, M_X is uniformly regular. Let $\{U_n\}$ be a sequence of cozero sets in X as in Proposition 2.8. If W is a nonempty cozero subset of X then $\mu(W) > 0$ for some $\mu \in M_X$ (since W has no isolated points). By Proposition 2.8, there is an n such that $U_n \subset W$ and $\mu(U_n) > 0$. Therefore $\{U_n\}$ is a pseudobase for X. □

By the above results, for every compact space X without isolated points, if M_X is c.s. then M_X is c.d. and X is separable. As we shall see later (Examples 3.10) these properties are distinct.

We now restrict ourselves to compact totally ordered spaces. Recall that a space X is a totally ordered space if its topology is the interval topology of some total ordering $\leqslant$ on X (see [18, Example 39]). The connection between the separability of such spaces and other countability properties has been studied by many authors (see e.g. [3], [9, Ch. 4], [19]). For our purpose we have the following.

THEOREM 3.6. *Let X be a compact totally ordered space without isolated points. The following are equivalent:*

(i) *M_X is* c.s.;
(ii) *M_X is* c.d.;
(iii) *X is separable.*

For the proof of the essential direction (ii)$\Rightarrow$(iii) we shall use the following two lemmas.

LEMMA 3.7. *If $f: X\to \mathbf{R}$ is continuous order preserving, then there is a pairwise disjoint family $\mathcal{J}$ of non-empty open intervals in X such that $X\setminus\bigcup\mathcal{J}$ is separable and f is constant on each element of $\mathcal{J}$ with different values on different elements of $\mathcal{J}$.*

Proof. For every $t\in f(X)$, $f^{-1}(\{t\})$ is a closed interval in X, that is, $f^{-1}(\{t\})=[x_t,y_t]$, where $x_t,y_t\in X$, $x_t\leqslant y_t$. Let $\mathcal{J}$ be the family of all non-empty intervals of the form (x_t,y_t). It suffices to show that the set $Y=\bigcup_{t\in f(X)}\{x_t,y_t\}$ is separable. Let C_1 be a countable dense subset of $f(X)$, C_2 be the (necessarily countable) set of all isolated from the left and all isolated from the right points of $f(X)$, and set $S=\bigcup_{t\in C_2\cup C_1}\{x_t,y_t\}$. We prove that S is dense in Y, that is, if $(x,y)\cap Y\neq\varnothing$ then $(x,y)\cap S\neq\varnothing$.

There is $t\in f(X)$ such that $x_t\in(x,y)$ or $y_t\in(x,y)$. Assume that $x_t\in(x,y)$ and set $t'=f(x)$, so that $t'<t$. If $(t',t)\cap f(X)=\varnothing$, then $t\in C_2$ and $x_t\in(x,y)\cap S$. If $(t',t)\cap f(X)\neq\varnothing$, then $(t',t)\cap C_1\neq\varnothing$ and $x_q\in(x,y)\cap S$ for all $q\in(t',t)\cap C_1$. Therefore $(x,y)\cap S\neq\varnothing$. The proof when $y_t\in(x,y)$ is similar (setting $t'=f(y)$). □

Let $\mathfrak{Z}$ be the family of all continuous order preserving real-valued functions on X.

LEMMA 3.8. *The set $\mathfrak{A}$ of all differences of elements of $\mathfrak{Z}$ is uniformly dense in $C(X)$.*

Proof. Since $\mathfrak{A}$ is clearly a subalgebra of $C(X)$, by the Stone–Weierstrass Theorem it is enough to show that $\mathfrak{A}$ separates points of X.

Let $x,y\in X$ with $x<y$. If (x,y) is non-empty, there is $\mu\in M_X$ such that $\mu(x,y)=\mu(X)=1$ (since (x,y) has no isolated points). Let F_μ be the distribution function of μ defined by $F_\mu(x)=\mu(\{y\in X: y\leqslant x\})$. Then F_μ is order preserving,

continuous (since μ is non-atomic) and $F_\mu(x)=0$, $F_\mu(y)=1$. If (x,y) is empty, then the distribution function F_{δ_y} of the measure δ_y belongs to $\mathfrak{I}$ and $F_{\delta_y}(x)=0$, $F_{\delta_y}(y)=1$. □

Proof of Theorem 3.6.

