ITERATES OF HOLOMORPHIC SELF-MAPS OF THE UNIT BALL IN $\mathbf{C}^N$

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Introduction. The sequence of iterates of a holomorphic map of the unit disc **D** into itself with no fixed points in **D** was studied by J. Wolff [7] and A. Denjoy [1]. They showed that for such a function the iterates converge, uniformly on compact subsets of **D**, to a unimodular constant. In Section 1 of this paper we consider the generalization of this question to holomorphic, fixed point free self-maps of the unit ball in $\mathbf{C}^N$. We will show that in this case also the sequence of iterates converges, uniformly on compact subsets of the ball, to a constant of norm 1. The basic tool we use is a theorem of W. Rudin [4] which characterizes the fixed point set of a holomorphic map of the ball into itself as an affine subset of the ball.

The one variable Denjoy–Wolff theorem is often stated to include holomorphic self-maps of the disc which fix one point in the disc, but which are not conformal automorphisms of the disc. In this case the entire sequence of iterates still converges to a constant, the interior fixed point. In Section 2 we consider the iteration of maps with fixed points in the ball in higher dimensions.

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1. Maps with no interior fixed points. Let B, or B_N if we wish to indicate the dimension explicitly, be the open unit ball in $\mathbf{C}^N$, in the Euclidean metric. Denote by $H(B;B)$ the family of all holomorphic maps of B into itself. For $f \in H(B;B)$ we denote the iterates of f by f_n:

$$f_1 = f, \qquad f_{n+1} = f \circ f_n \qquad n = 1, 2, 3, \ldots$$

Since $H(B;B)$ is a normal family, every sequence of iterates of f contains a subsequence which converges, uniformly on compact subsets of B. We will examine the possible subsequential limits of $\{f_n\}$ according to the fixed point character of f. Note that a subsequential limit of iterates of $f \in H(B;B)$ need not belong to $H(B;B)$. However the following lemma shows that this can only happen if the limit is a constant map of norm 1.

LEMMA 1.1. *Let $F: B \to \bar{B}$ be holomorphic. Then either $F(B) \subseteq B$ or $F(z) \equiv \zeta$ in ∂B, for all z in B.*

Proof. Suppose there is a z_0 in B with $F(z_0) = \zeta \in \partial B$. Set $G(z) = (1 + \langle z, \zeta \rangle)/2$, so G belongs to $A(B)$, the algebra of functions holomorphic in B and continuous

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on $\bar{B}$. Note that $G(\zeta)=1$ and $|G(z)|<1$ for all z in $\bar{B}\backslash\{\zeta\}$. Consider the holomorphic function $G\circ F$. Since $G\circ F(z_0)=1$ and $|G\circ F(z)|\leqslant 1$ for all z in B, the maximum modulus theorem implies $G\circ F$ is identically 1. So $F(z)\equiv\zeta$, for all z in B, as desired. □

We will find it convenient to use some facts from the theory of topological semigroups. Under the operation of composition and with the topology of uniform convergence on compact subsets of B, $H(B;B)$ becomes a topological semigroup [5]. For $f\in H(B;B)$ denote by $\Gamma(f)$ the closure, in the space of all holomorphic maps from B to $\mathbf{C}^N$ with the topology of uniform convergence on compact subsets of B, of the iterates of f. If $\Gamma(f)\subseteq H(B;B)$, then $\Gamma(f)$ is a compact topological semigroup and as such contains a unique idempotent [6]. Recall g is an *idempotent* if $g\circ g=g$. An idempotent in $H(B;B)$ is also called a *retraction* of B.

Next we give a precise statement of Rudin's theorem on the fixed point sets of maps in $H(B;B)$. This theorem is the key to the proof of our main theorem.

