EXAMPLES OF CELL-LIKE MAPS THAT ARE NOT SHAPE EQUIVALENCES

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1. Introduction. For the most part, cell-like maps behave in the expected fashion by being hereditary homotopy equivalences in the setting of absolute neighborhood retracts and by being hereditary shape equivalences in the general setting of metric spaces; see [7], [9], [11], [14], [15]. An exception is an example due to J. Taylor of a cell-like map F from a compactum T with nontrivial shape onto the Hilbert cube Q [16]. In this paper, we verify a suspicion that a careful analysis of this example would be beneficial by producing examples of:

(1) a cell-like map H from the compactum T onto Q such that the nondegeneracy set $N_H = \{q \in Q : H^{-1}(q) \neq \text{point}\}$ is a countable union of finite dimensional compacta;

(2) a map from a compactum onto Q whose point-inverses are finite dimensional absolute retracts and which is not a shape equivalence; and

(3) a locally contractible compactum Z which is not an ANR, compacta X and Y which are ANR's, and cell-like maps $g: X \to Z$ and $f: Z \to Y$ with the property that $N_g \cap f^{-1}(N_f) = \varnothing$.

A central feature is an analysis which includes a verification that the map F originally constructed by J. Taylor is a hereditary shape equivalence over its nondegeneracy set N_F and, of course, is a homeomorphism over $Q - N_F$.

A by-product of the techniques used to establish (3) is that if the compactum $T \subset Q$ is embedded as a Z-set, then the adjunction space $Q \cup_F Q$ is locally contractible. Moreover, the analysis in Section 7 produces a basis for $Q \cup_F Q$ consisting of contractible open sets. Since $Q \cup_F Q$ is not an ANR, as the induced map $\tilde{F}: Q \to Q \cup_F Q$ is cell-like but is not a hereditary shape equivalence ([9; Corollary 2]), Question (ANR 1) in [6] has a negative answer.

2. Preliminaries. In order to facilitate coping with the abundance of notation appearing as we simultaneously manipulate up to three inverse sequences, we adopt the following conventions, which will be adhered to scrupulously throughout the paper. The maps in an inverse sequence are denoted by lower case Greek letters. The inverse limit of an inverse sequence $\{X_n, \alpha_n\}$ is written $(X_n)_\infty$ and the compositions and induced maps are denoted, respectively, by $\alpha_{ij}: X_i \to X_j$ for $i \geqslant j$ (where $\alpha_{ii} = \text{identity}$ and $\alpha_{i,i-1} = \alpha_i$) and $\alpha_{\infty i}: (X_n)_\infty \to X_i$. Maps between inverse sequences are denoted by lower case Roman letters while the induced map between the limits is denoted by the capital of the same letter; for example, $\{g_n\}: \{X_n, \alpha_n\} \to \{Y_n, \beta_n\}$ and $G: (X_n)_\infty \to (Y_n)_\infty$.

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Normally, the kth-suspension of a compactum C might be denoted by $\Sigma^k X$ and might be defined to be the quotient space obtained from $I^k \times X$ by identifying each set $\{t\} \times X$ to be a point for $t \in \partial I^k$ (where $I = [-1,1]$ is an interval). In order to eliminate extraneous superscripting, we shall specify a positive integer r, set $I = [-1,1]^r$ equal to an r-cell, and let ΣX denote the rth-suspension of X (that is, the quotient space obtained from $I \times X$ by identifying each set $\{t\} \times X$ to a point for $t \in \partial I$). In turn, the $(k \cdot r)$th-suspension of X, which will be denoted $\Sigma^k X$, is the quotient space obtained from $I^k \times X$ by identifying each set $\{t\} \times X$ to a point for $t \in \partial I^k$.

Maintaining the notation just introduced, for a map $f: X \to Y$ between compacta, we define the $(k \cdot r)$th-suspension $\Sigma^k f: \Sigma^k X \to \Sigma^k Y$ of f as

$$\Sigma^k f([t_1, t_2, \ldots, t_k, x]) = [t_1, t_2, \ldots, t_k, f(x)],$$

where generally the brackets "[...]" are interpreted to mean "the equivalence class of ...".

We adopt below and subsequently the terminology that a *map* $f: X \to Y$ *is one-to-one over a subset* C *of* Y to specify that the restriction of f to $f^{-1}(C)$ is one-to-one.

LEMMA 2.1. *Suppose the map* $F: (X_n)_\infty \to (Y_n)_\infty$ *is induced by a map of inverse sequences* $\{f_n\}: \{X_n, \alpha_n\} \to \{Y_n, \beta_n\}$ *and suppose a subset* $C \subset (Y_n)_\infty$ *has the property that, for some* k, f_i *is one-to-one over* $\beta_{\infty i}(C)$ *whenever* $i \geq k$. *Then* F *is one-to-one over* C.

For a map $g: X \to Y$, we define sets $N_g = \{y \in Y : f^{-1}(y) \neq \text{point}\}$ and $H_g = g^{-1}(N_g)$, both of which occur in the literature under the name "nondegeneracy set of g". Both N_G and H_G are F_σ-subsets and, for X and Y compact, are σ-compacta.

