ON THE MEAN BOUNDARY BEHAVIOR AND THE TAYLOR COEFFICIENTS OF AN INFINITE BLASCHKE PRODUCT

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Introduction. If $\{z_n\}$ is a sequence (finite or infinite) of complex numbers of modulus less than 1 such that $\Sigma(1-|z_n|)<\infty$, then the Blaschke product

$$B(z)=\prod_n \frac{|z_n|}{z_n}\frac{z_n-z}{1-\bar{z}_n z}$$

converges uniformly on compact subsets of the unit disc U. We let $\mathcal{B}_\infty$ denote the set of Blaschke products whose zero sequences are infinite. In this paper we show that

$$\inf_{B\in\mathcal{B}_\infty}\ \overline{\lim_{r\to 1}}\,(1-r)^{-1}\int_{-\pi}^{\pi}(1-|B(re^{i\theta})|)^2\frac{d\theta}{2\pi}=\max_{0\leqslant x<1}\,(1+\sqrt{1-x})\,{}_2F_1({}^{1/2}{}_2{}^{1/2};x)=\gamma_0$$

where ${}_2F_1({}^{1/2}{}_2{}^{1/2};x)$ is a hypergeometric function. Using one of Gauss' identities for the hypergeometric functions and tables for the complete elliptic integrals (see [2, pp. 608–609]), we obtain the estimate $\gamma_0\geqslant .285$.

We have two applications of this result. In [2], Newman and Shapiro showed that if $B(z)=\sum_{n=0}^\infty a_n z^n\in\mathcal{B}_\infty$ then $\overline{\lim}_{n\to\infty}\, n|a_n|\geqslant 1/\pi=.3183\cdots$. We improve this to show that if $B(z)=\sum_{n=0}^\infty a_n z^n\in\mathcal{B}_\infty$ then $\overline{\lim}_{n\to\infty}\, n|a_n|\geqslant\sqrt{\gamma_0/2}\geqslant .37749$. In the other direction we modify a method of Newman and Shapiro [2] to show that if $\epsilon>0$ is given, there is a $B(z)=\sum_{n=0}^\infty a_n z^n\in\mathcal{B}_\infty$ such that $\overline{\lim}\, n|a_n|\leqslant 2/e+\epsilon=.735\cdots+\epsilon$. We conjecture that if $B(z)=\sum_{n=0}^\infty a_n z^n\in\mathcal{B}_\infty$ then $\overline{\lim}_{n\to\infty}\, n|a_n|\geqslant 2/e$.

It is well known that if $B(z)=\sum_{n=0}^\infty a_n z^n\in\mathcal{B}_\infty$ then $\sum_{n=1}^\infty n|a_n|^2=\infty$. As another application of our main theorem we improve this to show that if $B(z)=\sum_{n=0}^\infty a_n z^n\in\mathcal{B}_\infty$ then

$$\overline{\lim_{n\to\infty}}\sum_{k\in I_n} k|a_k|^2\geqslant\frac{\gamma_0}{8}\geqslant .0356,$$

where $I_n=\{k: k \text{ is an integer, } 2^n\leqslant k<2^{n+1}\}$.

We wish to thank Dick Askey for identifying a certain series that arose in our calculations as a hypergeometric function, and for showing us how to use Gauss' formula and the tables to evaluate it.

1. We begin with a lemma. For $0<r<1$ define $\varphi_r(z)=(z-r)/(1-rz)$.

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LEMMA 1.

$$\lim_{r\to 1}\sup_{0\leqslant\rho<1}(1-\rho)^{-1}\int_{-\pi}^{\pi}(1-|\varphi_r(\rho e^{i\theta})|)^2\frac{d\theta}{2\pi}$$
$$=\max_{0\leqslant\alpha\leqslant 1}\frac{16\alpha}{(1+\alpha)^3}\frac{1}{\pi}\int_0^\infty\frac{dx}{(1+x^2)\{\sqrt{1+x^2}+\sqrt{[(1-\alpha)/(1+\alpha)]^2+x^2}\}^2}.$$

Proof. Let γ denote the expression occurring on the right-hand side of the equality asserted in Lemma 1. (We shall show later than γ and γ_0 of the Introduction are the same.)

We note that

$$(1-|\varphi_r(\rho e^{i\theta})|)^2$$
$$=\left(\frac{|1-r\rho e^{i\theta}|^2-|r-\rho e^{i\theta}|^2}{|1-r\rho e^{i\theta}|(|1-r\rho e^{i\theta}|+|r-\rho e^{i\theta}|)}\right)^2$$
$$=\frac{(1-r^2)^2(1-\rho^2)^2}{[(1-r\rho)^2+4r\rho\sin^2\theta/2]\times\{\sqrt{(1-r\rho)^2+4r\rho\sin^2\theta/2}+\sqrt{[(r-\rho)/(1-r\rho)]^2+4r\rho\sin^2\theta/2}\}^2}.$$