(i)⇒(ii). This is true in general (Theorem 3.4).

(ii)⇒(iii). Let $\{f_n\}$ be a sequence in $C(X)$ which satisfies the definition of the c.d. property for M_X. By Lemma 3.8, we can assume that $f_n \in \mathfrak{I}$. For every n let $\mathcal{J}_n$ be the family of open intervals corresponding to f_n as in Lemma 3.7. We set $Y_n = X \setminus \bigcup \mathcal{J}_n$ and $Y = \bigcup_n Y_n$. It is enough to show that Y is dense in X.

Let (x,y) be a non-empty interval in X. There is $\mu_1 \in M_X$ with $\mu_1(x,y) = \mu_1(X) = 1$ and let $\mu_2 = \delta_z$, where $z \in (x,y)$. Since $\mu_2 \notin M_X$, there is an n such that $\int f_n \, d\mu_1 \neq f_n(z)$. If $I \cap (x,y) = \varnothing$ for all $I \in \mathcal{J}_n$, then $(x,y) \subset Y_n \subset Y$. On the other hand, if $I \cap (x,y) \neq \varnothing$ for some $I \in \mathcal{J}_n$, then at least one of the endpoints of I belongs to (x,y) (since otherwise $(x,y) \subset I$ and $\int f_n \, d\mu_1 = f_n(z)$). It follows that Y is dense in X.

(iii)⇒(i). Let T be the set of all left limit points of X together with the point $x_0 = \min X$. We define $f: X \to T$ by $f(x) = \sup\{y \in X : y < x\}$, if $x \neq x_0$ and $f(x_0) = x_0$. Consider T as a totally ordered space with the restriction of the ordering. (Notice that this topology on T is coarser than the relative topology it inherits from X.) Clearly, f is order preserving, continuous and onto.

Since X is compact and separable, so is T. Moreover, using the fact that X has no isolated points it is not hard to see that T is connected. It follows from a well-known result that T is order isomorphic to the closed unit interval in $\mathbf{R}$. For every closed interval I in X, the set $f^{-1}(f(I))$ differs from I in at most two points. Therefore, using Proposition 2.1 ((iv)⇒(i)), we conclude that M_X is c.s. □

REMARK. The assumption in Theorem 3.6 that X has no isolated points cannot be dropped since there are compact non-separable totally ordered spaces with $M_X = \{0\}$ (so (i)⇏(iii)). However, conditions (i) and (ii) are equivalent for any compact totally ordered space X. Indeed, if M_X is c.d. and Y is the perfect kernel of X (possibly $Y = \varnothing$), then M_Y is also c.d. By Theorem 3.6, M_Y is c.s. and so separable metrizable (Theorem 3.4). Since M_X is homeomorphic to M_Y, M_X is also separable metric and so c.s.

COROLLARY 3.9. *If X is a compact, connected totally ordered space and M_X is* c.d. (or c.s.), *then X is order isomorphic to the unit interval, provided that X has more than one point.*

It is clear that conditions (i)–(iii) of Theorem 3.6 are satisfied when X is a compact metric space. Moreover, for every compact space without isolated points, (i)⇒(ii) and (i)⇒(iii) (Theorem 3.4 and Corollary 3.5). The next examples show that no other implication between these conditions is valid.

EXAMPLES 3.10. (a) Let $Y = [0,1] \times \{0,1\}$ with the lexicographic ordering. Then Y is a separable compact totally ordered space without isolated points. Thus Y satisfies conditions (i)–(iii) of Theorem 3.6, but is not metrizable.

(b) The space $X=[0,1]^c$ is separable, but M_X is not c.d. Indeed, if f is a continuous function from X onto a compact metric space T, then f is determined by countably many indices [15]. Thus $f^{-1}(\{t\})$ is not scattered for any $t\in T$ and Proposition 3.2 yields that M_X is not c.d.