A map between metric spaces $f: X \to Y$ is a *shape equivalence* provided, for each polyhedron P, $f^\#: [Y,P] \to [X,P]$ is a bijection of sets (each of $[Y,P]$ and $[X,P]$ denotes the set of homotopy classes of maps and $f^\#([\alpha]) = [\alpha \circ f^\#]$). A proper map between metric spaces $f: X \to Y$ is a *hereditary shape equivalence* provided the restriction of f is a shape equivalence between $f^{-1}(A)$ and A for each closed subset $A \subset Y$. For the most part, we shall be concerned with maps between compacta. The exceptions are in the next two sections, in which we consider the restriction, mapping H_f to N_f, of a particular map $f: X \to Y$ between compacta. Even though f is not a shape equivalence, the restriction turns out to be a hereditary shape equivalence, a property that can be detected in either of two ways. The first uses a result implicit in [9; Corollary 8] that a map restricting to yield a hereditary shape equivalence from $f^{-1}(A_i)$ to A_i, for each of countably many compact sets, also restricts to yield a hereditary shape equivalence from $f^{-1}(\bigcup A_i)$ to $\bigcup A_i$. (While the statement in [9] requires each A_i to be finite dimensional, the proof uses only the facts that each restriction $f|_{f^{-1}(A_i)}$ is a hereditary shape equivalence and that the induced map from the double mapping cylinder of each restriction $f|_{f^{-1}(A_i)}$ to A_i is a hereditary shape equivalence, a proof of which can be found in the Appendix of [10].) The second relies on [9; Corollary 4] that a hereditary shape equivalence restricts to yield a shape equivalence from $f^{-1}(A)$ to A for an arbitrary subset A. The reader is referred to [12], [9], and [5] for in-depth treatments of shape theory, while we permit ourselves the freedom to use basic and standard results from shape theory.

The next result is useful for detecting that maps between inverse sequences induce shape equivalences; the omitted proof consists of a straightforward verification of the above definition. We thank L. Husch for acquainting us with this particular information.

LEMMA 2.2. *Suppose the map F: $(X_n)_\infty \to (Y_n)_\infty$ is induced by a map of inverse sequences $\{f_n\}: \{X_n, \alpha_n\} \to \{Y_n, \beta_n\}$, where the X_n's and Y_n's are* ANR*'s, and suppose for every k there is an $i \geqslant k$ and a map $g: Y_i \to X_k$ such that the diagram*

$$\begin{array}{ccc} X_i & \xrightarrow{\alpha_{ik}} & X_k \\ {\scriptstyle F_i}\downarrow & \overset{g}{\nearrow} & \downarrow{\scriptstyle F_n} \\ Y_i & \xrightarrow[\beta_{ik}]{} & Y_i \end{array}$$

homotopy commutes; that is, $g \circ f_i$ is homotopic to α_{ik} and $f_k \circ g$ is homotopic to β_{ik}. Then F is a shape equivalence.

PROPOSITION 2.3. *Suppose $I = [-1,1]^r$ is an r-cell, X is a compact* ANR, *and $\alpha: I \times X \to X$ is a null-homotopic map, and let $\{p_n\}: \{I^n \times X, \alpha_n\} \to \{I^n, \pi_n\}$ be the map of inverse sequences determined by specifying*

(a) *$\alpha_1 = \alpha$ and, for $k \geqslant 2$, $\alpha_k(t_1, \ldots, t_k, x) = (t_1, \ldots, t_{k-1}, \alpha(t_k, x))$;*
(b) *$\pi_1(I) =$ point and, for $k \geqslant 2$, $\pi_k(t_1, \ldots, t_k) = (t_1, \ldots, t_{k-1})$; and*
(c) *$p_0(X) =$ point and, for $k \geqslant 1$, $p_k(t_1, \ldots, t_k, x) = (t_1, \ldots, t_k)$.*

Then the induced map P: $(I^n \times X)_\infty \to (I^n)_\infty$ is a hereditary shape equivalence.

Proof. Suppose that C is a closed subset of $(I^n)_\infty$. Set $C_n = \pi_{\infty n}(C)$ and use $\hat{\alpha}_n$, $\hat{\pi}_n$, $\hat{p}_n$, and $\hat{P}$ to denote various restrictions. Then the map of inverse systems $\{\hat{p}_n\}: \{C_n \times X, \hat{\alpha}_n\} \to \{C_n, \hat{\pi}_n\}$ induces the map $\hat{P}: P^{-1}(C) \to C$. We shall show that $\hat{P}$ is a shape equivalence by verifying that the hypothesis of Lemma 2.2 holds.

Specify a null-homotopy $\{h_s\}_{0 \leqslant s \leqslant 1}: I \times X \to X$ with $h_0 = \alpha$ and $h_1(I \times X) = \{x_0\}$. Given an integer k, define $g: C_{k+1} \to C_k \times X$ by specifying that

$$g((t_1, \ldots, t_{k+1})) = (t_1, \ldots, t_k, x_0).$$

We easily conclude that $\hat{p}_k \circ g = \hat{\pi}_{k+1}$. A homotopy $\{q_s\}_{0 \leqslant s \leqslant 1}: C_{k+1} \times X \to C_k \times X$ from $\hat{\alpha}_{k+1}$ to $g \circ \hat{p}_{k+1}$ is determined by setting $q_s((t_1, \ldots, t_{k+1}, x)) = (t_1, \ldots, t_k, h_s(t_{k+1}, x))$. □

REMARK. The limit $(I^n)_\infty$ specified in the preceding proposition is canonically homeomorphic to the Hilbert cube I^∞ considered as a countable product of r-cells. According to our description, I^0 is a point, say c_0, so a typical element $(q_i) \in (I^n)_\infty$ has coordinates $q_0 = c_0$, $q_1 = (t_1)$, $q_2 = (t_1, t_2), \ldots,$ $q_i = (t_1, t_2, \ldots, t_i), \ldots$. This allows us to split $(I^n)_\infty$ into a *pseudo-boundary* B which consists of points (q_i) for which there is $k \geqslant 1$ such that q_k has 1 or -1 as a coordinate (i.e., $q_k \in \partial(I^k \times \cdots \times I^0)$) and a *pseudo-interior* $s = (I^n)_\infty - B$.

3. Scrutinizing the Taylor example. The focus of this section is on an example due to J. Taylor [16] of a cell-like map which is not a shape equivalence. At the heart of this example is the existence of a map from the rth-suspension of a compact polyhedron to itself, say $\beta: \Sigma L \to L$, such that every finite composition of maps in the sequence

$$\cdots \to \Sigma^n L \xrightarrow{\Sigma^{n-1}\beta} \Sigma^{n-1} L \to \cdots \to \Sigma L \xrightarrow{\beta} L$$

is essential. Such examples have been constructed by Adams [1] and Toda [17], [18]. For our purposes, we need specifics about neither the map β nor the polyhedron L. However, we shall need to have on hand a detailed description of the example constructed by J. Taylor and such a description is contained in the theorem below. It functions here as the central technical result.