Next, we integrate the above equation, let $x=\theta/(1-r\rho)$, $1-\rho=\alpha(1-r)$, and use the inequality $\sin^2\theta/2\leqslant\theta^2/4$ to obtain

$$(1-\rho)^{-1}\int_{-\pi}^{\pi}(1-|\varphi_r(\rho e^{i\theta})|)^2\frac{d\theta}{2\pi}$$
$$\geqslant\frac{\alpha(1+r)^2(2-\alpha+\alpha r)^2}{(1+\alpha r)^3}$$
$$\times\frac{1}{\pi}\int_0^{\pi/(1-r)(1+\alpha r)}\frac{dx}{(1+x^2)\{\sqrt{1+x^2}+\sqrt{[(1-\alpha)/(1+\alpha r)]^2+x^2}\}^2}.$$

Since $1-\rho=\alpha(1-r)$,

$$\sup_{0\leqslant\rho<1}(1-\rho)^{-1}\int_{-\pi}^{\pi}(1-|\varphi_r(\rho e^{i\theta})|)^2\frac{d\theta}{2\pi}$$
$$\geqslant\frac{\alpha(1+r)^2(2-\alpha+\alpha r)^2}{(1+\alpha r)^3}$$
$$\times\frac{1}{\pi}\int_0^{\pi/(1-r)(1+\alpha r)}\frac{dx}{(1+x^2)\{\sqrt{1+x^2}+\sqrt{[(1-\alpha)/(1+\alpha r)]^2+x^2}\}^2}$$

for any α with $0<\alpha\leqslant 1/(1-r)$, in particular, with $0<\alpha\leqslant 1$. Taking $\underline{\lim}_{r\to 1}$ on both sides and using the monotone convergence theorem on the right-hand side, we have

$$\varliminf_{r\to 1}\sup_{0\leqslant\rho<1}(1-\rho)^{-1}\int_{-\pi}^{\pi}(1-|\varphi_r(\rho e^{i\theta})|)^2\frac{d\theta}{2\pi}$$
$$\geqslant\frac{16\alpha}{(1+\alpha)^3}\frac{1}{\pi}\int_0^\infty\frac{dx}{(1+x^2)\{\sqrt{1+x^2}+\sqrt{[(1-\alpha)/(1+\alpha)]^2+x^2}\}^2}$$

for any α with $0<\alpha\leqslant 1$. But the expression on the right-hand side can easily be seen to be decreasing when $\alpha\geqslant 1$, so we have

$$\varliminf_{r\to 1}\sup_{0\leqslant\rho<1}(1-\rho)^{-1}\int_{-\pi}^{\pi}(1-|\varphi_r(\rho e^{i\theta})|)^2\frac{d\theta}{2\pi}\geqslant\gamma.$$

Next, we estimate from above. Since for a fixed r $(0<r<1)$,

$$(1-\rho)^{-1}\int_{-\pi}^{\pi}(1-|\varphi_r(\rho e^{i\theta})|)^2\frac{d\theta}{2\pi}\to 0 \quad \text{as } \rho\to 1,$$

this expression takes its maximum at some $\rho(r)$ with $0\leqslant\rho(r)<1$; $\rho(r)$ may not be uniquely determined but we claim that $\rho(r)\to 1$ as $r\to 1$. If not, there exists a sequence $r_n\nearrow 1$ such that $\rho(r_n)\leqslant c<1$. Then

$$\begin{aligned}
&\sup_{0\leqslant\rho<1}(1-\rho)^{-1}\int_{-\pi}^{\pi}(1-|\varphi_{r_n}(\rho e^{i\theta})|)^2\frac{d\theta}{2\pi}\\
&=\frac{1}{1-\rho(r_n)}\int_{-\pi}^{\pi}(1-|\varphi_{r_n}(\rho(r_n)e^{i\theta})|)^2\frac{d\theta}{2\pi}\\
&\leqslant\frac{1}{1-c}\int_{-\pi}^{\pi}(1-|\varphi_{r_n}(\rho(r_n)e^{i\theta})|)^2\frac{d\theta}{2\pi}\to 0 \quad (n\to\infty).
\end{aligned}$$

This contradicts the fact that

$$\varliminf_{r\to 1}\sup_{0\leqslant\rho<1}(1-\rho)^{-1}\int_{-\pi}^{\pi}(1-|\varphi_r(\rho e^{i\theta})|)^2\frac{d\theta}{2\pi}>0,$$

which was established in the first part of the lemma.

Next we estimate from above. Let $0<\delta<1$ and determine $\epsilon=\epsilon(\delta)$ so that $\sin x\geqslant\delta x$ if $0\leqslant x\leqslant\epsilon/2$ and $\sin x\leqslant\delta x$ if $\epsilon/2\leqslant x<\pi/2$. For $0\leqslant\theta\leqslant\epsilon$,

$$\begin{aligned}
(1-r\rho)^2+4r\rho\sin^2\frac{\theta}{2}&\geqslant(1-r\rho)^2+4r\rho\left(\delta\cdot\frac{\theta}{2}\right)^2\\
&\geqslant r\rho\delta^2\left\{\frac{(1-r\rho)^2}{r\rho\delta^2}+\theta^2\right\}\geqslant r\rho\delta^2\{(1-r\rho)^2+\theta^2\}\\
&=r\rho\delta^2(1-r\rho)^2\left\{1+\left(\frac{\theta}{1-r\rho}\right)^2\right\},
\end{aligned}$$

and similarly

$$(r-\rho)^2+4r\rho\sin^2\frac{\theta}{2}\geqslant r\rho\delta^2(1-r\rho)^2\left\{\left(\frac{r-\rho}{1-r\rho}\right)^2+\left(\frac{\theta}{1-r\rho}\right)^2\right\}.$$