THEOREM 1.2. [4; Sec. 8.2.3, p. 166]. *If $F\colon B\to B$ is holomorphic, then the fixed point set of F is an affine subset of B; that is, the intersection of B with $c+L$, where $c\in\mathbf{C}^N$ and L is a complex linear subspace of $\mathbf{C}^N$.*

Denote by $\operatorname{Aut}B$ the group of biholomorphic maps (*automorphisms*) of B onto itself. These maps take affine subsets of B onto affine subsets [4, Sec. 2.4.2, p. 33]. Moreover, since $\operatorname{Aut}B$ acts transitively on B [4, Sec. 2.2.3, p. 31], if A is an affine subset of B there is a $\Psi\in\operatorname{Aut}B$ so that

$$\Psi(A)=\{(z_1,z_2,\ldots,z_N)\in B \text{ with } z_i=0 \text{ for } i=r+1,\ldots,N\}.$$

To see this, first map some point of A to the origin, so that the image of A is the intersection of B with a complex linear subspace of $\mathbf{C}^N$. Now apply a unitary transformation. Thus $\Psi(A)\cong B_r$, the unit ball in $\mathbf{C}^r$. We will say $f\in H(B;B)$ is an automorphism of A if $\Psi\circ f\circ\Psi^{-1}$ is an automorphism when restricted to $\Psi(A)$.

Before stating our main result we need to develop a several variable analogue of a theorem which in the disc is due to J. Wolff [8]. To facilitate the statement of this theorem we introduce some notation. Let

$$e_1=(1,0,\ldots,0)=(1,0')\in\partial B.$$

For $\lambda>0$,

$$E(e_1,\lambda)=\{z=(z_1,z_2,\ldots,z_N) \text{ so that } |1-z_1|^2<\lambda(1-|z|^2)\}.$$

Some computation shows that $E(e_1,\lambda)$ is the set of points $(z_1,z_2,\ldots,z_N)=(z_1,z')$ in $\mathbf{C}^N$ satisfying $|z_1-(1-c)|^2+c|z'|^2<c^2$ where $c=\lambda/(1+\lambda)$. Thus $E(e_1,\lambda)$ is an ellipsoid in B, centered at $e_1/(1+\lambda)$ and containing e_1 in its boundary. For an arbitrary ζ in ∂B, $E(\zeta,\lambda)$ is the analogous ellipsoid in B, centered at $\zeta/(1+\lambda)$ and containing ζ in its boundary.

THEOREM 1.3. *If f is in $H(B;B)$ and fixed point free, then there is a unique point $\zeta\in\partial B$ such that each ellipsoid $E(\zeta,\lambda)$ is mapped into itself by f and every iterate of f.*

Proof. Choose $r_n \uparrow 1$. Let $a_n \in B$ be a fixed point of the map $r_n f\colon r_n\bar{B} \to r_n\bar{B}$. Passing to a subsequence if necessary, assume $a_n \to \zeta \in \bar{B}$. Since f has no fixed points in B, $\zeta \in \partial B$. Without loss of generality assume $\zeta = e_1$. Then $a_n \to e_1$, $f(a_n) = a_n/r_n \to e_1$, and

$$\frac{1-|f(a_n)|}{1-|a_n|} = \frac{1-(|a_n|/r_n)}{1-|a_n|} < 1.$$

Again passing to a subsequence if necessary we have

$$\lim_{n\to\infty} \frac{1-|f(a_n)|}{1-|a_n|} = \alpha \leqslant 1.$$

By Julia's lemma [4, Sec. 8.5.3, p. 175]

$$\frac{|1-f_1(z)|^2}{1-|f(z)|^2} \leqslant \alpha \frac{|1-z_1|^2}{1-|z|^2}$$

(here $f=(f_1, f_2, \ldots, f_N)$). Geometrically this means $f(E(e_1, \lambda)) \subseteq E(e_1, \alpha\lambda) \subseteq E(e_1, \lambda)$ since $\alpha \leqslant 1$, as desired.

To see that ζ is unique suppose we have another point ζ' in ∂B with the property that each ellipsoid $E(\zeta', \lambda)$ is mapped into itself by f. Choose λ_1 and λ_2 so that $E(\zeta', \lambda_1)$ and $E(\zeta, \lambda_2)$ are tangent to each other at the point z in B. Then $f(z)$ is in $\bar{E}(\zeta', \lambda_1) \cap \bar{E}(\zeta, \lambda_2) = \{z\}$, contradicting the hypothesis that f is fixed point free. □

NOTATION. We call the point ζ of Theorem 1.3 the Denjoy–Wolff point of f. The constant map $g(z) \equiv \zeta$ for z in B will be denoted $\zeta(f)$.

