ABSOLUTE VALUES OF HYPONORMAL OPERATORS WITH ASYMMETRIC SPECTRA

C. R. Putnam

1. Let H be an infinite dimensional, separable Hilbert space. Let the bounded operator T on H, with the Cartesian representation $T=A+iB$, be completely hyponormal, so that

$$T^*T-TT^*=D, \quad \text{equivalently}, \quad AB-BA=-\tfrac{1}{2}iD, \quad D\geqslant 0, \tag{1.1}$$

and T has no nontrivial reducing subspace on which it is normal. In addition, it will henceforth be supposed that T has a polar factorization

$$T=UP, \quad U \text{ unitary and } P=|T|=(T^*T)^{1/2}. \tag{1.2}$$

Such a factorization exists (and is unique) if and only if 0 is not in the point spectrum of T^*; see [9], p. 277.

Let A be a selfadjoint operator with the spectral family $\{E_t\}$. The set of vectors x in H for which $\|E_t x\|^2$ is an absolutely continuous function of t is a subspace, $H_a(A)$, and the operator A is said to be absolutely continuous in case $H=H_a(A)$. (See, e.g., Kato [2], p. 516.) Similar concepts can be defined for a unitary operator.

If T is completely hyponormal then its real and imaginary parts are absolutely continuous; further, if T has a factorization (1.2) then U is also absolutely continuous. (See [3], p. 42; [8], p. 193.) Simple examples show, however, that the absolute value of T, that is $P=|T|=(T^*T)^{1/2}$, need not be absolutely continuous or even have an absolutely continuous part. In fact, it has recently been shown ([1]) that any nonnegative operator P for which $\sigma(P)$ contains at least two points, $0\notin\sigma_p(P)$, and for which neither $\max\sigma(P)$ nor $\min\sigma(P)$ belongs to $\sigma_p(P)$ with a finite multiplicity, is the absolute value of some completely hyponormal operator T with a factorization (1.2).

In case T is completely hyponormal and if the spectrum of $|T|=(T^*T)^{1/2}$ has Lebesgue linear measure zero (and whether or not T has a factorization (1.2)) then $\sigma(T)$ is radially symmetric. In fact, $\sigma(T)$ is the closure of a countable number, finite or infinite, of pairwise disjoint open annuli centered at the origin; see [6], p. 426. On the other hand, if T is completely hyponormal and if there exists some open wedge

$$W=\{z: z=re^{it},\ r>0,\ -\pi<a<t<b\leqslant\pi\} \tag{1.3}$$

which does not intersect $\sigma(T)$, then necessarily $|T|$ is absolutely continuous. This follows from [6], p. 424, since, in the above case where $W\cap\sigma(T)$ is empty, necessarily T has a factorization (1.2). This last assertion follows from the fact

Received July 7, 1982.
This work was supported by a National Science Foundation research grant.
Michigan Math. J. 30 (1983).

that 0 cannot belong to $\sigma_p(T^*)$; see Radjabalipour [11], p. 385, Lemma 4. If T is completely hyponormal and if T has a polar factorization (1.2) then, even if every set W of (1.3) intersects $\sigma(T)$, one may, under certain circumstances, still conclude that $|T|$ is absolutely continuous, or at least has an absolutely continuous part $(H_a(|T|)\neq\{0\})$; see [9], [10].

The above results suggest that there is a close correlation between the radial asymmetry of the spectrum of a completely hyponormal T and the existence of an absolutely continuous part for the absolute value $|T|$. It may be noted, however, that in all of the above results, the sufficiency conditions on $\sigma(T)$ which assure the existence of an absolutely continuous part of $|T|$ involve hypotheses which, for instance, preclude the existence of an open connected set contained in $\sigma(T)\cap\{z: |z|>0\}$ and separating 0 from ∞, for example, an annulus $\{z: 0\leqslant r_1<|z|<r_2\}$ lying in $\sigma(T)$. The hypothesis of the following result, on the contrary, does not preclude this possibility.