For $\epsilon\leqslant\theta\leqslant\pi$,

$$(1-r\rho)^2+4r\rho\sin^2\theta/2\geqslant 4r\rho\sin^2\epsilon/2=r\rho\delta^2\epsilon^2,$$

and similarly

$$(r-\rho)^2+4r\rho\sin^2\theta/2\geqslant r\rho\delta^2\epsilon^2.$$

We use these inequalities to estimate

$$(1-\rho)^{-1}\int_{-\pi}^{\pi}(1-|\varphi_r(\rho e^{i\theta})|)^2\frac{d\theta}{2\pi}$$
$$=\frac{(1-r^2)^2(1-\rho^2)^2}{(1-\rho)(r\rho)^2(1-r\rho)^4}\frac{1}{\pi}\left\{\int_0^{\epsilon}+\int_{\epsilon}^{\pi}\right\}$$
$$\leqslant\frac{(1-r^2)^2(1-\rho^2)^2}{(1-\rho)(r\rho)^2(1-r\rho)^4}$$
$$\times\frac{1}{\pi}\int_0^{\epsilon}\frac{d\theta}{\{1+[\theta/(1-r\rho)]^2\}\times\{\sqrt{1+[\theta/(1-r\rho)]^2}+\sqrt{[(r-\rho)/(1-r\rho)]^2+[\theta/(1-r\rho)]^2}\}^2}$$
$$+\frac{(1-r^2)^2(1-\rho^2)^2}{4(1-\rho)(r\rho)^2\delta^4\epsilon^4}=(\mathrm{I})+(\mathrm{II}).$$

To estimate (I), set $x=\theta/(1-r\rho)$ and $1-\rho=\alpha(1-r)$.

$$(\mathrm{I})=\frac{(1+r)^2(2-\alpha+\alpha r)^2\cdot\alpha}{(r\rho)^2(1+\alpha r)^3\delta^4}$$
$$\times\frac{1}{\pi}\int_0^{\epsilon/(1-r\rho)}\frac{dx}{(1+x^2)\{\sqrt{1+x^2}+\sqrt{[(1-\alpha)/(1+\alpha r)]^2+x^2}\}^2}$$
$$\leqslant\frac{(1+\alpha)^3}{(r\rho)^2(1+\alpha r)^3\delta^4}\frac{16}{(1+\alpha)^3}$$
$$\times\frac{1}{\pi}\int_0^{\infty}\frac{dx}{(1+x^2)\{\sqrt{1+x^2}+\sqrt{[(1-\alpha)/(1+\alpha)]^2+x^2}\}^2}$$
$$\leqslant\frac{1}{(r\rho)^2}\cdot\left(\frac{1+\alpha}{1+\alpha r}\right)^3\cdot\frac{1}{\delta^4}\cdot\gamma$$
$$\leqslant\frac{1}{(r\rho)^2}\cdot\frac{1}{r^3}\cdot\frac{1}{\delta^4}\cdot\gamma,$$

since $(1+\alpha)/(1+\alpha r)\leqslant 1/r$.

$$(\mathrm{II})\leqslant\frac{1-\rho}{(r\rho)^2\delta^4\epsilon^4}.$$

Combining (I) and (II), we obtain

$$\sup_{0\leqslant\rho<1}(1-\rho)^{-1}\int_{-\pi}^{\pi}(1-|\varphi_r(\rho e^{i\theta})|)^2\frac{d\theta}{2\pi}$$
$$=(1-\rho(r))^{-1}\int_{-\pi}^{\pi}(1-|\varphi_r(\rho(r)e^{i\theta})|)^2\frac{d\theta}{2\pi}$$
$$\leqslant\frac{1}{(r\rho(r))^2}\cdot\frac{1}{r^3}\cdot\frac{1}{\delta^4}\cdot\gamma+\frac{1-\rho(r)}{(r\rho(r))^2\delta^4\epsilon^4}.$$

Since $\rho(r)\to 1$ as $r\to 1$, we have

$$\varlimsup_{r\to 1}\sup_{0\leqslant\rho<1}(1-\rho)^{-1}\int_{-\pi}^{\pi}(1-|\varphi_r(\rho e^{i\theta})|)^2\frac{d\theta}{2\pi}\leqslant\frac{1}{\delta^4}\cdot\gamma.$$

But since δ is arbitrary in $0<\delta<1$, we can replace the right-hand side by γ. This completes the proof of Lemma 1. □

THEOREM 1.

$$\inf_{B\in\mathcal{B}_\infty} \overline{\lim_{r\to 1}}\,(1-\rho)^{-1}\int_{-\pi}^{\pi}(1-|B(\rho e^{i\theta})|)^2\frac{d\theta}{2\pi}=\gamma.$$