(c) Let Y be the one-point compactification of a discrete uncountable space and $X=[0,1]\times Y$. Then X is not separable but M_X is c.d. (by Proposition 3.2, using the projection from X onto $[0,1]$).

(d) Let Y be a separable compact non-metrizable totally ordered space without isolated points (Example (a)) and set $X=Y\times Y$. Then X is separable and M_X is c.d. (by Theorem 3.6 and Corollary 3.3). However, M_X is not c.s., as the following proposition due to S. Argyros shows.

PROPOSITION 3.11. *Let X, Y be compact spaces such that $M_X\neq\{0\}$ and $M_{X\times Y}$ is* c.s. *Then Y is metrizable.*

Proof. There are two sequences $\{f_n\}$ in $C(X)$ and $\{g_n\}$ in $C(Y)$ such that the sequence $\{f_n\cdot g_n\}$ in $C(X\times Y)$ separates $M_{X\times Y}$. Let y_1, y_2 be two distinct points of Y and $\mu\in M_X$, $\mu\neq 0$. We set $\mu_1=\mu\times\delta_{y_1}$ and $\mu_2=\mu\times\delta_{y_2}$. Since $\mu_1,\mu_2\in M_{X\times Y}$, there is an n such that $\mu_1(f_n\cdot g_n)\neq\mu_2(f_n\cdot g_n)$, that is, $g_n(y_1)\neq g_n(y_2)$. This shows that $\{g_n\}$ separates points of Y, so Y is metrizable. □

The above proposition can also be proved more easily using Theorem 3.4. Indeed if μ is a non-zero measure in M_X the space $M=\{\mu\times\delta_y: y\in Y\}$ is contained in $M_{X\times Y}$ and so is metrizable. Since Y is homeomorphic to M, Y is also metrizable.

We wish to thank Professor S. Argyros for useful communications regarding this section.

4. Spaces of measures. Up to now we have treated the c.s. and c.d. properties for certain sets of measures. Here we examine these properties for the spaces of measures $M_s(X)$, $s=\sigma$, τ or t. It turns out that the c.s. property for $M_s(X)$ is equivalent to the separability of $C(X)$ in the weak topology $\sigma((C(X),M_s(X))$. Since the separability is possessed or not by all locally convex topologies with the same dual, the weak topology can be replaced by the strict topology β_s when X is completely regular.

THEOREM 4.1. *For any space X and $s=\sigma$, τ or t the following are equivalent:*

(i) *$M_s(X)$ is* c.s.;

(ii) *$M_s^+(X)$ is* c.s.;

(iii) *$C(X)$ is separable in the weak topology $\sigma(C(X),M_s(X))$;*

(iv) *$M_s(X)$ is separably submetrizable;*

(v) *every element of $M_s(X)$ is G_δ.*

Moreover, conditions (i)–(v) *for $s=t$ are equivalent to:*

(vi) *$M_t^+(X)$ is* c.s. *and* c.d. *in $M_\sigma^+(X)$;*

(vii) *X is separably submetrizable.*

Proof. (i)⇔(ii). Assume that $M_s^+(X)$ is c.s. and let $\{f_n\}$ be a sequence in $C(X)$ separating measures in $M_s^+(X)$. Let μ, ν be in $M_s(X)$ with $\mu(f_n)=\nu(f_n)$

for all n. Then $(\mu-\nu)^+(f_n)=(\mu-\nu)^-(f_n)$. Since $(\mu-\nu)^+$ and $(\mu-\nu)^-$ are in $M_s^+(X)$, it follows that $(\mu-\nu)^+=(\mu-\nu)^-$, so $\mu-\nu=0$. The converse is trivial.

(i)$\Leftrightarrow$(iii). Using the Hahn–Banach Theorem it is easy to see that a sequence $\{f_n\}$ in $C(X)$ separates measures in $M_s(X)$ if and only if the linear span of $\{f_n\}$ is dense in $C(X)$ for the topology $\sigma(C(X),M_s(X))$.