THEOREM 1. *Let T be completely hyponormal with the polar factorization* (1.2), *where*

$$U=\int_{-\pi}^{\pi} e^{it}\,dG_t \quad \text{and} \quad |T|^2=P^2=\int_0^{\infty} t\,dF_t. \tag{1.4}$$

Suppose that

$$r>0,\ r\notin\sigma(T) \quad \text{and} \quad z_1\in\sigma(T),\ z_1\neq r,\ |z_1|=r. \tag{1.5}$$

Suppose that there exists a Borel subset α of $(-\pi,\pi]$ for which

$$\int_\alpha t^{-2}\,dt<\infty \quad \text{and} \quad G(\alpha)=I. \tag{1.6}$$

Then there exists an open interval Δ containing r^2 with the property that

$$\{0\}\neq F(\Delta)H\subset H_a(P^2). \tag{1.7}$$

Obviously, the existence of an absolutely continuous part for P^2 is equivalent to the existence of such a part for P. Further, the role of the direction determined by the ray $\arg z=0$ in the hypotheses (1.5) and (1.6) above could be played by any direction determined by $\arg z=c$ $(=\text{const})$, where $-\pi<c\leqslant\pi$, and (1.5) and (1.6) are replaced by corresponding hypotheses. This is clear if one substitutes $e^{ic}T$ and $e^{ic}U$ for T and U in (1.2).

2. Proof of Theorem 1. In view of (1.1) and (1.2),

$$P^2-UP^2U^*=D\geqslant 0. \tag{2.1}$$

If

$$G_n=G((-\pi,-1/n)\cup(1/n,\pi]),\quad n=1,2,\ldots, \tag{2.2}$$

then it may be supposed that $G_n\neq 0$ for all n. (In fact, otherwise, some open wedge W of (1.3) does not intersect $\sigma(T)$, and, as noted earlier, $|T|$ is absolutely continuous.) A multiplication by G_n on the left and right of each side of (2.1) yields

$$G_nP^2G_n - UG_n(G_nP^2G_n)U^*G_n = G_nDG_n. \tag{2.3}$$

If one regards the selfadjoint operators $G_nP^2G_n$ and G_nDG_n, as well as the unitary operator $U_n = UG_n$, as acting on the space $G_nH(\neq \{0\})$, then it is clear that $1 \notin \sigma(U_n)$ and hence the Cayley transform

$$A_n = i(1+U_n)(1-U_n)^{-1}, \quad \text{where} \quad U_n = UG_n, \tag{2.4}$$

is bounded. A straightforward calculation (cf. [3], p. 16) shows that

$$A_nP_n^2 - P_n^2A_n = \tfrac{1}{2}i(A_n+i)(G_nDG_n)(A_n-i), \tag{2.5}$$

where $(A_n+i)(G_nDG_n)(A_n-i) \geqslant 0$, $P_n \geqslant 0$ and

$$P_n^2 = G_nP^2G_n = \int_0^\infty t\,dF_t^n. \tag{2.6}$$

Let $0<a<b$ and suppose that (cf. (1.5))

$$[a^{1/2}, b^{1/2}] \cap \sigma(T) \text{ is empty}, \quad r \in (a^{1/2}, b^{1/2}) \tag{2.7}$$

(that is, $r^2 \in \Delta = (a,b)$) and $a^{1/2}, b^{1/2} \notin \sigma_p(P)$ (that is, $a, b \notin \sigma_p(P^2)$).