Proof. Take $B\in\mathcal{B}_\infty$ and let $z_n=r_ne^{i\theta_n}$ be its zeros, $r_n\nearrow 1$. Fix r, $0<r<1$, and choose n so large that $r_n=r_n(r)>r$. Then we have

$$\sup_{r<\rho<1}(1-\rho)^{-1}\int_{-\pi}^{\pi}(1-|B(\rho e^{i\theta})|)^2\frac{d\theta}{2\pi}\geqslant\sup_{r<\rho<1}(1-\rho)^{-1}\int_{-\pi}^{\pi}(1-|\varphi_{r_n}(\rho e^{i\theta})|)^2\frac{d\theta}{2\pi}$$
$$=\sup_{0<\rho<1}(1-\rho)^{-1}\int_{-\pi}^{\pi}(1-|\varphi_{r_n}(\rho e^{i\theta})|)^2\frac{d\theta}{2\pi}.$$

Hence,

$$\overline{\lim_{\rho\to 1}}\,(1-\rho)^{-1}\int_{-\pi}^{\pi}(1-|B(\rho e^{i\theta})|)^2\frac{d\theta}{2\pi}$$
$$=\lim_{r\to 1}\sup_{\rho>r}(1-\rho)^{-1}\int_{-\pi}^{\pi}(1-|B(\rho e^{i\theta})|)^2\frac{d\theta}{2\pi}$$
$$\geqslant\lim_{r\to 1}\sup_{0<\rho<1}(1-\rho)^{-1}\int_{-\pi}^{\pi}(1-|\varphi_r(\rho e^{i\theta})|)^2\frac{d\theta}{2\pi}=\gamma,$$

by Lemma 1.

Next we show that the infimum is actually attained. Define sequences $r_n\nearrow 1$ and $\rho_n\nearrow 1$ such that

(1) $$\rho_n<r_n<\rho_{n+1},$$

(2) $$1-r_{n+1}<\tfrac{1}{2}(1-r_n),$$

(3) $$(1-\rho_n)\left(\sum_{k<n}\frac{1}{1-r_k}\right)<\frac{1}{n},$$

(4) $$\frac{(1-r_n)^2}{(1-\rho_n)^3}<\frac{1}{n},\quad\text{and}$$

(5) $$\frac{(1-r_n)}{(1-\rho_n)}\sum_{k<n}\frac{1}{1-r_k}<\frac{1}{n}.$$

The existence of such sequences can be shown inductively. Suppose that $r_1,\dots,r_{n-1}$ and $\rho_1,\dots,\rho_{n-1}$ have been defined satisfying (1)–(5). Choose ρ_n so that

(1) $$r_{n-1}<\rho_n<1,\quad\text{and}$$

(3) $$(1-\rho_n)\left(\sum_{k<n}\frac{1}{1-r_k}\right)<\frac{1}{n}.$$

Once ρ_n is defined, we can choose r_n so that (1), (2), (4), and (5) are satisfied.

Now, let B be the Blaschke product whose zeros are $r_1,r_2,\dots$. Suppose $\rho_N\leqslant\rho<\rho_{N+1}$. We estimate

$$1-|B(\rho e^{i\theta})|\leqslant(1-|\varphi_{r_N}(\rho e^{i\theta})|)+\sum_{k\neq N}(1-|\varphi_{r_k}(\rho e^{i\theta})|).$$

Note that

$$\begin{aligned}1-|\varphi_{r_k}(\rho e^{i\theta})| &\leqslant 1-|\varphi_{r_k}(\rho e^{i\theta})|^2\leqslant\frac{(1-\rho^2)(1-r_k^2)}{(1-r_k\rho)^2}\\ &\leqslant\frac{4}{r_1}\,\frac{(1-\rho)(1-r_k)}{[(1-r_k)+(1-\rho)]^2}.\end{aligned}$$

The last inequality follows from

$$1-r_k\rho=(1-r_k)+r_k(1-\rho)\geqslant r_1[(1-r_k)+(1-\rho)].$$

So,

$$\begin{aligned}\sum_{k<N}(1-|\varphi_{r_k}(\rho e^{i\theta})|) &\leqslant\frac{4}{r_1}\sum_{k<N}\frac{(1-\rho)(1-r_k)}{[(1-r_k)+(1-\rho)]^2}\\ &\leqslant\frac{4}{r_1}(1-\rho)\sum_{k<N}\frac{1}{1-r_k},\end{aligned}$$

and

$$\begin{aligned}\sum_{k>N}(1-|\varphi_{r_k}(\rho e^{i\theta})|) &\leqslant\frac{4}{r_1}\sum_{k>N}\frac{(1-\rho)(1-r_k)}{[(1-r_k)+(1-\rho)]^2}\\ &\leqslant\frac{4}{r_1}\,\frac{1}{1-\rho}\sum_{k>N}(1-r_k)\leqslant\frac{4}{r_1}\,\frac{1}{1-\rho}\,2(1-r_{N+1})\end{aligned}$$

by (2). Combining these estimates, we have

$$1-|B(\rho e^{i\theta})|\leqslant(1-|\varphi_{r_N}(\rho e^{i\theta})|)+C\left[(1-\rho)\sum\frac{1}{1-r_k}+\frac{1-r_{N+1}}{1-\rho}\right],$$