THEOREM 3.1. *Suppose $\beta: \Sigma L \to L$ is a map from the rth-suspension of a compact polyhedron to itself such that every finite composition of maps in the sequence $\{\Sigma^n L, \Sigma^n \beta\}$ is essential and suppose the maps of inverse sequences $\{g_n\}: \{I^n \times L, \alpha_n\} \to \{\Sigma^n L, \beta_n\}$ and $\{f_n\}: \{\Sigma^n L, \beta_n\} \to \{I^n, \pi_n\}$ are determined by specifying*

(a) *$g_0: L \to L$ is the identity map and, for $k \geqslant 1$, $g_k: I^k \times L \to \Sigma^k L$ is the natural quotient map;*
(b) *$\alpha_1 = \beta \circ g_1$ and, for $k \geqslant 2$, $\alpha_k((t_1, \ldots, t_k, x)) = (t_1, \ldots, t_{k-1}, \alpha_1(t_k, x))$;*
(c) *$\beta_1 = \beta$ and, for $k \geqslant 2$, $\beta_k = \Sigma^k \beta$, that is,*

$$\beta_k([(t_1, \ldots, t_k, x)]) = [(t_1, \ldots, t_{k-1}, \beta(t_k, x))];$$

(d) *$\pi_1(I) =$ point and, for $k \geqslant 2$, $\pi_k(t_1, \ldots, t_k) = (t_1, \ldots, t_{k-1})$; and*
(e) *$f_0(L) =$ point and, for $k \geqslant 1$, $f_k([t_1, \ldots, t_k, x)] = (t_1, \ldots, t_k)$.*

Then the induced maps G: $(I^n \times L)_\infty \to (\Sigma^n L)_\infty$ and F: $(\Sigma^n L)_\infty \to (I^n)_\infty$ satisfy:

(1) *both G and F are cell-like but neither is a shape equivalence;*
(2) *the composition $F \circ G$ is a hereditary shape equivalence;*
(3) *G restricts to a hereditary shape equivalence from H_G to N_G;*
(4) *F restricts to a hereditary shape equivalence from H_F to N_F;*
(5) *$N_G \cap H_F = \varnothing$; in fact, $N_G \subset F^{-1}(B)$ while $H_F \subset F^{-1}(s)$, where B and s are the pseudo-boundary and pseudo-interior of $(I^n)_\infty$, respectively;*
(6) *G restricts to a hereditary shape equivalence from $G^{-1}((\Sigma^n L)_\infty - H_F)$ to $(\Sigma^n L)_\infty - H_F$;*
(7) *F restricts to a hereditary shape equivalence from $F^{-1}((I^n)_\infty - F(N_G))$ to $(I^n)_\infty - F(N_G)$.*

Proof. Since the restriction of α_1 to $\{t\} \times L$ is a constant map for $t \in \partial I$, we can conclude

(*) the map $\alpha_1: I \times L \to L$ is null-homotopic.

It should be evident that each f_n is one-to-one over ∂I^n and that each g_n is one-to-one over $f_n^{-1}((I^n - \partial I^n) \times L)$. Lemma 2.1 can be used to conclude that

(**) F is one-to-one over B and G is one-to-one over $F^{-1}(s)$.

An immediate consequence of Property (**) is that Condition (5) holds. The composition $F \circ G$ is induced by the map of inverse sequences

$$\{f_n \circ g_n\} : \{I^n \times L, \alpha_n\} \to \{I^n, \pi_n\}$$

and, in the presence of Property (*), Proposition 2.3 applies and establishes Condition (2).

Observe that, for any subset $K \subset (\Sigma^n L)_\infty - H_F$, in the commutative diagram

$$\begin{array}{ccccc} & & G^{-1}(K) & & \\ & \swarrow^{G|_{G^{-1}(K)}} & & \searrow^{F \circ G|_{G^{-1}(K)}} & \\ K & & \xrightarrow[F|_K]{\hspace{3em}} & & F(K) \end{array}$$

the restriction $F|_K$ is a homeomorphism and, consequently, Condition (6) follows from Condition (2). Similarly, Condition (7) follows from Condition (2). Conditions (3) and (4) follow from Conditions (6) and (7), respectively. □

The shape equivalence $F \circ G$ establishes that $(I^n \times L)_\infty$ and $(I^n)_\infty$ have the same shape and, since the latter is contractible, both have the shape of a point. The assumption that finite compositions of suspensions of the map β are essential assures that each $\beta_{\infty i} : (\Sigma^n L)_\infty \to \Sigma^i L$ represents a nonzero element of $[(\Sigma^n L)_\infty, \Sigma^i L]$ and, consequently, $(\Sigma^n L)_\infty$ does not have trivial shape. As a result, while both G and F are cell-like, neither is a shape equivalence.

REMARK. The map $F: (\Sigma^n L)_\infty \to (I^n)_\infty$ is exactly the example constructed by J. Taylor in [16] and has the pleasant feature that its image is the Hilbert cube. No doubt, the above proof obscures the reason that F is cell-like. In Section 5, we shall analyse the individual point-inverses of F and, in doing so, reproduce the simple argument from [16] establishing that F is cell-like.

4. Example with countable dimensional nondegeneracy set. Kozlowski in [9] and Mogilski and Roslaniec in [14] have investigated conditions on the nondegeneracy set of a cell-like map that force the map to be a hereditary shape equivalence. The properties of the examples described in Theorem 3.1 impose severe limitations on extending their results. The example presented in the next theorem places further restrictions and answers negatively Question (sc2) posed in [6]. The discussion in Section 2 points out that any cell-like mapping D between compacta for which N_D is a countable union of finite dimensional compacta restricts to a hereditary shape equivalence from H_D to N_D.