A multiplication on the left and right of each side of (2.5) by $F^n(\Delta)$ leads to

$$\begin{aligned}(F^n(\Delta)A_nF^n(\Delta))(F^n(\Delta)P_n^2) - (F^n(\Delta)P_n^2)(F^n(\Delta)A_nF^n(\Delta))\\ = \tfrac{1}{2}iF^n(\Delta)(A_n+i)G_nDG_n(A_n-i)F^n(\Delta).\end{aligned} \tag{2.8}$$

If

$$S_n = P_n^2 + iA_n \quad \text{and} \quad S_n^\Delta = F^n(\Delta)P_n^2 + iF^n(\Delta)A_nF^n(\Delta), \tag{2.9}$$

then both S_n (as an operator on G_nH) and S_n^Δ (as an operator on $F^n(\Delta)G_nH \equiv F^n(\Delta)H$) are completely hyponormal. In fact, that the assumed complete hyponormality of $T=UP$ (on H) implies that of U_nP_n (on G_nH), where $P_n = (G_nP^2G_n)^{1/2}$, is essentially contained in the proof of Lemma 8 of [4]. The complete hyponormality of S_n^Δ (on $F^n(\Delta)H$) then follows from Lemma 5 of [4]. In addition, by Theorem 3 of [4], one has $\sigma(S_n^\Delta) \subset \sigma(S_n)$; in fact, by [5], p. 695,

$$\sigma(S_n^\Delta) \cap \{z\colon \mathrm{Re}(z) \in \Delta\} = \sigma(S_n) \cap \{z\colon \mathrm{Re}(z) \in \Delta\}. \tag{2.10}$$

Next, it will be shown that

$$\|S_n^\Delta\| \leqslant \text{const} < \infty \quad (n=1,2,\ldots), \tag{2.11}$$

where "const" is independent of n. To see this, note that if (2.11) is false then there exist points z_n of $\sigma(S_n^\Delta)$ belonging to the closure of the open strip $\{z\colon \mathrm{Re}(z) \in \Delta\}$ and satisfying $|\mathrm{Im}(z_n)| \to \infty$ as $n \to \infty$. Since each S_n^Δ is completely hyponormal then any nonempty intersection of $\sigma(S_n^\Delta)$ with an open disk has positive Lebesgue (area) measure ([4], Theorem 4) and, consequently, it can be assumed that $\mathrm{Re}(z_n)$ belongs to the open interval Δ. Further (cf. (2.10)) it can clearly be supposed that each $z_n = a_n + ib_n$ lies in the boundary of the set $\sigma(S_n)$.

Consequently, for each $n=1,2,\ldots$, there exist real $a_n \geq 0$ and b_n, where $a_n \in \Delta$ and $|b_n| \to \infty$, and a sequence of unit vectors $\{x_1^n, x_2^n, \ldots\}$ satisfying the (strong) limit relations

$$(2.12) \qquad (P_n^2 - a_n)x_k^n \to 0 \quad \text{and} \quad (A_n - b_n)x_k^n \to 0, \quad \text{as } k \to \infty,$$

for each fixed n. In view of (2.4), $U_n = (A_n - i)(A_n + i)^{-1}$ and hence

$$(2.13) \qquad [U_n - (1+c_n)]x_k^n \to 0 \quad (k \to \infty)$$

for each fixed $n=1,2,\ldots$, where $c_n \to 0$ as $n \to \infty$. By (2.12) and (2.13), it is seen that if $T_n = U_n P_n$ $(P_n = (G_n P^2 G_n)^{1/2})$ then $a_n^{1/2}(1+c_n) \in \sigma(T_n)$. But $\sigma(T_n) \subset \sigma(T)$ ([8], p. 192) and, since $a_n \in \Delta$ and $c_n \to 0$ as $n \to \infty$, it is clear that the first condition of (2.7) is violated. This completes the proof of (2.11).

Next, in view of (2.1),

$$(2.14) \qquad \|G(\delta)D^{1/2}\|^2 \leq \text{const}|\delta|,$$

where δ denotes any subinterval of $[-\pi, \pi]$; see [3], pp. 20, 22. Consequently, by (1.6), $\|(1-U)^{-1}D^{1/2}\|^2 \leq \text{const} \int_\alpha |1-e^{it}|^{-2}\,dt \leq \text{const} \int_\alpha t^{-2}\,dt < \infty$, so that

$$(2.15) \qquad AD^{1/2} \text{ is bounded,} \quad \text{where } A = i(1+U)(1-U)^{-1}.$$