A consequence of Theorem 1.3 is the following:

COROLLARY 1.4. *Let $f \in H(B; B)$ be fixed point free. Then $\Gamma(f)$ contains at most one constant map, which can only be $\zeta(f)$.*

Proof. Let ζ be the Denjoy–Wolff point of f. Suppose there is a sequence $\{n_i\}$ so that $f_{n_i} \to w \in \bar{B}$. If $w \neq \zeta$ we can find a small neighborhood V of w in B disjoint from some ellipsoid $E(\zeta, \lambda)$. By Theorem 1.3, if z is any point in $E(\zeta, \lambda)$, then the image point $f_n(z)$ is in $E(\zeta, \lambda)$ for all $n \geqslant 1$. Thus $f_{n_i}(z) \notin V$ for any i, so $f_{n_i}(z) \not\to w$. Therefore the only constant function which may appear in $\Gamma(f)$ is $\zeta(f)$. □

We can now state our main theorem.

THEOREM A. *Let f be in $H(B; B)$ and suppose f has no fixed points in B. Then $f_n \to \zeta(f)$.*

We give the proof of Theorem A in several steps, beginning with the following proposition.

PROPOSITION 1.5. *Let f be an arbitrary map in $H(B; B)$. If there is a nonconstant map among the subsequential limits of $\{f_n\}$, then $\Gamma(f)$ contains a nonconstant idempotent.*

Proof. We suppose there is a nonconstant map g and a sequence $\{n_i\}$ so that $f_{n_i}\to g$. Note that $g(B)\subseteq B$. Set $m_i=n_{i+1}-n_i$. Choose a convergent subsequence of $\{f_{m_i}\}$, say $f_{m_{i_k}}\to h$. On the one hand $f_{m_{i_k}}\circ f_{n_{i_k}}\to h\circ g$. But also $f_{m_i}\circ f_{n_i}=f_{n_{i+1}}\to g$. So $h\circ g=g$ which implies that h is the identity map on the range of g, which consists of more than one point. By Theorem 1.2 the fixed point set of h is an affine subset A of B. The dimension of A is $\geqslant 1$ and the range of h contains A.

Now suppose that the range of h properly contains A. Then the above argument, applied to h instead of g, produces another subsequential limit of $\{f_n\}$ which is the identity on an affine subset A' of B containing the range of h. Moreover the dimension of A' is strictly greater than the dimension of A. Choose from among the subsequential limits of $\{f_n\}$ the map H with fixed point set of maximal dimension. For this map H we must have range $H=$ fixed point set of H, since otherwise there would be another subsequential limit with a fixed point set of larger dimension. Thus H is an idempotent, and since the dimension of the fixed point set of H is $\geqslant 1$, H is nonconstant. □

Our next goal is to establish Theorem A for automorphisms of B with no fixed points in B. If f is in Aut B then f is continuous from $\bar{B}$ to $\bar{B}$ and thus has a fixed point in $\bar{B}$. The automorphisms of B with no fixed points in B fix either exactly one point or exactly two points of ∂B [4, Sec. 2.4.6, p. 33]. The case of two fixed points in ∂B is easy to handle.

PROPOSITION 1.6. *Let $f\in$ Aut B fix precisely two points of ∂B. Then f_n converges to one of these fixed points.*

Proof. Suppose that f fixes ζ_1 and ζ_2 in ∂B. Consider the complex line L through ζ_1 and ζ_2. Since an automorphism takes complex lines to complex lines, f maps $L\cap B$ onto $L\cap B$. Now the Denjoy–Wolff theorem in one variable implies that the iterates of f restricted to $L\cap B$ converge to one of the fixed points, say ζ_1. By Lemma 1.1, every convergent subsequence of $\{f_n\}$ must converge to ζ_1. This implies that $f_n\to\zeta_1$, since $H(B;B)$ is a normal family. Clearly ζ_1 must be the Denjoy–Wolff point of f. □