(i)$\Rightarrow$(iv)$\Rightarrow$(v) are easy to see. If the zero measure is G_δ in $M_s(X)$, then there is a sequence $\{f_n\}$ in $C(X)$ such that

$$\{0\}=\bigcap_{n,m}\{\mu\in M_s(X):|\mu(f_n)|<1/m\}.$$

It follows that $\{f_n\}$ separates measures in $M_s(X)$, so (v)$\Rightarrow$(i). Therefore the first five conditions are equivalent. Notice also that the equivalence (i)$\Leftrightarrow$(iv) follows from Proposition 2.4 (i).

(i)$\Rightarrow$(vii). The set of Dirac measures on X as a subset of $M_s(X)$ is c.s., so X must be separably submetrizable.

(vii)$\Rightarrow$(vi). We have that there is a continuous one-to-one function $f: X\to \mathbf{R}^{\mathbf{N}}$. By a well-known result $f_*: M_t^+(X)\to M_t^+(\mathbf{R}^{\mathbf{N}})$ is also one-to-one, from which the c.s. property for $M_t^+(X)$ follows (Proposition 2.1). An easy way to see that $M_t^+(X)$ is also c.d. is as follows: let $\mu\in M_\sigma^+(X)$ and $\nu\in M_t^+(X)$ with $f_*(\mu)=f_*(\nu)$. By Propositions 2.1 and 2.2, it is enough to show that $\mu=\nu$. If K is a compact subset of X, then $K=f^{-1}(f(K))$ where $f(K)$ is a zero set in $\mathbf{R}^{\mathbf{N}}$. Thus K is also a zero set in X and

$$\mu(K)=f_*(\mu)(f(K))=f_*(\nu)(f(K))=\nu(K).$$

Therefore μ and ν coincide on compact subsets. If C is a σ-compact subset of X with $\nu(C)=\nu(X)$, then since $\mu(X)=\nu(X)$ we have $\mu(C)=\mu(X)$. Thus $\mu\in M_t^+(X)$ and by the property of regularity $\mu=\nu$.

Finally, it is obvious that (vi)$\Rightarrow$(ii) for $s=t$, completing the proof of the theorem. □

REMARK. Summers, using a Stone–Weierstrass type theorem, has proved the following equivalences from Theorem 4.1:

(a) (iii)$\Leftrightarrow$(iv) for $s=t$; and

(b) (vii)$\Leftrightarrow$(iv) for $s=t$,

(see [20, Theorems 2.1 and 4.3]). Further, he asks whether (a) and (b) for $s=\tau$ hold (see also [22, Problem 13.3]). Clearly, Theorem 4.1 contains a positive answer to (a). We shall give some partial positive answers to (b) (4.2 and 4.5), though the answer in general is probably negative.

PROPOSITION 4.2. *If the Baire σ-algebra $\mathcal{B}(X)$ on X is countably generated, then $M_\sigma(X)$ is* c.s. *and so $C(X)$ is separable in $\sigma(C(X),M_\sigma(X))$.*

Proof. This follows from Proposition 2.1. □

In particular, Proposition 4.2 holds when X is the Sorgenfrey line, or a separable metric space (see [22, Theorem 13.2]). Further, Proposition 4.2 holds when X is a cosmic space (i.e., regular continuous image of a separable metric space). To check that $\mathcal{B}(X)$ is countably generated, let T be a separable metric space, $f: T \to X$ continuous onto and $\mathcal{B}$ a countable base for the topology of T. Then using the fact that X is hereditarily Lindelöf and regular, it is easy to see that $\{\overline{f(B)}: B \in \mathcal{B}\}$ is a countable family of zero sets generating $\mathcal{B}(X)$.