Note that since U is absolutely continuous, then 1 is not in its point spectrum, so that A exists (possibly as an unbounded selfadjoint operator). Since $(AD^{1/2})^* \supset D^{1/2}A$ (cf. [12], p. 29) it is clear from (2.15) and (2.5) that $AP^2x - P^2Ax = \frac{1}{2}i(A+i)D(A-i)x$ holds for all x in D_A. Since, by (2.15), $\|(A+i)D(A-i)x\| \leq \text{const}\|x\|$ for all x in D_A, then $(A+i)D(A-i) \subset M$, where M is bounded and, consequently,

$$(2.16) \qquad AP^2x - P^2Ax = \tfrac{1}{2}iMx, \quad x \in D_A.$$

Next, it will be shown that

$$(2.17) \qquad AF(\Delta) \text{ is bounded,}$$

where $\{F_t\}$ is the spectral family of P^2 and Δ is defined in (2.7). Since A is selfadjoint, and hence closed, it is sufficient, by the closed graph theorem, to show that

$$(2.18) \qquad R_{F(\Delta)} \subset D_A.$$

In view of (2.7), one can choose an open interval $\Delta_1 \supset \Delta^- = [a, b]$ so that

$$\text{dist}(\Delta_1, \sigma(T)) > 0 \quad \text{and} \quad \text{dist}(\Delta_1^c, \Delta) = d > 0,$$

where $\Delta_1^c = (-\infty, \infty) - \Delta_1$. Then $AF^n(\Delta) = F^n(\Delta_1^c)AF^n(\Delta) + F^n(\Delta_1)AF^n(\Delta)$ and (cf. (2.9) and (2.11)) $\|F^n(\Delta_1)A_nF^n(\Delta)\| = \|F^n(\Delta_1)A_nF^n(\Delta_1)F^n(\Delta)\| \leq$ const. Also, by (2.8) and the estimate of [7], p. 196,

$$\|F^n(\Delta_1^c)A_nF^n(\Delta)\| \leq \|F^n(\Delta_1^c)\tfrac{1}{2}(A_n+i)G_nD^{1/2}D^{1/2}G_n(A_n-i)F^n(\Delta)\|/d.$$

In view of (2.15), the right side of this relation is majorized by a constant (independent of n) and so, since $AF^n(\Delta) = A_nF^n(\Delta)$, (2.19) $$\|AF^n(\Delta)\| \leqslant \text{const} < \infty.$$

On letting $n \to \infty$ and noting that $P_n^2 \to P^2$ (strongly) and that neither end point of Δ belongs to the point spectrum of P^2 (cf. (2.7)), then $F^n(\Delta) \to F(\Delta)$ strongly, a result due to Rellich; see [12], p. 56. It follows from (2.19) that if N is any fixed positive integer, and if $x \in H$, then $\|G_N AF^n(\Delta)x\| = \|AG_N F^n(\Delta)x\| \leqslant \text{const}\|x\| < \infty$, where "const" is independent of both N and n. On first letting $n \to \infty$ and $N \to \infty$, one obtains (2.18) and, as noted, also (2.17).

It now readily follows from (2.16), (2.17) and (2.18) that

(2.20) $$L(P^2F(\Delta)) - (P^2F(\Delta))L = \tfrac{1}{2}iF(\Delta)MF(\Delta), \quad L = F(\Delta)AF(\Delta),$$

where L and M are bounded. Since $M \geqslant 0$ then $F(\Delta)MF(\Delta) \geqslant 0$ and it follows from (2.20) (cf. (1.1)) that $P^2F(\Delta) + iL$ is hyponormal on $F(\Delta)H$. In order to prove that $P^2F(\Delta)$ is absolutely continuous (that is, (1.7)), it is sufficient to show that $P^2F(\Delta) + iL$ is completely hyponormal on $F(\Delta)H$ (see [3], p. 42).