The case of one fixed point in ∂B requires more work. We will assume, without loss of generality, that the fixed point is $e_1=(1,0')$. To study automorphisms of the disc it is convenient to transfer to the upper half plane via the biholomorphic map $z\to i(1+z)/(1-z)$. A similar device is available in several variables. Let $\Omega\subset\mathbf{C}^N$ be the region (the *Siegel upper half-space*) consisting of those points (w_1,w') with $\operatorname{Im} w_1>|w'|^2$, where $w'=(w_2,\ldots,w_N)$, $|w'|^2=|w_2|^2+\cdots+|w_N|^2$. Define Φ, the Cayley transform, on $\mathbf{C}^N\backslash\{z_1=1\}$ by $\Phi(z)=i(e_1+z)/(1-z_1)$. Then Φ is a biholomorphic map of B onto Ω [4; Sec. 2.3.1, p. 31]. Moreover if $\bar{\Omega}=\Omega\cup\partial\Omega$, where $\partial\Omega=\{(w_1,w')$ such that $\operatorname{Im} w_1=|w'|^2\}$, and $\bar{\Omega}\cup\{\infty\}$ is the one-point compactification of $\bar{\Omega}$, then defining $\Phi(e_1)=\infty$ extends Φ to a homeomorphism of $\bar{B}$ onto $\bar{\Omega}\cup\{\infty\}$. The automorphisms of B with fixed point set $\{e_1\}$ correspond to the automorphisms of Ω with fixed point set $\{\infty\}$.

An example of a class of such automorphisms are the Heisenberg translations, defined as follows. For each $b=(b_1,b')$ in $\partial\Omega$ set

$$h_b(w_1,w')=(w_1+b_1+2i\langle w',b'\rangle, w'+b').$$

The Heisenberg translations form a subgroup of $\operatorname{Aut}\Omega$, and for $b\neq 0$ each h_b fixes ∞ only [4; Sec. 2.3.3, p. 32]. By a Heisenberg translation of B we shall mean an automorphism of B of the form $\Phi^{-1}\circ h_b\circ\Phi$, where Φ is the Cayley transform and h_b is as above.

It is easy to see that the iterates of a Heisenberg translation converge to e_1, since for any $0\neq b\in\partial\Omega$, $(h_b)_n\to\infty$. However, in contrast to the situation in one variable, not every automorphism of Ω with fixed point set precisely $\{\infty\}$ is a Heisenberg translation. For example, if $\lambda=(\lambda_2,\ldots,\lambda_N)$ where $|\lambda_i|=1$ and if $b\neq 0$ is real then $g_{b,\lambda}(w_1,w')\equiv(w_1+b,\lambda_2w_2,\ldots,\lambda_Nw_N)$ is an automorphism of Ω fixing $\{\infty\}$ only. Note that $g_{b,\lambda}$ fixes setwise the image under Φ of the complex line through 0 and e_1, namely $\{(w_1,w')\in\Omega$ with $w'=0\}$. We will see that any automorphism of B with fixed point set $\{e_1\}$ which is not a Heisenberg translation of B must fix setwise some nonempty, proper affine subset of B. A map f is said to fix a set S *setwise* if $f(S)\subseteq S$. In this situation we will also say f fixes S *as a set.*

THEOREM 1.7. *Let $G\in\operatorname{Aut}B$ fix e_1 only. Write $G=(G_1,G_2,\ldots,G_N)$. If*

$$(*)\qquad |1-G_1(z)|^2/(1-|G(z)|^2)=|1-z_1|^2/(1-|z|^2)$$

holds for every z in B, then either G is a Heisenberg translation of B or G fixes as a set a proper, nonempty, affine subset of B.

REMARK. David Ullrich has pointed out to me that condition $(*)$ of Theorem 1.7 *must* hold for an automorphism of B with fixed point set precisely $\{e_1\}$. We give a proof of this fact at the end of Section 1. Note that $(*)$ has a simple geometric meaning: the boundary of each ellipsoid $E(e_1,\lambda)$ is mapped into itself by G.

Before giving a proof of Theorem 1.7 we will establish the following corollary.