THEOREM 4.1. *There is a cell-like map D from a compactum onto the Hilbert cube which is not a shape equivalence and for which N_D is the countable union of finite dimensional compact subsets.*

The example is constructed by composing the map $F\colon (\Sigma^n L)_\infty \to (I^n)_\infty$ with a map of the type described next.

PROPOSITION 4.2. *Suppose that A is a σ-compact subset of the pseudo-interior s of the Hilbert cube $[-1,1]^\infty$. Then there is a map $C\colon [-1,1]^\infty \to [-1,1]^\infty$ such that each point inverse of C is convex, $H_C \subset s$, $A \subset H_C$, and N_C is the countable union of finite dimensional compacta.*

Proof. For a number $0<\epsilon<1$, we shall write $Q[\epsilon]$ to denote the subset $[-1+\epsilon, 1-\epsilon]^\infty$ of $[-1,1]^\infty$ and $(t_1,\ldots,t_m)\times Q[\epsilon]$ to denote the subset

$$(t_1,\ldots,t_m)\times[-1+\epsilon,1-\epsilon]\times[-1+\epsilon,1-\epsilon]\times\cdots$$

of $[-1,1]^\infty$. Choose numbers $1>\epsilon_1>\epsilon_2>\cdots>0$ and a homeomorphism $h\colon [-1,1]^\infty \to [-1,1]^\infty$ with $h(s)=s$ so that $h(A)\subset \bigcup Q[\epsilon_i]$. The map C is the composition of h with the map associated with the upper semicontinuous decomposition G of $[-1,1]^\infty$ which we describe next.

Associate to each point $(t_n)\in Q[\epsilon_1]$ the subset $R((t_n))=Q[\epsilon_1]$. For $i\geqslant 2$, associate to a point $(t_n)\in Q[\epsilon_i]-Q[\epsilon_{i-1}]$ the subset $R((t_n))=(t_1,\ldots,t_m)\times Q[\epsilon_i]$, where m is chosen to be minimal with respect to

(a) $m\geqslant i$,
(b) $R((t_n))\cap Q[\epsilon_{i-1}]=\varnothing$, and
(c) $\operatorname{diam} R((t_n))\leqslant d(R((t_n)),Q[\epsilon_{i-1}])$.

Consequences of these conditions are

(1) $R((t_n))\cap R((s_n))\neq\varnothing$ if and only if $R((t_n))=R((s_n))$,
(2) if a sequence $(t_n)_j$ of points in $Q[\epsilon_i]-Q[\epsilon_{i-1}]$ converges to a point of $Q[\epsilon_{i-1}]$, then the diameters of the sets $R((t_n)_j)$ converge to zero,
(3) for a sequence of points $(t_n)_j\in Q[\epsilon_j]-Q[\epsilon_{j-1}]$, the diameters of the sets $R((t_n)_j)$ converge to zero, and
(4) for each point $(t_n)\in Q[\epsilon_i]-Q[\epsilon_{i-1}]$, there is an integer M and a neighborhood U of (t_n) such that if $(s_n)\in U\cap(Q[\epsilon_i]-Q[\epsilon_{i-1}])$ and $R((s_n))=(s_1,\ldots,s_r)\times Q[\epsilon_i]$ while $R((t_n))=(t_1,\ldots,t_m)\times Q[\epsilon_i]$, then $m\leqslant r\leqslant M$.

It follows from these four conditions that the decomposition G whose elements are the sets $R((t_n))$ for $(t_n)\in\bigcup Q[\epsilon_i]$ and singletons from $[-1,1]^\infty-\bigcup Q[\epsilon_i]$ is upper semicontinuous. Let $C'\colon [-1,1]^\infty\to[-1,1]^\infty/G$ be the associated map and set $C=C'\circ h$.

Clearly, the point inverses of C are convex and $A\subset\bigcup Q[\epsilon_i]\subset H_C\subset s$. For a neighborhood U as in Condition (4), we can write $C(U\cap(Q[\epsilon_i]-Q[\epsilon_{i-1}]))=F_m\cup\cdots\cup F_M$ where the set F_r consists of those points $C((s_n))$ for which $R((s_n))=(s_1,\ldots,s_r)\times Q[\epsilon_i]$. Since F_r naturally embeds in the r-cell $[-1,1]^r$, $\dim F_r\leqslant r$ and the Sum Theorem [8, p. 30] yields that

$$\dim C(U\cap(Q[\epsilon_i]-Q[\epsilon_{i-1}]))<\infty$$

(a more careful analysis shows that $\dim C(U\cap(Q[\epsilon_i]-Q[\epsilon_{i-1}]))\leqslant M$). We conclude that $C(\bigcup Q[\epsilon_i])$, which is equal to $C(H_C)$, is a countable union of finite dimensional compacta.

It remains to verify that $[-1,1]^\infty/G$ is homeomorphic to the Hilbert cube. A possible approach would be to show directly that the decomposition G is shrinkable. We choose to observe that the convexity of the elements of G permits easy verification of the hypothesis of Theorem 5 in [9] whose conclusion is that $[-1,1]^\infty/G$ is an ANR, and, in turn, to appeal to Theorem 3 in [19] to conclude that $[-1,1]^\infty/G \simeq [-1,1]^\infty$. □

Proof of Theorem 4.1. Let $F\colon (\Sigma^n L)_\infty \to (I^n)_\infty$ be as in Theorem 3.1 and let $C\colon (I^n)_\infty \to (I^n)_\infty$ be as just described with $N_F \subset H_C$. With regard to the map $D = C \circ F$, we have $N_D \subset N_C$ and, consequently, N_D is a countable union of finite dimensional compacta while D is not a shape equivalence since $(\Sigma^n L)_\infty$ has nontrivial shape. However, D is a cell-like map since either $y \notin N_C$ and $D^{-1}(y) = F^{-1}(C^{-1}(y))$ is a cell-like set as F is a cell-like map or $y \in N_C$ and $D^{-1}(y) = F^{-1}(C^{-1}(y))$ is a cell-like set as $C^{-1}(y) \subset s$ is a cell-like set and Conditions (5) and (7) of Theorem 3.1 yield that $F|_{F^{-1}(C^{-1}(y))}$ is a (hereditary) shape equivalence. □