It is more interesting to examine the c.s. and c.d. properties in spaces where the σ-algebra of Baire sets is not countably generated. If $M_\tau^+(X)$ is c.s. then $\operatorname{card}(X) \leqslant \operatorname{card}(M_\tau^+(X)) \leqslant c$. We shall prove that for a metric space X, (i) $M_\tau^+(X)$ is c.s. if and only if $\operatorname{card}(X) \leqslant c$, and (ii) $M_\tau^+(X)$ is c.d. if and only if either $\operatorname{card}(X) \leqslant c$ or $M_\sigma^+(X) = M_\tau^+(X)$. In fact, these are true for a wider class of spaces (Corollary 4.5 and Theorem 4.6).

We recall that a space X is *developable* if there is a sequence $\{\mathfrak{U}_n\}_n$ of open covers of X such that for each $x \in X$

$$\bigcup \{U \in \mathfrak{U}_n : x \in U\}, \quad n = 1, 2, \ldots$$

is a neighborhood base for x.

It is clear that

(i) every metric space is developable;
(ii) a subspace of a developable space is developable;
(iii) every closed subset of a developable space X is G_δ, so every Borel measure on X is regular.

We shall also need the following result [4, Theorem 9]: every open cover of a developable space has a σ-discrete closed refinement which covers the space.

LEMMA 4.3. *Let X be a developable space with* $\operatorname{card}(X) \leqslant c$. *Then there is a countably generated σ-algebra of Borel sets containing all subsets of X which are second countable and G_δ in X.*

Proof. Let $\{\mathfrak{U}_n\}$ be a sequence of open covers of X witnessing to the developability of X. For every n let $\mathfrak{F}_n$ be a σ-discrete closed refinement of $\mathfrak{U}_n$ which covers X and set $\mathfrak{F} = \bigcup_n \mathfrak{F}_n$. Then $\mathfrak{F}$ is a σ-discrete closed covering of X with the property that every open set in X is a union of some members of $\mathfrak{F}$. Indeed, if V is open in X and $x \in V$, there is an n such that $\bigcup \{U \in \mathfrak{U}_n : x \in U\} \subset V$. Let $F \in \mathfrak{F}_n$ with $x \in F$. Since $\mathfrak{F}_n$ refines $\mathfrak{U}_n$, there is $U \in \mathfrak{U}_n$ with $F \subset U$. It follows that $F \subset V$.

We write $\mathfrak{F} = \bigcup_n \mathfrak{A}_n$ where each $\mathfrak{A}_n$ is discrete. For every n the family $\mathfrak{A}_n \cup \{X - \bigcup \mathfrak{A}_n\}$ is a partition of X with cardinality $\leqslant c$, so we can find a real-valued function f_n on X such that f_n is constant on each element of the partition and takes different values on different elements of the partition. Since $\mathfrak{A}_n$ is a discrete family of closed sets, $f_n^{-1}(B)$ is a Borel set in X for any $B \subset \mathbf{R}$. In particular, f_n is Borel measurable. We define $F: X \to \mathbf{R}^{\mathbf{N}}$ by $F(x) = (f_n(x))_{n \in \mathbf{N}}$ and set $\mathcal{B} = F^{-1}(\mathcal{B}(\mathbf{R}^{\mathbf{N}}))$.

We now show that $\mathcal{B}$ has the desired properties. Clearly, $\mathcal{B}$ is a countably generated σ-algebra of Borel sets and $\mathfrak{F} \subset \mathcal{B}$. Let Y be a second countable subset of X with $Y = \bigcap_{n \in \mathbf{N}} V_n$, each V_n open in X. For every $x \in Y$, there is a sequence $\{F_{x,n}\}_n$ of elements of $\mathfrak{F}$ with $x \in F_{x,n} \subset V_n$. We have $Y = \bigcap_n \bigcup_{x \in Y} F_{x,n}$. Moreover, for every n the family $\{F_{x,n} : x \in Y\}$ is σ-discrete and each of its elements has non-empty intersection with Y. Since Y is second countable, it follows that these families are countable. Therefore $Y \in \mathcal{B}$. □