COROLLARY 1.8. *If $G\in\operatorname{Aut}B_N$ fixes e_1 only then $G_n\to e_1$.*

Proof. If condition $(*)$ of Theorem 1.7 fails to hold for some point w in B then by Theorem 1.3 we must have

$$|1-G_1(w)|^2/(1-|G(w)|^2)=\beta|1-w_1|^2/(1-|w|^2)$$

for some $\beta<1$. Suppose further that G_n does not converge to $e_1=\zeta(G)$. By Corollary 1.4 and Proposition 1.5 $\Gamma(G)$ contains a nonconstant idempotent. Moreover, by a theorem of H. Cartan [3; p. 78] the nonconstant subsequential limits of the iterates of an automorphism must again be automorphisms. Since the only idempotent which is an automorphism is the identity map I on B, $\Gamma(G)$ contains I. Thus there is a sequence $\{n_i\}$ so that $G_{n_i}\to I$. In particular $G_{n_i}(w)\to w$. But this cannot be, for w lies in the boundary of $E(e_1,\lambda)$ where $\lambda=|1-w_1|^2/(1-|w|^2)$ and $G_n(w)$ is in $\bar{E}(e_1,\beta\lambda)\subseteq\operatorname{int}E(e_1,\lambda)$ for every $n\geqslant 1$.

This contradicts our assumption that G_n does not converge to e_1.

We suppose now that G satisfies $(*)$ at every point of B and apply Theorem 1.7. If G is a Heisenberg translation $\Phi^{-1}\circ h_b\circ\Phi$, then $G_n \to e_1$ since $(h_b)_n \to \infty$. We finish the remaining case by induction. Note that the corollary is true for $N=1$ and assume it holds for $k<N$. We are left to consider the possibility that G fixes setwise a nonempty, proper, affine subset A of B of dimension $k<N$. Now $\tilde{G}=G|_A$ is an automorphism of $A\cong B_k$ fixing e_1 only. By induction $\tilde{G}_n \to e_1$ and by Lemma 1.1 $G_n \to e_1$. □

In the proof of Theorem 1.7 we will transfer back and forth between the ball B and the Siegel upper half-space Ω via the Cayley transform Φ. For the proof of Theorem 1.7 we will use lower case letters to denote automorphisms of Ω and the corresponding capital letters for the associated automorphism of B obtained by composition on the right and left by Φ and Φ^{-1} respectively.

Proof of Theorem 1.7. Let $G\in \operatorname{Aut} B_N$ fix e_1 only and satisfy $(*)$. If $\Phi(z)=w$ then $\operatorname{Im} w_1-|w'|^2=(1-|z|^2)/|1-z_1|^2$. Thus the boundary of the ellipsoid $E(e_1,\lambda)$ is mapped by Φ to $\{(w_1,w')\in\Omega$ such that $\operatorname{Im} w_1-|w'|^2=1/\lambda\}$. Condition $(*)$ for $G\in\operatorname{Aut} B_N$ becomes, for the function $g=\Phi\circ G\circ\Phi^{-1}$,

$$(**)\qquad \operatorname{Im} g_1(w)-|g'(w)|^2=\operatorname{Im} w_1-|w'|^2$$

where $g=(g_1,g_2,\dots,g_N)=(g_1,g')$.

Set $G(0)=\alpha$ so $g(i,0')=\Phi(\alpha)=(a_1,a')$. Now $\operatorname{Im} a_1-|a'|^2=1$ since g satisfies $(**)$. Write $a_1=c+i(1+|a'|^2)$ where c is real. We claim that there is a Heisenberg translation of Ω taking (a_1,a') to $(i,0')$. To see this consider the point $(c+i|a'|^2,a')$ in $\partial\Omega$. The Heisenberg translation associated to this point takes $(i,0')$ to (a_1,a'). Its inverse is a Heisenberg translation having the desired property; we denote it simply by h_b. (A computation shows that $b=(-c+i|a'|^2,\,-a')$).

Now $h_b\circ g$ is an automorphism of Ω fixing ∞ and $(i,0')$. The corresponding automorphism F of B fixes 0 and e_1, and is just $H_b\circ G$. Note that F is unitary. Moreover, since F fixes e_1, F fixes as a set the orthogonal complement of the complex line through e_1, namely the set $\{z_1=0\}$. Thus $F(z_1,z_2,\dots,z_N)=F(z_1,z')=(z_1,Uz')$ where U is a unitary operator on $\mathbf{C}^{N-1}$. An easy computation shows that

$$F\circ\Phi^{-1}(w_1,w')=\left(\frac{w_1-i}{w_1+i},\,\frac{2}{w_1+i}Uw'\right)=\Phi^{-1}\circ F(w_1,w')$$