where C is a constant which does not depend on N and $\rho e^{i\theta}$. Let

$$\epsilon_N(\rho)=(1-\rho)\sum_{k<N}\frac{1}{1-r_k}+\frac{1-r_{N+1}}{1-\rho}.$$

Since

$$(1-|B(\rho e^{i\theta})|)^2\leqslant(1-|\varphi_{r_N}(\rho e^{i\theta})|)^2+C^2\epsilon_N(\rho)^2+2C\epsilon_N(\rho)(1-|\varphi_{r_N}(\rho e^{i\theta})|),$$

we have

$$\begin{aligned}&(1-\rho)^{-1}\int_{-\pi}^{\pi}(1-|B(\rho e^{i\theta})|)^2\frac{d\theta}{2\pi}\\ &\leqslant\sup_{0\leqslant t<1}(1-t)^{-1}\int_{-\pi}^{\pi}(1-|\varphi_{r_N}(te^{i\theta})|)^2\frac{d\theta}{2\pi}\\ &\quad+\frac{C^2\epsilon_N(\rho)^2}{1-\rho}+\frac{2C\epsilon_N(\rho)}{1-\rho}\int_{-\pi}^{\pi}(1-|\varphi_{r_N}(\rho e^{i\theta})|^2)\frac{d\theta}{2\pi}.\end{aligned}$$

Now,

$$\begin{aligned}\frac{\epsilon_N(\rho)^2}{1-\rho} &= (1-\rho)\Big(\sum_{k<N}\frac{1}{1-r_k}\Big)^2+\Big(\frac{1-r_{N+1}}{1-\rho}\Big)\Big(\sum_{k<N}\frac{1}{1-r_k}\Big)+\frac{(1-r_{N+1})^2}{(1-\rho)^3}\\ &\leqslant (1-\rho_N)\Big(\sum_{k<N}\frac{1}{1-r_k}\Big)^2+\Big(\frac{1-r_{N+1}}{1-\rho_{N+1}}\Big)\Big(\sum_{k<N}\frac{1}{1-r_k}\Big)+\frac{(1-r_{N+1})^2}{(1-\rho_{N+1})^3}\\ &< 3/N\end{aligned}$$

by (2), (3), and (5).

$$\begin{aligned}\frac{\epsilon_N(\rho)}{1-\rho}\int_{-\pi}^{\pi}(1-|\varphi_{r_N}(\rho e^{i\theta})|^2)\frac{d\theta}{2\pi} &\leqslant \frac{\epsilon_N(\rho)}{1-\rho}\,\frac{(1-r_N^2)(1-\rho^2)}{1-r_N^2\rho^2}\\ &\leqslant 2\epsilon_N(\rho)=2\Big(\frac{\epsilon_N(\rho)^2}{1-\rho}\Big)^{1/2}(1-\rho)\leqslant 2\sqrt{3/N}\end{aligned}$$

by the estimate above. Combining these, we have

$$\begin{aligned}(1-\rho)^{-1}\int_{-\pi}^{\pi}(1-|B(\rho e^{i\theta})|)^2\frac{d\theta}{2\pi} &\leqslant \sup_{0\leqslant t<1}(1-t)^{-1}\int_{-\pi}^{\pi}(1-|\varphi_{r_N}(te^{i\theta})|)^2\frac{d\theta}{2\pi}\\ &\quad+\Big(\frac{3C^2}{N}+\frac{4C\sqrt{3}}{\sqrt{N}}\Big),\end{aligned}$$

for $\rho_N\leqslant\rho<\rho_{N+1}$.

From Lemma 1, given $\epsilon>0$, there exists N_0 such that if $N\geqslant N_0$,

$$\sup_{0\leqslant\rho<1}(1-\rho)^{-1}\int_{-\pi}^{\pi}(1-|\varphi_{r_N}(\rho e^{i\theta})|)^2\frac{d\theta}{2\pi}<\gamma+\epsilon.$$

So,

$$(1-\rho)^{-1}\int_{-\pi}^{\pi}(1-|B(\rho e^{i\theta})|)^2\frac{d\theta}{2\pi}\leqslant\gamma+\epsilon+C_1\cdot\frac{1}{\sqrt{N}},$$

if $N>N_0$, $\rho_N<\rho<1$, where C_1 is a constant independent of N and ρ. Since ϵ is arbitrary, we obtain

$$\varlimsup_{\rho\to1}(1-\rho)^{-1}\int_{-\pi}^{\pi}(1-|B(\rho e^{i\theta})|)^2\frac{d\theta}{2\pi}\leqslant\gamma.$$

This completes the proof. □

2. The fact that $\gamma=\gamma_0$ is a consequence of Lemma 2.

$$\begin{aligned}&\frac{16\alpha}{(1+\alpha)^3}\cdot\frac{1}{\pi}\int_0^{\infty}\frac{dx}{(1+x^2)\{\sqrt{1+x^2}+\sqrt{[(1-\alpha)/(1+\alpha)]^2+x^2}\}^2}\\ &\quad=(1+\sqrt{1-u})[{}_2F_1({}^{1/2}{}_2{}^{1/2};u)-1],\end{aligned}$$

where $u=4\alpha/(1+\alpha)^2$.