To this end, let Γ be a subspace of $F(\Delta)H$ which reduces $P^2F(\Delta) + iL$, that is, is invariant under both L and $P^2F(\Delta)$, and for which $(F(\Delta)MF(\Delta))\Gamma = \{0\}$. If $x \in \Gamma$, then $x = F(\Delta)x$ and $\|M^{1/2}x\|^2 = (Mx, x) = 0$, so that $Mx = 0$. Also, by (2.17) and (2.16), $AP^2x = P^2Ax$. Since Γ is invariant under $P^2F(\Delta)$ then Γ is also invariant under P^2 and hence, again by (2.17), $AP^4x = AP^2P^2x = P^2AP^2x = P^4Ax$ and, similarly, $AP^{2n}x = P^{2n}Ax$ $(n = 0, 1, 2, \ldots)$ for all x in Γ. Thus, $Af(P^2)x = f(P^2)Ax$, where $f(t)$ denotes any polynomial. However, $Af(P^2)x = Af(P^2)F(\Delta)x = AF(\Delta)f(P^2)x$. Since $AF(\Delta)$ is bounded, one need only choose a sequence of polynomials $\{f_n(t)\}$ for which $f_n(P^2) \to F(\Delta)$ (strongly) to conclude that $Ax = AF(\Delta)x = F(\Delta)Ax$, that is, Γ is invariant under A and hence reduces U. Consequently, Γ reduces $T = UP$ and $T \mid \Gamma$ is normal. In view of the supposed complete hyponormality of T then $\Gamma = \{0\}$ and, in particular, $P^2F(\Delta) + iL$ is completely hyponormal, and hence $P^2F(\Delta)$ is absolutely continuous, as was to be shown. □

3. In this section there will be given some generalizations of Theorem 1.

THEOREM 2. *Let T be completely hyponormal with the polar factorization* (1.2) *where* (1.4) *and* (1.5) *are assumed. In addition, suppose that* (2.15) *holds. Then there exists an open interval Δ containing r^2 for which* (1.7) *holds.*

The proof of Theorem 2 follows from that of Theorem 1 if it is noted that the hypothesis (1.6) of Theorem 1 was used only to establish (2.15) in the proof of that theorem.

Somewhat more general than Theorem 2 is the following.

THEOREM 3. *Let T be completely hyponormal with the polar factorization* (1.2) *where* (1.4) *and* (1.5) *are assumed. In addition, suppose that* (2.19) *holds, where A is the Cayley transform of U, that is, $A = i(1+U)(1-U)^{-1}$, and where $\{F_t^n\}$ is the spectral family of P_n^2 of* (2.6). *Then there exists an open interval Δ containing r^2 for which* (1.7) *holds.*

Proof of Theorem 3. First, there will be proved a simple

LEMMA. *Let A be any selfadjoint operator and let $x_n \in D_A$. Suppose that, as $n \to \infty$, $x_n \to x$ (weakly) and $Ax_n \to y$ (weakly). Then $x \in D_A$ and $y = Ax$.*

Proof. Let $u \in D_A$. Then

$$(Ax_n, u) \to (y, u) \quad \text{and} \quad (Ax_n, u) = (x_n, Au) \to (x, Au),$$

so that $(x, Au) = (y, u)$ for all u in D_A. Hence $x \in D_{A^*}$ and, since $A = A^*$, $y = Ax$ and the Lemma is proved.

It will next be shown that if x is any vector in H, if A is defined as in Theorem 3, and if Δ is chosen as in (2.7), then $F(\Delta)x \in D_A$ and, as $n \to \infty$,

$$AF^n(\Delta)x \to AF(\Delta)x \text{ (weakly).} \tag{3.1}$$

To see this, note that, by (2.19), $\{AF^n(\Delta)x\}$ is a bounded sequence of vectors and hence $AF^{n_k}(\Delta)x \to \text{limit} \equiv y$ (weakly) for some subsequence $\{n_k\}$ of $\{n\}$. Hence, if u is any vector in the domain of A, then, as $k \to \infty$, $(AF^{n_k}(\Delta)x, u) \to (y, u)$, and $(AF^{n_k}(\Delta)x, u) = (F^{n_k}(\Delta)x, Au) \to (F(\Delta)x, Au)$, again using (2.7) and the Rellich result. Consequently, $F(\Delta)x \in D_A$ (using $A = A^*$) and $AF(\Delta)x = y$. This argument shows that every weakly convergent subsequence of the bounded sequence $\{AF^n(\Delta)x\}$ must have the same limit (namely, $F(\Delta)x$), and so (3.1) follows. (Incidentally, this argument gives another proof of (2.17) as a consequence of (2.19).)