on $\mathbf{C}^N\backslash\{w_1=-i\}$. Therefore the automorphism f of Ω defined by $f=\Phi\circ F\circ\Phi^{-1}$ coincides with the original unitary map F on Ω;

$$f(w)=(w_1,Uw')\qquad (w=(w_1,w')\in\Omega).$$

At this point we consider two cases. If every eigenvalue of U is 1 then U, and hence F, is the identity. Thus $G=H_b^{-1}$ is a Heisenberg translation of B and we are done. So we suppose that U has an eigenvalue $e^{i\theta}\neq 1$. We will show that this implies that G fixes setwise a proper affine subset of B. It is sufficient to show that $g=\Phi\circ G\circ\Phi^{-1}$ fixes setwise a proper affine subset of Ω, since Φ preserves affine sets.

Choose $0 \neq \Lambda = (\lambda_2, \lambda_3, \ldots, \lambda_N)$ so that $\Lambda(U) = e^{i\theta}\Lambda$ where (U) denotes the matrix of the operator U relative to the standard basis on $\mathbf{C}^{N-1}$. Recall that $g = \Phi \circ G \circ \Phi^{-1} = \Phi \circ H_b^{-1} \circ F \circ \Phi^{-1} = h_b^{-1} \circ f$, where h_b^{-1} is the Heisenberg translation associated to the point $(c + i|a'|^2, a_2, \ldots, a_N)$ in $\partial\Omega$. Let A be the column vector $(a_2, \ldots, a_N)^t$ so that $\Lambda A = \sum_{i=2}^N \lambda_i a_i$. Now consider the set

$$\mathfrak{F} = \left\{ (w_1, w_2, \ldots, w_N) \in \Omega \text{ with } \sum_{i=2}^{N} \lambda_i w_i = \Lambda A/(1 - e^{i\theta}) \right\}.$$

$\mathfrak{F}$ is a nonempty, proper affine subset of Ω. We claim that g fixes $\mathfrak{F}$ as a set. To see this choose $(w_1, w_2, \ldots, w_N)$ in $\mathfrak{F}$. Now $g(w_1, w_2, \ldots, w_N) = h_b^{-1} \circ f(w_1, w') = h_b^{-1}(w_1, Uw')$. Writing $W' = (w_2, w_3, \ldots, w_N)^t$ we see that the last $N-1$ coordinates of $h_b^{-1}(w_1, Uw')$ are $((U)W' + A)^t$. To check that $g(w_1, w_2, \ldots, w_N)$ is in $\mathfrak{F}$ we compute

$$\begin{aligned} \Lambda((U)W' + A) &= e^{i\theta}\Lambda W' + \Lambda A \\ &= e^{i\theta}\Lambda A/(1 - e^{i\theta}) + \Lambda A \\ &= \Lambda A/(1 - e^{i\theta}). \end{aligned}$$

Therefore $g(w_1, w')$ is in $\mathfrak{F}$, as desired. □

A final observation before the proof of Theorem A is the following.

LEMMA 1.9. *If $f \in H(B; B)$ is such that $f_{n_i} \to I$, the identity map on B, for some sequence $\{n_i\}$, then $f \in \operatorname{Aut} B$.*

Proof. We may assume $f_{n_i - 1} \to g$. Then $f_{n_i - 1} \circ f \to g \circ f$. Since $f_{n_i} \to I$ we have $g \circ f = I$. In particular g is in $H(B; B)$ and therefore we also have $f_{n_i} = f \circ f_{n_i - 1} \to f \circ g$. So $f \circ g = g \circ f = I$ as desired. □

Proof of Theorem A. Proposition 1.6 and Corollary 1.8 together establish Theorem A for automorphisms of B with no fixed points in B. Now suppose f is an arbitrary fixed point free map in $H(B; B)$. If every subsequential limit of $\{f_n\}$ is constant then by Corollary 1.4 $f_n \to \zeta(f)$, uniformly on compact subsets of B, and we are done. Hence we suppose there is a nonconstant map among the subsequential limits of $\{f_n\}$. By Proposition 1.5 there is a sequence $\{n_i\}$ and a nonconstant idempotent h so that $f_{n_i} \to h$. Let A be the fixed point set of h, an affine set of dimension $\geqslant 1$.