5. An AR-map that is not a shape equivalence. We shall call a proper map each of whose point-inverses is an absolute retract an AR-*map*. In [13], van Mill solved a problem posed by Borsuk [3] by exhibiting an AR-map between compacta which is not a shape equivalence. The original approach of van Mill relied on specifics about the example constructed by J. Taylor (see Section 3) while its extension in [10] showed that any cell-like map which is not a hereditary shape equivalence gives rise to an AR-map which is not a hereditary shape equivalence. Neither approach produced such an AR-map having finite dimensional point-inverses, a feature which seemed plausible since the point-inverses of the map $F\colon (\Sigma^n L)_\infty \to (I^n)_\infty$ have dimension $\leqslant \dim L$. The example which we shall construct realizes this expectation for it is an AR-map which is not a hereditary shape equivalence and whose point-inverses have dimension $\leqslant \dim L + 1$.

Let $\{X_n, \sigma_n\}$ be an inverse sequence of compacta and define

$$\sigma_n^*\colon X_n \times \{n-1\} \to X_{n-1} \times \{n-1\}$$

by requiring that $\sigma_n^*(x, n-1) = (\sigma_n(x), n-1)$. Set $T_0 = X_0 \times \{0\}$; set

$$T_1 = (X_1 \times [0,1]) \cup_{\sigma_1^*} T_0$$

and define $\tau_1\colon T_1 \to T_0$ be requiring that τ_1 equals the identity on T_0 and that $\tau_1(x,s) = (\sigma_1(x), 0)$ for $0 \leqslant s \leqslant 1$; recursively set

$$T_n = (X_n \times [n-1, n]) \cup_{\sigma_n^*} T_{n-1}$$

and define $\tau_n\colon T_n \to T_{n-1}$ by requiring that τ_n equals the identity on T_{n-1} and that $\tau_n(x,s) = (\sigma_n(x), n-1)$ for $n-1 \leqslant s \leqslant n$. ($T_n$ is the compactum obtained from T_{n-1} by attaching the mapping cylinder of the map σ_n to T_{n-1} along the common subspace X_{n-1}, and the map τ_n is the strong deformation retraction obtained by "pushing down" the mapping cylinder.)

The limit of the inverse sequence $\{T_n, \tau_n\}$ is denoted $\mathfrak{Z}\mathrm{el}(\{X_n\})$ and is an "infinite mapping cylinder" with a copy of $(X_n)_\infty$ attached at its end. More precisely, there is a natural projection $p: \mathfrak{T}\mathrm{el}(\{X_n\}) \to [0,\infty]$ and canonical identifications $p^{-1}(n) \cong X_n \times \{n\}$, $p^{-1}([n-1,n]) \cong X_n \times [n-1,n]$, and $p^{-1}(\infty) \cong (X_n)_\infty$. These identifications "name" the points of $\mathfrak{T}\mathrm{el}(\{X_n\})$; that is, (x,s) is considered an element of $\mathfrak{T}\mathrm{el}(\{X_n\})$ provided either $s=\infty$ and $x \in (X_n)_\infty$ or $s \in [n-1,n]$ and $x \in X_n$.

We record for later use standard facts, proofs of which can be extracted from the development of the theory of ANR's found in [3].

LEMMA 5.1. *Let $\{X_n, \sigma_n\}$ be an inverse sequence of compact* ANR*'s.*
(i) $\mathfrak{T}\mathrm{el}(\{X_n\})$ *is an* ANR.
(ii) $\mathfrak{T}\mathrm{el}(\{X_n\}) \cup_{X_0 \times \{0\}} \mathrm{Cone}(X_0)$ *is an* AR.

We shall also need the following result, which is implicitly contained in [10].

PROPOSITION 5.2. *Let $F: X \to Y$ be a cell-like map between compacta, let W be a compactum containing X, let $\tilde{F}: W \to Y$ be a hereditary shape equivalence extending F, and let e denote the induced map from W to the adjunction space $W \cup_F Y$. Then the map $\tilde{F} \circ e^{-1}: W \cup_F Y \to Y$ is cell-like, and it is a hereditary shape equivalence if and only if F is a hereditary shape equivalence.*

THE EXAMPLE. The starting point is the map $F: (\Sigma^n L)_\infty \to (I^n)_\infty$ constructed in Section 3 and the following description of its point-inverses.

For a point $(q_n) \in (I^n)_\infty$, $F^{-1}((q_n))$ is the limit of the inverse sequence $\{g_n(q_n \times L), \beta_n'\}$ where β_n' is the restriction of β_n. The set $g_n(q_n \times L)$ is either a point or a copy of L; consequently, $\dim F^{-1}((q_n)) \leqslant \dim L$ and

$$\dim \mathfrak{T}\mathrm{el}(\{g_n(q_n \times L)\}) \leqslant \dim L + 1.$$

Since β_n' is essentially the map β restricted to $g_1(t_1 \times L)$, where $q_n = (t_1, \ldots, t_n)$, and this restriction is null-homotopic, we conclude that β_n' is null-homotopic and we have fulfilled an earlier promise to reproduce the original argument in [16] establishing that F is cell-like.