THEOREM 4.4. *Let X be a developable space with* $\operatorname{card}(X) \leqslant c$. *Then there is a countably generated σ-algebra $\mathcal{B}$ of Borel sets in X such that if μ, ν are two non-negative Borel measures on X and ν is τ-additive, then*

$$\mu(B)=\nu(B) \quad \text{for all } B\in\mathcal{B} \Rightarrow \mu=\nu.$$

Proof. Let $\mathcal{B}$ be as in Lemma 4.3 and assume that $\mu(B)=\nu(B)$ for all $B\in\mathcal{B}$. Let $\{\mathcal{U}_n\}$ be a sequence of open covers of X witnessing to the developability of X. Applying the τ-additivity of ν, for each cover $\mathcal{U}_n$ we find a countable $\mathcal{V}_n\subset\mathcal{U}_n$ such that $\nu(\bigcup\mathcal{V}_n)=\nu(X)$. Then $Y=\bigcap_{n=1}^{\infty}\bigcup\mathcal{V}_n$ is G_δ, second countable and $\nu(Y)=\nu(X)$ (see Prikry [13]). For every closed subset F of X we have that $F\cap Y$ is second countable and G_δ in X, so $F\cap Y\in\mathcal{B}$ and $\mu(F\cap Y)=\nu(F\cap Y)$. In particular $\mu(Y)=\nu(Y)=\nu(X)=\mu(X)$, so μ is τ-additive. Since μ, ν coincide on the closed sets, by the property of regularity $\mu=\nu$. □

The following corollary is immediate from the above theorem and Propositions 2.1 and 2.2.

COROLLARY 4.5. *Let X be a metric space, or more generally a developable space with $\mathcal{B}(X)=\mathcal{B}_0(X)$, such that* $\operatorname{card}(X)\leqslant c$. *Then $M_\tau^+(X)$ is* c.s. *and* c.d. *in $M_\sigma^+(X)$, so $C(X)$ is separable in $\sigma(C(X),M_\tau(X))$.*

By the above corollary, if $\operatorname{card}(X)\leqslant c$ then $M_\tau^+(X)$ is c.d. in $M_\sigma^+(X)$. Trivially this is also true when $M_\sigma^+(X)=M_\tau^+(X)$. The next theorem shows that these are the only possible cases. Its proof makes use of $\{0,1\}$-measurable and real-valued measurable cardinals (see [9, Ch. 5]).

THEOREM 4.6. *Let X be a metric space or more generally a developable space with $\mathcal{B}(X)=\mathcal{B}_0(X)$. Then $M_\tau^+(X)$ is* c.d. *in $M_\sigma^+(X)$ if and only if either* $\operatorname{card}(X)\leqslant c$ *or $M_\sigma^+(X)=M_\tau^+(X)$.*

Proof. We give a proof for the general case of a developable space with $\mathcal{B}(X)=\mathcal{B}_0(X)$. We assume that $M_\tau^+(X)$ is c.d. in $M_\sigma^+(X)$. By Proposition 2.2, there is a separable metric space T and $f: X\to T$ continuous, onto such that for every $\mu\in M_\sigma^+(X)$ and $\nu\in M_\tau^+(X)$,

$$f_*(\mu)=f_*(\nu)\Rightarrow\mu\in M_\tau^+(X).$$

Claim 1. $\operatorname{Card}(X)<$ the least $\{0,1\}$-measurable cardinal.

Suppose that the claim is false, so there is a non-zero $\{0,1\}$-valued measure λ defined on all subsets of X and vanishing on singletons. Set $\mu=\lambda|_{\mathcal{B}(X)}\in M_\sigma^+(X)$. Then $f_*(\mu)\in M_\sigma^+(T)$ is a non-zero $\{0,1\}$-valued measure, so there is some $t\in T$ such that $f_*(\mu)=\delta_t$. If $x\in f^{-1}(\{t\})$, then $f_*(\mu)=f_*(\delta_x)$, thus $\mu\in M_\tau^+(X)$. Since μ is also $\{0,1\}$-valued and $\mu(\{x\})=\lambda(\{x\})=0$ (because $\{x\}\in\mathcal{B}(X)$) for every $x\in X$, this leads to a contradiction.