Proof. We set $x=\tan t$ and obtain

$$\int_0^\infty \frac{dx}{(1+x^2)\{\sqrt{1+x^2}+\sqrt{[(1-\alpha)/(1+\alpha)]^2+x^2}\}^2}$$
$$=\int_0^{\pi/2} \frac{dt}{(\sec t+\sqrt{\sec^2 t-4\alpha/(1+\alpha)^2})^2}=\int_0^{\pi/2}\left(\frac{\cos t}{1+\sqrt{1-u\cos^2 t}}\right)^2 dt.$$

We rewrite the integrand successively by rationalizing the denominator, squaring out, and expanding into series. We obtain successively

$$\begin{aligned}\left(\frac{\cos t}{1+\sqrt{1-u\cos t}}\right)^2 &= \frac{1}{u^2}\left(\frac{1-\sqrt{1-u\cos^2 t}}{\cos t}\right)^2\\ &= \frac{2}{u^2}\,\frac{1-\sqrt{1-u\cos^2 t}}{\cos^2 t}-\frac{1}{u}\\ &= \frac{2}{u^2}\sum_{n=1}^\infty \binom{\frac{1}{2}}{n}(-1)^{n+1}u^n\cos^{2(n-1)}t-\frac{1}{u}\\ &= \sum_{n=2}^\infty \binom{\frac{1}{2}}{n}(-1)^{n+1}u^{n-2}\cos^{2(n-1)}t.\end{aligned}$$

Now, integrating with respect to t term by term and using the formula

$$\int_0^{\pi/2}\cos^{2n} t\,dt=\frac{2n-1}{2n}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2},$$

we obtain

$$\int_0^{\pi/2}\left(\frac{\cos t}{1+\sqrt{1-u\cos^2 t}}\right)^2 dt=\frac{\pi}{2}\sum_{n=1}^\infty \frac{[1\cdot 3\cdot\cdots\cdot(2n-1)]^2}{(n+1)!\,n!\,2^{2n}}\cdot u^{n-1}.$$

So,

$$\begin{aligned}&\frac{16\alpha}{(1+\alpha)^3}\cdot\frac{1}{\pi}\int_0^\infty \frac{dx}{(1+x^2)\{\sqrt{1+x^2}+\sqrt{[(1-\alpha)/(1+\alpha)]^2+x^2}\}^2}\\ &\quad=\frac{2}{1+\alpha}\sum_{n=1}^\infty \frac{[1\cdot 3\cdot\cdots\cdot(2n-1)]^2}{(n+1)!\,n!\,2^{2n}}\cdot u^n\\ &\quad=(1+\sqrt{1-u})\left\{\sum_{n=0}^\infty \frac{[(2n)!]^2}{(n+1)!\,(n!)^3 2^{4n}}\cdot u^n-1\right\}.\end{aligned}$$

But,

$$\begin{aligned}\sum_{n=0}^\infty \frac{[(2n)!]^2}{(n+1)!\,(n!)^3 2^{4n}}\cdot u^n&=\sum_{n=0}^\infty \frac{(1)_{2n}^2}{(2)_n(1)_n^3 2^{4n}}u^n\\ &=\sum_{n=0}^\infty \frac{(\frac{1}{2})_n^2(1)_n^2 2^{4n}}{(2)_n(1)_n^3 2^{4n}}u^n=\sum_{n=0}^\infty \frac{(\frac{1}{2})_n^2}{(2)_n}\frac{u^n}{n!}\\ &={}_2F_1({}^{1/2}{}_2{}^{1/2};u).\end{aligned}$$

We used the standard notation $(a)_n=[\Gamma(a+n)]/[\Gamma(a)]$. This completes the proof. □

In order to estimate γ_0 from below we use a formula of Gauss' [1, p. 558, 15.2.20] to show that

$$ {}_2F_1({}^{1/2}{}_2{}^{1/2};u)=\frac{4}{\pi}\left\{\frac{1}{u}E(u)-\left(\frac{1}{u}-1\right)K(u)\right\}, $$

where $E(u)=\int_0^{\pi/2}(1-u\sin^2\theta)^{1/2}\,d\theta$ and $K(u)=\int_0^{\pi/2}(1-u\sin^2\theta)^{-1/2}\,d\theta$. Next, using the tables for the functions E and K [1, pp. 608–609], we find that the tabulated value of u $(0\leqslant u\leqslant 1)$ that makes the expression $(1+\sqrt{1-u})\times\{{}_2F_1({}^{1/2}{}_2{}^{1/2};u)-1\}$ the largest is $u=.99$. In this way we obtain the estimate $\gamma_0\geqslant .285$.

We point out that $u=1$ corresponds to $\alpha=1$ and in this case the left-hand member of the equality of Lemma 2 is

$$ \frac{2}{\pi}\int_0^\infty\frac{dx}{(1+x^2)[x+\sqrt{1+x^2}]^2}=\frac{2}{\pi}\int_0^{\pi/2}\left(\frac{\cos t}{1+\sin t}\right)^2dt $$
$$ =\frac{2}{\pi}\int_0^{\pi/2}\frac{1-\sin\theta}{1+\sin\theta}\,d\theta=\frac{4}{\pi}-1\geqslant .273. $$

In other words we obtain the estimate $\gamma_0\geqslant 4/\pi-1\geqslant .273$ without recourse to the tables.