Specify a set-valued function R from $(I^n)_\infty$ to the compact subsets of

$$\mathfrak{T}\mathrm{el}(\{\Sigma^n L\}) \cup_{L \times \{0\}} \mathrm{Cone}(L)$$

by requiring that

$$R((q_n)) = \mathfrak{T}\mathrm{el}(\{g_n(q_n \times L)\}) \cup_{L \times \{0\}} \mathrm{Cone}(L)$$

for $(q_n) \in (I^n)_\infty$. It is easily checked that R is upper semicontinuous. Denote the graph of R by W; that is,

$$W = \{R((q_n)) \times \{(q_n)\} : (q_n) \in (I^n)_\infty\}.$$

Identify $(\Sigma^n L)_\infty$ with the subspace

$$\{(x, F(x)) : x \in (\Sigma^n L)_\infty\} \subset W$$

(that is, the graph of F) and define $\tilde{F}: W \to (I^n)_\infty$ by setting

$$\tilde{F}(R((q_n)) \times \{(q_n)\}) = (q_n).$$

The map $\tilde{F}$ is an AR-map (see Lemma 5.1) extending F and is a hereditary shape equivalence since, identifying $(I^n)_\infty$ with $(I^n)_\infty \times \{\text{cone point of } C(L)\}$, the "natural push" down the "infinite mapping cylinder" structure establishes that $\tilde{F}$ is a strong deformation retraction. Denoting the induced map from W to the adjunction space $W \cup_F (I^n)_\infty$ by e, Proposition 5.2 applies to show that the AR-map $\tilde{F} \circ e^{-1}: W \cup_F (I^n)_\infty \to (I^n)_\infty$ is not a hereditary shape equivalence, and evidently the point-inverses of $\tilde{F} \circ e^{-1}$ have dimension $\leqslant \dim L + 1$.

A careful inspection of the proof of Theorem 1.1 in [10] reveals that $W \cup_F (I^n)_\infty$ has the same shape as the double mapping cylinder of F, which we denote by $\mathrm{DM}(F)$ and which is the quotient space obtained from $(\Sigma^n L)_\infty \times [-1,1]$ by identifying each of the sets $F^{-1}((q_n)) \times \{-1\}$ and $F^{-1}((q_n)) \times \{1\}$ to points for $(q_n) \in (I^n)_\infty$. The map from $\mathrm{DM}(F)$ to the first suspension of $(\Sigma^n L)_\infty$, obtained by identifying the copies of $(I^n)_\infty$ at the -1 and 1 levels to points, is a hereditary shape equivalence (by virtue of having only two nondegenerate point-inverses), revealing that $\mathrm{DM}(F)$ has the same shape as the first suspension of $(\Sigma^n L)_\infty$. This suspension has nontrivial shape since the rth-suspension $\Sigma(\Sigma^n L)_\infty$ has nontrivial shape, being canonically homeomorphic to $(\Sigma^n L)_\infty$. We conclude that $W \cup_F (I^n)_\infty$ has nontrivial shape, establishing:

THEOREM 5.3. *There is an* AR-*map f from a compactum X onto the Hilbert cube Q which is not a shape equivalence and for which*

$$\sup\{\dim f^{-1}(q) : q \in Q\} < \infty.$$

6. A locally contractible compactum which is not an ANR. A finite dimensional compactum is an ANR if and only if it is locally contractible; currently there is no comparable characterization known for infinite dimensional ANR's. An example due to Borsuk [3] established that local contractibility does not suffice, for it is a locally contractible compactum which has nonzero homology in every dimension and, consequently, cannot be an ANR. We present another example of a locally contractible compactum which is not an ANR: it possesses properties not possible in Borsuk's example.

THEOREM 6.1. *There are compacta and maps*

$$X \xrightarrow{g} Z \xrightarrow{f} Y$$

satisfying:

(i) *g and f are cell-like and $N_g \cap H_f = \varnothing$;*
(ii) *X and Y are* ANR*'s but Z is not an* ANR;
(iii) *Z is locally contractible.*

The spaces, maps, and Conclusion (i). Starting with the spaces and maps

$$(I^n \times L)_\infty \xrightarrow{G} (\Sigma^n L)_\infty \xrightarrow{F} (I^n)_\infty$$

described in Section 3, we obtain

(†) $$\mathfrak{Z}\,\mathrm{el}(\{I^n \times L\}) \xrightarrow{\bar G} \mathfrak{Z}\,\mathrm{el}(\{I^n \times L\}) \cup_G (\Sigma^n L)_\infty \xrightarrow{\bar F} \mathfrak{Z}\,\mathrm{el}(\{I^n \times L\}) \cup_{F\circ G} (I^n)_\infty,$$

where $\bar G$ agrees with G over $(\Sigma^n L)_\infty$ and is one-to-one over the complement of $(\Sigma^n L)_\infty$, and where $\bar F$ agrees with F over $(I^n)_\infty$ and is one-to-one over the complement of $(I^n)_\infty$. We shall show that the spaces and maps of (†) satisfy the conclusion of Theorem 6.1. Condition (i) is an immediate consequence of F and G being cell-like (see Condition (1) of Theorem 3.1) and $N_G \cap H_F$ being empty (see Condition (5) of Theorem 3.1).

Conclusion (ii). Lemma 5.1 states that $\mathfrak{Z}\,\mathrm{el}(\{I^n \times L\})$ is an ANR. In order to proceed, we appeal to [9; Corollary 2] which states that the image of a cell-like map defined on an ANR is an ANR if and only if the cell-like map is a hereditary shape equivalence. The composition $\bar F \circ \bar G$ is a hereditary shape equivalence since it agrees with the hereditary shape equivalence $F \circ G$ over $(I^n)_\infty$ and $(I^n)_\infty$ is a closed subset containing $N_{\bar F \circ \bar G}$ (see [9; Lemma 8]). Consequently, $\mathfrak{Z}\,\mathrm{el}(\{I^n \times L\}) \cup_{F\circ G} (I^n)_\infty$ is an ANR. Conversely, since $\bar G$ agrees with G over $(\Sigma^n L)_\infty$ and G is not a shape equivalence, $\mathfrak{Z}\,\mathrm{el}(\{I^n \times L\}) \cup_G (\Sigma^n L)_\infty$ is not an ANR.