Next, if $x \in H$ then an application of (2.5) to $F^n(\Delta)x$ yields

$$AF^n(\Delta)P_n^2x - P_n^2AF^n(\Delta)x = \tfrac{1}{2}i(A+i)(G_nDG_n)(A-i)F^n(\Delta)x. \tag{3.2}$$

Since $P_n^2x \to P^2x$ (strongly) then, by an argument similar to that used in proving (3.1), one sees that $AF^n(\Delta)P_n^2x \to AF(\Delta)P^2x$ (weakly). Also, since the P_n^2 are selfadjoint, it is clear that the left side of (3.2) converges weakly to

$$AF(\Delta)P^2x - P^2AF(\Delta)x.$$

Since D is selfadjoint then (3.1) implies that

$$y_n = (G_nDG_n)(A-i)F^n(\Delta)x \to y = D(A-i)F(\Delta)x \text{ (weakly),}$$

and it follows from (3.2) and the Lemma that $y \in D_A$ and that

$$(A+i)y_n \to (A+i)y \text{ (weakly).}$$

Consequently, if M denotes the operator

$$M = (A+i)D(A-i), \tag{3.3}$$

then $R_{F(\Delta)} \subset D_M$. Thus, both operators $AF(\Delta)$ and $MF(\Delta)$ are bounded and $AF(\Delta)P^2 - P^2AF(\Delta) = \frac{1}{2}iMF(\Delta)$. Hence, if $L = F(\Delta)AF(\Delta)$, then

$$L(P^2F(\Delta)) - (P^2F(\Delta))L = \tfrac{1}{2}iF(\Delta)MF(\Delta).$$

Note that this result corresponds to (2.20), but the present M of (3.3) may not, as before, have a bounded extension. The remainder of the proof is similar to that of Theorem 1 following formula line (2.20) and will be omitted. □

4. THEOREM 4. *Let T be completely hyponormal with the polar factorization* (1.2), *where* (1.4) *and* (1.5) *are assumed. As above, let $\{F_t^n\}$ be the spectral family of $P_n^2 = G_n P^2 G_n$. Suppose that Δ satisfies* (2.7) *and that the sequence of operators*

(4.1) $\{D^{1/2}(A-I)F^n(\Delta)\}$ *does not converge strongly to* 0 *as $n \to \infty$.*

Then $P^2F(\Delta)$, as an operator on $F(\Delta)H$, contains an absolutely continuous part.

Proof. As proved earlier (see (2.11)), for $n = 1, 2, \ldots,$

(4.2) $\|F^n(\Delta)AF^n(\Delta)\| \leqslant \text{const} < \infty$ ("const" independent of n).

If $L_n = F^n(\Delta)AF^n(\Delta)$ and $B_n = D^{1/2}(A-I)F^n(\Delta)$ then (2.8) becomes

(4.3) $$L_n P_n^2 F^n(\Delta) - P_n^2 F^n(\Delta) L_n = \tfrac{1}{2} i B_n{}^* B_n.$$

In view of (4.1) there exists a vector x in H and a subsequence $\{n_k\}$ of $\{n\}$ for which

(4.4) $$\|B_{n_k} x\| \geqslant \text{const} > 0 \quad (k = 1, 2, \ldots).$$

By (4.2), there exists a subsequence $\{m_k\}$ of $\{n_k\}$ for which $L_{m_k} \to \text{limit} \equiv L$ (weakly) and so, by (4.3), (2.7) and the selfadjointness of P_n^2 and $F^n(\Delta)$,