Claim 2. If $M_\sigma^+(X)\neq M_\tau^+(X)$, then there is a real-valued measurable cardinal $\leqslant\operatorname{card}(X)$.

This is well known when X is metrizable (see [21, Part I, Theorem 28]) and we include a proof for the sake of completeness. Using a standard argument there is a non-zero $\mu \in M_\sigma^+(X)$ and an open covering $\mathcal{V}$ of X such that $\mu(U)=0$ for every $U \in \mathcal{V}$. Let $\bigcup_{n=1}^\infty \mathcal{C}_n$ be a σ-discrete closed refinement of $\mathcal{V}$. Fix an n such that $\mu(\bigcup \mathcal{C}_n)>0$. Choose a subset D of X by picking one point from each member of $\mathcal{C}_n$. Set

$$\lambda(E)=\mu(\bigcup\{C \in \mathcal{C}_n : C \cap E \neq \varnothing\})$$

for every $E \subset D$. Then λ is a non-zero, σ-additive measure defined on all subsets of D and vanishing on singletons. Since $\operatorname{card}(D) \leqslant \operatorname{card}(X)$, the proof of claim 2 is complete.

Now assume (for the purpose of a contradiction) that $M_\sigma^+(X) \neq M_\tau^+(X)$ and $\operatorname{card}(X)>c$. Using claims 1 and 2, there is by Ulam's Theorem [9, p. 297] a real-valued measurable cardinal $\leqslant c$. Since $\operatorname{card}(X)>c$, there is $t \in T$ such that $\operatorname{card}(f^{-1}(\{t\}))>c$. Set $Y=f^{-1}(\{t\})$. As in the proof of Lemma 4.3, there are discrete families $\mathfrak{F}_n$, $n=1,2,\ldots$, of closed subsets of Y such that every open subset of Y is the union of some members of $\bigcup_{n=1}^\infty \mathfrak{F}_n$. For every n, let Y_n be a subset of $\bigcup \mathfrak{F}_n$ containing exactly one point from each member of $\mathfrak{F}_n$. Then each Y_n is closed and discrete and $\bigcup_{n=1}^\infty Y_n$ is dense in Y. Since Y is first countable, $\operatorname{card}(\bigcup_{n=1}^\infty Y_n) \geqslant c$. Fix an n such that $\operatorname{card}(Y_n) \geqslant c$ and let λ be a probability measure defined on all subsets of Y_n and vanishing on singletons. (Such λ exists because there is a real-valued measurable cardinal $\leqslant \operatorname{card}(Y_n)$.) Let

$$\mu(A)=\lambda(A \cap Y_n) \quad \text{for every } A \in \mathcal{B}(X).$$

Then $\mu \in M_\sigma^+(X)$, $f_*(\mu)=\delta_t=f_*(\delta_x)$, where $x \in f^{-1}(\{t\})$, which is a contradiction since $\mu \notin M_\tau^+(X)$. □

REMARKS. (a) An example of a developable non-metrizable space is the set of all finite subsets of **R** endowed with the Pixley–Roy topology [12]. Further, in [14] it is proved that assuming Martin's Axiom and the negation of the continuum hypothesis, there exists a subspace Y of this space with the same properties, which is also normal, so $\mathcal{B}(Y)=\mathcal{B}_0(Y)$.

For the special metric case Corollary 4.5 and Theorem 4.6 can be proved more directly by a similar method, using that (i) every metric space has a σ-discrete base and (ii) every τ-additive measure on a metric space is carried by a closed separable subset. In this case Corollary 4.5 contains a result of Summers [20, Theorem 3.2].