We claim that f maps A into A. To see this choose z_0 in A. Now

$$f_{n_i}(z_0) \to h(z_0) = z_0 \quad \text{and thus} \quad f(f_{n_i}(z_0)) \to f(z_0).$$

But $f(f_{n_i}(z_0)) = f_{n_i}(f(z_0)) \to h(f(z_0))$. So $f(z_0) = h(f(z_0))$; that is, $f(z_0)$ is in the fixed point set A of h, as desired.

Moreover, f_{n_i} restricted to A converges to the identity on A. Lemma 1.9, with A replacing B, implies that $\tilde{f} \equiv f|_A$ is an automorphism of A, which clearly has no interior fixed points. By Corollary 1.8, $\tilde{f}_n$ converges to a constant in ∂B. But this contradicts the fact that $\tilde{f}_{n_i}$ converges to the identity map on A. Thus the subsequential limits of $\{f_n\}$ must all be constant and we are done by Corollary 1.4. □

We finish this section with a proof of the fact that condition (*) of Theorem 1.7 must hold for any $G \in \operatorname{Aut} B$ with fixed point set $\{e_1\}$. As previously remarked this is equivalent to the following:

THEOREM 1.10. *Let* $g \in \operatorname{Aut} \Omega$ *fix* ∞ *only. Then for every* $w = (w_1, w')$ *in* Ω

$$(**) \qquad \operatorname{Im} g_1(w) - |g'(w)|^2 = \operatorname{Im} w_1 - |w'|^2.$$

Proof. Suppose $g(i, 0') = (a_1, a')$. Set $t = \operatorname{Im} a_1 - |a'|^2$. Since (a_1, a') is in Ω, t is positive. For $s > 0$ define $\delta_s \in \operatorname{Aut} \Omega$ by $\delta_s(w_1, w') = (s^2 w_1, s w')$. If $s \neq 1$ the fixed point set of δ_s is $\{0, \infty\}$. Consider the automorphism $\delta_s \circ g$ where $s = 1/\sqrt{t}$. The image of $(i, 0')$ under this map is

$$(t^{-1} a_1, t^{-1/2} a')$$

and $\operatorname{Im}(t^{-1} a_1) - t^{-1}|a'|^2 = 1$. Thus, as in the proof of Theorem 1.7, there is a Heisenberg translation h_c^{-1} so that $h_c^{-1} \circ \delta_s \circ g$ fixes $(i, 0')$ and ∞. Moreover we must have, for some unitary operator U,

$$h_c^{-1} \circ \delta_s \circ g(w_1, w') = (w_1, Uw')$$

so that

$$\begin{aligned} g(w_1, w') &= \delta_{\sqrt{t}} \circ h_c(w_1, Uw') \\ &= (t(w_1 + c_1 + 2i\langle Uw', c'\rangle), \sqrt{t}(Uw' + c')). \end{aligned}$$

If $t = 1$ we have $g(w_1, w') = h_c(w_1, Uw')$ and an easy computation shows that g satisfies (**). Suppose that $t \neq 1$. We will show that this contradicts the hypothesis that g fixes ∞ only by producing a point in $\partial\Omega$ fixed by g.

If $t \neq 1$ we may solve $\sqrt{t}(Uw' + c') = w'$, since $(U - t^{-1/2} I)$ is nonsingular. Let v' denote the solution. If $v_1 = \alpha + i|v'|^2$ where α is real, then (v_1, v') will be in $\partial\Omega$. We claim we may choose α so that $g(v_1, v') = (v_1, v')$. By our choice of v' we have

$$g(v_1, v') = (t(\alpha + i|v'|^2 + c_1 + 2i\langle Uv', c'\rangle), v').$$

We wish to have $t(\alpha + i|v'|^2 + c_1 + 2i\langle Uv', c'\rangle) = \alpha + i|v'|^2$. Since (v_1, v') is in $\partial\Omega$ and g is an automorphism, $g(v_1, v')$ lies in $\partial\Omega$. Thus for any real α,

$$\operatorname{Im} t(\alpha + i|v'|^2 + c_1 + 2i\langle Uv', c'\rangle) = |v'|^2 = \operatorname{Im}(\alpha + i|v'|^2).$$