(4.5) $$LP^2F(\Delta) - P^2F(\Delta)L = \tfrac{1}{2} iQ,$$

where $B_{m_k}{}^* B_{m_k} \to Q \geqslant 0$ (weakly). Moreover, $Q \neq 0$. Otherwise, $B_{m_k}{}^* B_{m_k} \to 0$ (weakly) and so, $\|B_{m_k} x\|^2 = (B_{m_k}{}^* B_{m_k} x, x) \to 0$, in contradiction to (4.4). It now follows from (4.5) and [3], p. 42, that $P^2F(\Delta)$ has an absolutely continuous part. □

5. Remarks. We do not know whether (4.1) is actually implied by the hypotheses (1.4), (1.5) and (2.7) of Theorem 4. In any case, (4.2) must hold. In the event that (4.2) can be strengthened to (2.19), then, in fact (as a consequence of the proof of Theorem 3), (4.1) of Theorem 4 follows automatically from (1.4), (1.5) and (2.7).

In view of the above results, we offer the following

CONJECTURE. *Let T be completely hyponormal with a spectrum $\sigma(T)$ which is not radially symmetric, so that there exists some circle $\{z: |z| = r\}$ intersecting both $\sigma(T)$ and its complement in nonempty sets. In addition, suppose that T has a polar factorization* (1.2). *Then $H_a(|T|) \neq \{0\}$.*

If the conjecture is true it would follow, for instance, that if T is any completely hyponormal operator for which

(5.1) $$\sigma_p(T^*) \text{ is empty},$$

then $T + z$ has an absolutely continuous part for all complex z except possibly one.

This would follow from the facts that, by (5.1), $T+z$ has a polar factorization for all z (cf. the discussion in Section 1) and that $\sigma(T+z)$, even for any bounded T, can be radially symmetric for at most one z.

At present, we do not even have a counterexample to rule out the possibility of a sharpened form of the above conjecture obtained by omitting the assumption that T has a polar factorization (1.2). If this sharpened conjecture is true then the hypothesis (5.1) could be omitted in the remarks of the preceding paragraph. Actually, it is not hard to show that the sharpened version of the conjecture does hold if $T=V+z$, where $z\neq 0$ and V is the simple unilateral shift. In fact, $|V+z|$ is even absolutely continuous for all $z\neq 0$.

REFERENCES

1. K. F. Clancey and C. R. Putnam, *Nonnegative perturbations of selfadjoint operators,* J. Funct. Anal., to appear.
2. T. Kato, *Perturbation theory for linear operators,* Springer, New York, 1966.
3. C. R. Putnam, *Commutation properties of Hilbert space operators and related topics,* Ergebnisse der Math., 36, Springer, New York, 1967.
4. ———, *An inequality for the area of hyponormal spectra,* Math. Z. 116 (1970), 323-330.
5. ———, *A similarity between hyponormal and normal spectra,* Illinois J. Math. 16 (1972), 695-702.
6. ———, *Spectra of polar factors of hyponormal operators,* Trans. Amer. Math. Soc. 188 (1974), 419-428.
7. ———, *A norm inequality in hyponormal operator theory,* Michigan Math. J. 22 (1975), 195-200.
8. ———, *A polar area inequality for hyponormal spectra,* J. Operator Theory 4 (1980), 191-200.
9. ———, *Absolute continuity of polar factors of hyponormal operators,* Amer. J. Math., Suppl., 1981, 277-283.
10. ———, *The spectrum of the absolute value of a hyponormal operator,* Operator Theory: Advances and Applications, 4 (1982), 449-461 (Toeplitz Centennial).
11. M. Radjabalipour, *On subnormal operators,* Trans. Amer. Math. Soc. 211 (1975), 377-389.
12. B. Sz.-Nagy, *Spektraldarstellung linearer Transformationen des Hilbertschen Raumes,* Ergebnisse der Math., 39, Springer, Berlin, 1967.

Department of Mathematics
Purdue University
West Lafayette, Indiana 47907