(b) Theorem 4.4 and Corollary 4.5 are not valid for M_σ^+ (in the presence of real-valued measurable cardinals) even for discrete spaces. Indeed, let $\kappa \leqslant c$ be a cardinal such that there exists a probability measure λ defined on all subsets of κ and vanishing on singletons, such that the measure algebra of λ is not generated by countably many of its members. (It is an unpublished result of D. H. Fremlin that such a measure exists if $\kappa=c$ is real-valued measurable.) Let $\mathcal{B}$ be any countably generated σ-algebra of subsets of κ. Without loss of generality we can assume that λ is homogeneous and, using Maharam's representation theorem of measure algebras, we can find an uncountable family $\{B_\alpha\}_{\alpha \in \rho}$ of stochastically independent subsets of κ of λ-measure 1/2 which generates the measure algebra of λ. Let E be a countable subset of ρ such that $\mathcal{B}$ is "contained" in the measure algebra generated by $\{B_\alpha\}_{\alpha\in E}$. If $\alpha\notin E$ and μ, ν are the restrictions of λ to B_α and $\kappa\backslash B_\alpha$, respectively, then μ and ν coincide on $\mathcal{B}$, but $\mu\neq\nu$. We are grateful to Professor Karel Prikry for communicating to us the content of this paragraph.

This discussion suggests the following:

QUESTION. If X is a metric space with $M_\sigma^+(X)$ c.s., is it then true that $M_\sigma^+(X)=M_\tau^+(X)$?

By Corollary 4.5, if the answer to this question is "yes" then $M_\sigma^+(X)$ is c.s. if and only if $\operatorname{card}(X)\leqslant c$ and $M_\sigma^+(X)=M_\tau^+(X)$.

(c) Motivated by Theorem 4.4, one may introduce new notions by replacing in the definitions of the c.s. and c.d. properties the continuity (or, equivalently, the Baire measurability) of the sequence $\{f_n\}$ by the Borel measurability. Of course, if $\mathcal{B}(X)=\mathcal{B}_0(X)$ there is nothing new, though in general we obtain two much weaker notions. Indeed, let Y be a discrete space with $\operatorname{card}(Y)=\aleph_1$ and $X=Y\cup\{\infty\}$ the one-point compactification of Y. Then the set $\{\delta_\infty\}$ is not c.d. in $M_\sigma^+(X)$ (Proposition 2.3 (ii)) and $M_\sigma^+(X)$ is not c.s. since X is not metrizable. However, because the cardinality of X is $\leqslant c$ and every subset of X is a Borel set, there is a countably generated σ-algebra of Borel sets containing all singletons. Since X is scattered, it follows that $M_\sigma^+(X)$ is c.s. in the Borel sense. Moreover, we show by an example that the analogue of Theorem 3.6 and Corollary 3.9 for the c.s. and c.d. properties in the Borel sense are not correct.

EXAMPLE 4.7. The extended long line L^* [18, Example 46] is constructed from the space of all ordinals $\leqslant\omega_1$ by placing between each ordinal $\alpha<\omega_1$ and its successor a copy I_α of the open unit interval $(0,1)$. Consider L^* with its natural ordering. Then L^* is a compact non-separable connected totally ordered space. We show that the set M_{L^*} of all non-atomic measures on L^* is c.s. and c.d. in the Borel sense.

Let $\{U_{\alpha,n}\}_n$ be a countable base for the open sets in I_α. For every n, $\{U_{\alpha,n}:\alpha<\omega_1\}$ is a pairwise disjoint family of open sets in L^*. Since the cardinal of each of these families is $\leqslant c$, as in the proof of Lemma 4.3, we can find a countably generated σ-algebra $\mathcal{B}$ of Borel sets such that $U_{\alpha,n}\in\mathcal{B}$ for all α and n. Now let $\nu\in M_{L^*}$ and $\mu\in M_\sigma^+(L^*)$ such that $\mu(B)=\nu(B)$ for all $B\in\mathcal{B}$. Since $\nu\in M_{L^*}$, there is a closed separable subset C of L^* with full ν-measure ([1, Theorem 3.4] and [16, Theorem 3.2]). Using the separability of C we deduce that $C\backslash\{\omega_1\}$ is bounded by an $\alpha_0<\omega_1$. Thus every Borel subset of C differs from a member of $\mathcal{B}$ by a countable set. It now follows easily that $\mu=\nu$.

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