Conclusion (iii). We shall find it convenient to make the additional assumption that the map $\beta: \Sigma L \to L$, which forms the heart of the constructions in Section 3, be "pointed"; that is, there is a point $w \in L$ for which $\beta(\Sigma w) = w$, where $\Sigma w \subset \Sigma L$ is an r-cell, being the rth-suspension of a point. For now we view $(I^n)_\infty$ as a subset of $(I^n \times L)_\infty$ (respectively, $(\Sigma^n L)_\infty$) by identifying a point $(q_n) \in (I^n)_\infty$, where $q_0 = c_0$, $q_1 = (t_1), \ldots, q_n = (t_1, t_2, \ldots, t_n), \ldots$, with the point $(q_n') \in (I^n \times L)_\infty$ (respectively, $([q_n']) \in (\Sigma^n L)_\infty$) determined by setting $q_0' = w$, $q_1' = (t_1, w), \ldots, q_n' = (t_1, t_2, \ldots, t_n, w), \ldots$. With these identifications, both $F \circ G$ and F are retractions. There is an induced extension of $F \circ G$ (respectively, F) to a retraction of $\mathfrak{Z}\,\mathrm{el}(\{I^n \times L\})$ (respectively, $\mathfrak{Z}\,\mathrm{el}(\{I^n \times L\} \cup_G (\Sigma^n L)_\infty)$ onto $\mathfrak{Z}\,\mathrm{el}(\{I^n\})$; recalling that the s-coordinate parameterizes the "infinite mapping cylinder", for $n \geqslant s > n-1$ we see that a point $((t_1, \ldots, t_n, l), s)$ (respectively, $([(t_1, \ldots, t_n, l)], s))$ is sent to the point $((t_1, \ldots, t_n), s)$. The extension is denoted by $\mathfrak{Z}(F \circ G)$ (respectively, $\mathfrak{Z}(F)$).

For the most part, the space

$$\mathfrak{Z}\,\mathrm{el}(\{I^n \times L\}) \cup_G (\Sigma^n L)_\infty$$

is locally contractible since it contains a "large" open set which is an ANR; specifically, the restriction of $\bar G$ yields a homeomorphism between

$$\mathfrak{Z}\,\mathrm{el}(\{I^n \times L\}) - \mathrm{Cl}(H_{\bar G})$$

and

$$[\mathfrak{Z}\,\mathrm{el}(\{I^n \times L\}) \cup_G (\Sigma^n L)_\infty] - (I^n)_\infty.$$

Consequently, it remains to exhibit local contractibility at points of $(I^n)_\infty$.

For an integer k and an open set $U \subset I$ we set $U(k)$ equal to the open subset of $\mathfrak{Z}\,\mathrm{el}(\{I^n\})$ consisting of points whose kth-coordinate lies in U and whose "mapping cylinder" coordinate is $> k-1$; specifically,

$$U(k)=\tau_{\infty k}^{-1}((I_1\times\cdots\times I_{k-1}\times U)\times[k,k-1)),$$

where $\mathfrak{Z}\mathrm{el}(\{I^n\})$ is the limit of the inverse sequence $\{T_n,\tau_n\}$ as described in Section 5.

An integer k, an open set $U\subset I$, and a map $h\colon U\to U$ with $h(U\cap\partial I)\subset U\cap\partial I$ determine a map

$$\mathfrak{Z}(h)\colon\mathfrak{Z}(F)^{-1}(U(k))\to\mathfrak{Z}(F)^{-1}(U(k))$$

by setting

$$\mathfrak{Z}(h)((t_1,\ldots,t_n,l),s))=((t_1,\ldots,t_{k-1},h(t_k),t_{k+1},\ldots,t_n,l),s)$$

for points not in $(\Sigma^nL)_\infty$, where $n\geqslant s>n-1$, and by setting

$$\mathfrak{Z}(h)(([q_n]))=([q_n'])$$

for points in $(\Sigma^nL)_\infty$, where, for $q_n=(t_1,\ldots,t_n,l_n)$,

$$q_n'=(t_1,\ldots,t_{k-1},h(t_k),t_{k+1},\ldots,t_n,l_n)$$

for $n\geqslant k$ and recursively, $q_{k-1}'=\alpha_k(q_k')$, $q_{k-2}'=\alpha_{k-1}(q_{k-1}'),\ldots,$ $q_0'=\alpha_1(q_1')$.

The map $\mathfrak{Z}(h)$ leaves the subset $(I^n)_\infty\cap[\mathfrak{Z}(F)^{-1}(U(k))]$ invariant and, since it changes only kth-coordinates of points in this subspace, the distance it moves these points is limited by the choice of k. In contrast, $\mathfrak{Z}(h)$ may move points of $(\Sigma^nL)_\infty$ a large distance regardless of the choice of k.

Guided by the next result we will shortly complete the verification that $\mathfrak{Z}\mathrm{el}(\{I^n\times L\})\cup_G(\Sigma^nL)_\infty$ is locally contractible at points of $(I^n)_\infty$.

LEMMA 6.2. *Suppose that a point x in a metric space X has the property that, for each $\epsilon>0$, there is an open set U containing x and a homotopy $\{h_t\colon U\to X\}_{0\leqslant t\leqslant 1}$ with $h_0=\mathrm{Id}_U$, $h_1(U)$ contained in an* ANR, *and*

$$\mathrm{Diam}\{h_t(x):0\leqslant t\leqslant 1\}<\epsilon.$$

Then X is locally contractible at x.

Proof. Choose a neighborhood V of x so that

$$\mathrm{Diam}\{h_t(y):y\in V\text{ and }0\leqslant t\leqslant 1\}<\epsilon$$

and so that $h_1(V)$ is contractible in the ϵ-neighborhood of $h_1(x)$. Then V is contractible in the 2ϵ-neighborhood of x.