Thus (v_1, v') will be a fixed point of g if α is chosen in $\mathbf{R}$ to satisfy

$$\operatorname{Re} t(\alpha + i|v'|^2 + c_1 + 2i\langle Uv', c'\rangle) = \alpha = \operatorname{Re}(\alpha + i|v'|^2)$$

or

$$t\alpha + \operatorname{Re} t(c_1 + 2i\langle Uv', c'\rangle) = \alpha.$$

Since $t \neq 1$ we may solve this equation for real α. Thus the assumption that $t \neq 1$ implies that the fixed point set of g contains more than one point, contradicting the hypothesis. □

2. Maps which fix an interior point. We consider the case of $f \in H(B; B)$ fixing at least one point of B. Several remarks can be made about the sequence of iterates of f; we collect these comments together in:

THEOREM B. *Let $f \in H(B;B)$ have a fixed point in B. Then either*

(1) *There is a constant function $g(z) \equiv z_0 \in B$ in $\Gamma(f)$. In this case $f_n \to g$, and the fixed point set of f is of course precisely $\{z_0\}$, or*

(2) *There is a sequence $\{m_i\}$ such that f_{m_i} converges to a nonconstant idempotent h. The fixed point set of h is an affine subset of B which may be strictly larger than the fixed point set of f, even if f is not in* Aut *B. Moreover, if f is not an automorphism of B then every subsequential limit of $\{f_n\}$ is degenerate in the sense that its range is contained in an affine subset of B of lower dimension than B.*

Proof. Suppose there is a sequence $\{n_i\}$ such that f_{n_i} converges to a constant function g. Then clearly the fixed point set of f is precisely the range of g. We claim $f_n \to g$, for otherwise there is a sequence $\{m_i\}$ such that $f_{m_i} \to h$, where h is not a constant map. Without loss of generality $f_{m_i - n_i} \to k \in H(B;B)$. Then $f_{m_i-n_i} \circ f_{n_i} \to k \circ g$ and also $f_{m_i-n_i} \circ f_{n_i} = f_{m_i} \to h$. But $k \circ g$ is constant and h is not, which is a contradiction. This proves (1).

If there is no constant map in $\Gamma(f)$, then Proposition 1.5 shows that there is a nonconstant idempotent among the subsequential limits of $\{f_n\}$. Moreover, the proof of Proposition 1.5 shows that given any nonconstant subsequential limit G there is a subsequential limit H which is the identity map on the range of G. Thus if the affine subset of B of smallest dimension containing the range of G is all of B, then the identity map on B is a subsequential limit of $\{f_n\}$. This implies that f is an automorphism of B, by Lemma 1.9. □

For an example where the fixed point set of the limit function is strictly larger than the fixed point set of f, let g be a holomorphic function on the unit disc, with $|g| < 1$. Define f on B_2 by $f(z_1, z_2) = (-z_1, g(z_1)z_2)$. Thus $f \in H(B_2; B_2)$ and the fixed point set of f is $\{(0,0)\}$. Now $f_{2k}(z_1, z_2) = (z_1, g^k(z_1)g^k(-z_1)z_2)$ and $f_{2k} \to h$, where $h(z_1, z_2) = (z_1, 0)$.

We remark that Case (2) of Theorem B can only occur if f acts as an automorphism on some affine set in B of dimension ≥ 1.

3. Remarks on Theorems A and B. Some similar results have been obtained by Yoshisha Kubota [2], using different methods. He does not consider the fixed point free maps as a separate case, and his result does not show that in this situation the entire sequence of iterates converges to a point in ∂B.

Corollary 1.8 has also been independently obtained by David Ullrich. His argument, while similar in spirit to ours, uses the Iwasawa decomposition for $g \in \operatorname{Aut} \Omega$ as $g = \Psi \circ \delta_\lambda \circ h_b$ where h_b is a Heisenberg translation, $\delta_\lambda(w_1, w') = (\lambda^2 w_1, \lambda w')$ and Ψ is an automorphism of Ω fixing $(i, 0')$ in Ω. He shows that $\lambda = 1$ if g fixes ∞ only and that $\Psi(w_1, w') = (w_1, Uw')$ for some unitary operator U on $\mathbf{C}^{N-1}$. The remainder of the argument proceeds as before.

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