3. Now we give some applications.

LEMMA 3.

$$ \overline{\lim_{r\to 1}}\,(1-r)\sum_{k=1}^{\infty}k^2|a_k|^2r^{2(k-1)}\leqslant\frac{1}{2}\left(\overline{\lim_{k\to\infty}}\,k|a_k|\right)^2. $$

Proof. Let $c_0=\overline{\lim}_{k\to\infty}\,k|a_k|<\infty$ and let $\epsilon>0$. Take N so large that $k|a_k|<c_0+\epsilon$ whenever $k\geqslant N$. Then

$$ \sum_{k=1}^{\infty}k^2|a_k|^2r^{2(k-1)}\leqslant\sum_{k=1}^{N}k^2|a_k|^2+\sum_{k=N+1}^{\infty}k^2|a_k|^2r^{2(k-1)} $$
$$ \leqslant\sum_{k=1}^{N}k^2|a_k|^2+(c_0+\epsilon)^2\cdot\frac{r^{2N}}{1-r^2}. $$

So,

$$ \overline{\lim_{r\to 1}}\,(1-r)\sum_{k=1}^{\infty}k^2|a_k|^2r^{2(k-1)}\leqslant\tfrac{1}{2}(c_0+\epsilon)^2. $$

Since $\epsilon>0$ is arbitrary, we obtain the lemma. □

THEOREM 2. *If* $B(z)=\sum_{n=0}^{\infty}a_nz^n\in\mathcal{B}_\infty$, *then*

$$ \overline{\lim_{n\to\infty}}\,n|a_n|\geqslant\sqrt{\gamma_0/2}\geqslant .37749\cdots. $$

Proof. Note that

$$ 1-|B(re^{i\theta})|\leqslant\int_r^1|B'(te^{i\theta})|\,dt\quad\text{a.e. }\theta. $$

By the continuous form of Minkowski's inequality,

$$\left\{\int_{-\pi}^{\pi}(1-|B(re^{i\theta})|)^2\frac{d\theta}{2\pi}\right\}^{1/2}\leqslant\int_r^1 M_2(t,B')\,dt,$$

where

$$M_2(t,B')=\left(\frac{1}{2\pi}\int_0^{2\pi}|B'(te^{i\theta})|^2\,d\theta\right)^{1/2}=\left(\sum_{k=1}^{\infty}k^2|a_k|^2t^{2(k-1)}\right)^{1/2}.$$

Let $c_1=\overline{\lim}_{r\to 1}(1-t)M_2(t,B')^2$ and let $\epsilon>0$. Take r close enough to 1 that

$$(1-t)M_2(t,B')^2<c_1+\epsilon\quad\text{whenever } r\leqslant t<1.$$

So,

$$\begin{aligned}\left\{\int_{-\pi}^{\pi}(1-|B(re^{i\theta})|)^2\frac{d\theta}{2\pi}\right\}^{1/2}&\leqslant\int_r^1(1-t)^{-1/2}\{(1-t)M_2(t,B')^2\}^{1/2}\,dt\\&\leqslant(c_1+\epsilon)^{1/2}\int_r^1(1-t)^{-1/2}\,dt\\&=2(c_1+\epsilon)^{1/2}(1-r)^{1/2}.\end{aligned}$$

Since $\epsilon>0$ is arbitrary, we obtain

$$\overline{\lim_{r\to 1}}\,(1-r)^{-1}\int_{-\pi}^{\pi}(1-|B(re^{i\theta})|)^2\frac{d\theta}{2\pi}\leqslant 4c_1.$$

Combining this with Lemma 2 and Theorem 1, we obtain

$$\overline{\lim_{n\to\infty}}\; n|a_n|\geqslant\sqrt{\gamma_0/2}=.37749\cdots.$$

This completes the proof. □

We point out that if we use the estimate $\gamma_0\geqslant .273$ that can be obtained without use of the tables, we can conclude that

$$\overline{\lim_{n\to\infty}}\; n|a_n|\geqslant\sqrt{.273/2}=.3694\cdots,$$

an estimate which already improves the result of Newman and Shapiro mentioned in the Introduction.

THEOREM 3. *For any* $\epsilon>0$, *there exists a* $B(z)=\sum_{n=0}^{\infty}a_nz^n\in\mathcal{B}_\infty$ *such that*

$$\overline{\lim_{n\to\infty}}\; n|a_n|\leqslant\frac{2}{e}+\epsilon=.735\cdots+\epsilon.$$

Proof. Let $0<c<1$ and let $z_k=1-c^k$, $k=1,2,\ldots$. Let B be the Blaschke product with zeros z_k, $k=1,2,\ldots$. Then, as in [2], we have

$$\bar{a}_n=\sum_{k=1}^{\infty}\frac{z_k^{n-1}}{B'(z_k)}.$$

We estimate $|B'(z_k)|$ from below. Since

$$B'(z_k)=\frac{1}{1-z_k^2}\prod_{l\neq k}\frac{z_l-z_k}{1-z_kz_l},$$

$$
\begin{aligned}
|B'(z_k)| &= \frac{1}{1-z_k^2} \prod_{l\neq k} \left| \frac{z_l - z_k}{1 - z_k z_l} \right| \\
&\geqslant \frac{1}{1-z_k^2} \left(\prod_{l<k} \frac{c^l - c^k}{c^l + c^k} \right) \left(\prod_{l>k} \frac{c^k - c^l}{c^k + c^l} \right) \\
&= \frac{1}{1-z_k^2} \left(\prod_{l<k} \frac{1 - c^{k-l}}{1 + c^{k-l}} \right) \left(\prod_{l>k} \frac{1 - c^{l-k}}{1 + c^{l-k}} \right) \\
&\geqslant \frac{1}{1-z_k^2} \left(\prod_{l=1}^{\infty} \frac{1 - c^l}{1 + c^l} \right)^2 .
\end{aligned}
$$