Given a point $(q_n)\in(I^n)_\infty\subset(\Sigma^nL)_\infty$ and $\epsilon>0$, choose k so that points in $(I^n)_\infty$ differing only in their kth-coordinates are ϵ-close. Let $U=I-\{q\}$ where q is any point in $\partial I-\{q_k\}$ and let $\{h_t\colon U\to U\}$ be any homotopy satisfying $h_t(U\cap\partial I)\subset U\cap\partial I$ for $0\leqslant t\leqslant 1$, $h_0=\mathrm{Id}_U$, and $h_1(U)\subset\partial I$. The open set $\mathfrak{Z}(F)^{-1}(U(k))$ and induced homotopy $\{\mathfrak{Z}(h_t)\}$ (as described previously) satisfy the hypothesis of the Lemma 6.2 for the point (q_n), since the image of $\mathfrak{Z}(h_1)$ is contained in the ANR $\mathfrak{Z}\mathrm{el}(\{I^n\})\subset\mathfrak{Z}\mathrm{el}(\{I^n\times L\})\cup_G(\Sigma^nL)_\infty$. □

7. A basis of contractible open sets for $Q\cup_F Q$. In this section, we provide a negative answer to Question (ANR 1) in [6] by producing a compactum that has a basis of contractible open sets but is not an ANR. The space Z in Theorem 6.1 also has this property, the argument establishing this being slightly more complicated than that given below.

THEOREM 7.1. *If* $(\Sigma^n L)_\infty \subset Q$ *is embedded as a Z-set, then the adjunction space* $Q \cup_F Q$ *has a basis of contractible open sets but* $Q \cup_F Q$ *is not an* ANR.

Proof. First, we appeal to [9; Corollary 2] in order to conclude that $Q \cup_F Q$ is not an ANR (for any embedding $(\Sigma^n L)_\infty \subset Q$).

The Z-set unknotting result of R. D. Anderson [2] permits us to work with a particular Z-embedding of $(\Sigma^n L)_\infty$ into a Hilbert cube, namely, its natural embedding into the Hilbert cube that is the quotient space obtained from

$$[\mathfrak{Z}\mathrm{el}(\{\Sigma^n L\}) \cup_{L \times \{0\}} \mathrm{Cone}(L)] \times Q$$

by identifying each of the sets $\{x\} \times Q$ to a point for $x \in (\Sigma^n L)_\infty$. The quotient space is an AR and is a Q-manifold except possibly at points of the Z-set $(\Sigma^n L)_\infty$ (we are identifying $(\Sigma^n L)_\infty$ with its image) and, consequently, is a Q-manifold. Being contractible, it is homeomorphic to the Hilbert cube. The reader is referred to [4] and [19] for further details.

The adjunction space $Q \cup_F Q$ is a Q-manifold except at points of $Q = F((\Sigma^n L)_\infty)$ and, evidently, with the structure just described points of $F((\Sigma^n L)_\infty)$ have "small" contractible open neighborhoods provided they have such neighborhoods in the subspace

$$\mathfrak{Z}\mathrm{el}(\{\Sigma^n L\}) \cup_F Q.$$

Before establishing the latter, we introduce additional notation.

The points of $\mathfrak{Z}\mathrm{el}(\{\Sigma^n L\}) \cup_F Q$ split into types.

First Type. Points $(q_n) \in Q$ where $q_0 = c_0$, $q_1 = (c_0, t_1)$, $q_2 = (c_0, t_1, t_2), \ldots$, $q_n = (c_0, t_1, \ldots, t_n), \ldots$; see the remark following Proposition 2.3.

Second Type. These are points not in Q and these have the form

$$([(t_1, \ldots, t_n), l], s) \quad \text{where} \quad n-1 < s \leqslant n,$$

the brackets $[\ldots]$ denoting equivalence classes determined by the relation $(t_1, \ldots, t_n, l) \sim (t_1, \ldots, t_n, l')$ whenever $(t_1, \ldots, t_n) \in \partial I^n$.

As a result of mapping cylinder identifications, certain points of the Second Type have another "name"; specifically, $([t_1, \ldots, t_{n+1}, l], n)$ is identified with the point of the Second Type $([t_1, \ldots, t_n, \beta(t_{n+1}, l)], n)$. This observation assures the "well-definedness" of maps described subsequently.

For an integer $k > 1$ and a subset $C \subset I^k$, we name a set $C(k)$ that consists of points of the First Type for which $(t_1, \ldots, t_k) \in C$ and points of the Second Type for which $(t_1, \ldots, t_k) \in C$ and $s > k-1$.

It is not hard to show that $U(k)$ is an open subset of

$$\mathfrak{Z}\mathrm{el}(\{\Sigma^n L\}) \cup_F Q$$

whenever U is an open subset of I^k and that sets of this form can be used to form neighborhood bases for points of Q. In fact, the latter remains true if we restrict to those open subsets U satisfying:

(a) $U \cap (I^{k-1} \times \partial I)$ is contractible and is a strong deformation retract of U; and

(b) the strong deformation retraction $\{H_r: U \to U\}$ can be chosen so that $H_r(t_1,\dots,t_k) = (t_1,\dots,t_k) = (t_1,\dots,t_{k-1},h_r(t_k))$ and $H_r(U \cap \partial I^k) \subset U \cap \partial I^k$. For example, sets of the form $V \times [-1,1)$ and $V \times (-1,1]$ for V a contractible open subset of I^{k-1} are sufficient.

The final step is to show that $U(k)$ is contractible whenever U satisfies (a) and (b). A strong deformation retraction $\{H_r: U \to U\}$ as in (b) induces a strong deformation retraction $\{\mathfrak{I}(H_r): U(k) \to U(k)\}$ of $U(k)$ to $(U \cap (I^{k-1} \times \partial I))(k)$ by setting

$$\mathfrak{I}(H_r)((q_n)) = (q_n')$$

for points of the First Type where $q_n = q_n'$ for $n \leqslant k-1$ and

$$q_n' = (c_0, t_1, \dots, t_{k-1}, h_r(t_k), t_{k+1}, \dots, t_n)$$

for $n \geqslant k$ and by setting

$$\mathfrak{I}(H_r)([t_1,\dots,t_n,l],s) = ([t_1,\dots,t_{k-1},h_r(t_k),t_{k+1},\dots,t_n,l],s)$$

for points of the Second Type. The contraction is completed by observing that $(U \cap (I^{k-1} \times \partial I))(k)$ deforms down the "mapping cylinder lines" to a set homeomorphic to the contractible set $(U \cap (I^{k-1} \times \partial I)) \times (k-1,k]$. □

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