So,

$$
\begin{aligned}
(n+1)|a_{n+1}| &\leqslant \sum_{k=1}^{\infty} \frac{(n+1) z_k^n}{|B'(z_k)|} \\
&\leqslant \left(\prod_{l=1}^{\infty} \frac{1 + c^l}{1 - c^l} \right)^2 \sum_{k=1}^{\infty} (n+1) z_k^n (1 - z_k^2) \\
&\leqslant 2(n+1) \left(\prod_{l=1}^{\infty} \frac{1 + c^l}{1 - c^l} \right)^2 \sum_{k=1}^{\infty} (1 - c^k)^n c^k .
\end{aligned}
$$

In order to estimate the sum, we look at $u(t) = (1 - c^t)^n c^t$. We can easily see that $u(t)$ increases when $t \leqslant t_0 = \log_c(1/(n+1))$ and decreases when $t > t_0$. Then

$$
\begin{aligned}
\sum_{k=1}^{\infty} (1 - c^k)^n c^k &\leqslant \int_1^{\infty} (1 - c^t)^n c^t \, dt + \max_{1 \leqslant t < \infty} u(t) \\
&\leqslant -\frac{1}{(n+1) \log c} + \left(1 - \frac{1}{n+1} \right)^n \frac{1}{n+1} .
\end{aligned}
$$

So,

$$
(n+1)|a_{n+1}| \leqslant 2 \left(\prod_{l=1}^{\infty} \frac{1 + c^l}{1 - c^l} \right)^2 \left\{ -\frac{1}{\log c} + \left(1 - \frac{1}{n+1} \right)^n \right\} .
$$

Hence

$$
\overline{\lim_{n \to \infty}} \, n|a_n| \leqslant 2 \left(\prod_{l=1}^{\infty} \frac{1 + c^l}{1 - c^l} \right)^2 \left(-\frac{1}{\log c} + \frac{1}{e} \right) .
$$

Taking $c > 0$ small enough, we obtain

$$
\overline{\lim_{n \to \infty}} \, n|a_n| \leqslant 2/e + \epsilon . \qquad \square
$$

THEOREM 4. *If* $B(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{B}_{\infty}$ *then*

$$
\overline{\lim_{n \to \infty}} \left(\sum_{k \in I_n} k |a_k|^2 \right) \geqslant \frac{\gamma_0}{8} \geqslant .0356 .
$$

Proof. Let $c_1 = \overline{\lim}_{n \to \infty} (\sum_{k \in I_n} k |a_k|^2) < \infty$ and let $\epsilon > 0$. Take N large enough that

$$
\sum_{k \in I_n} k |a_k|^2 < c_1 + \epsilon \quad \text{if } n \geqslant N.
$$

Then

$$\sum_{k=1}^{\infty} k^2|a_k|^2 r^{2k} = \sum_{n=1}^{\infty} \sum_{k\in I_n} k^2|a_k|^2 r^{2k}$$
$$\leqslant \sum_{n=1}^{\infty} 2^{n+1} r^{2^{n+1}} \Big(\sum_{k\in I_n} k|a_k|^2 \Big)$$
$$\leqslant \sum_{n=1}^{N-1} 2^{n+1} \Big(\sum_{k\in I_n} k|a_k|^2 \Big) + 2(c_1+\epsilon) \sum_{n=N}^{\infty} 2^n r^{2^{n+1}}.$$

But

$$\sum_{n=N}^{\infty} 2^n r^{2^{n+1}} \leqslant \sum_{n=N}^{\infty} \int_{2^n}^{2^{n+1}} r^t\, dt = \int_{2^N}^{\infty} r^t\, dt$$
$$= \frac{r^{2^N}}{\log(1/r)} \leqslant \frac{1}{1-r}.$$

Hence

$$\overline{\lim_{r\to 1}}\,(1-r) \sum_{k=1}^{\infty} k^2|a_k|^2 r^{2k} \leqslant 2(c_1+\epsilon).$$

Since $\epsilon > 0$ is arbitrary, we obtain

$$\overline{\lim_{r\to 1}}\,(1-r) \sum_{k=1}^{\infty} k^2|a_k|^2 r^{2k} \leqslant 2c_1.$$

Now proceed as in the proof of Theorem 2 to show that

$$\overline{\lim_{r\to 1}}\,(1-r)^{-1} \int_{-\pi}^{\pi} (1-|B(re^{i\theta})|)^2 \frac{d\theta}{2\pi} \leqslant 8c_1.$$

So $c_1 \geqslant \gamma_0/8 \geqslant .356\cdots$. □

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