ON PARTIALLY TEICHMÜLLER BELTRAMI DIFFERENTIALS

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Introduction. If E is a measurable subset of a Riemann surface R, we consider two problems associated with quasiconformal mappings F from R to another Riemann surface R_1 with the property that F is conformal on $R-E$. The first problem is to characterize extremal mappings in the same Teichmüller class as F subject to the condition that they be conformal on $R-E$. Using a modified version of the "main inequality" of Reich and Strebel we can show that mappings F extremal in this sense have partially Teichmüller form. By this we mean there is an integrable holomorphic quadratic differential ϕ on the surface R_1 such that F^{-1} has Beltrami coefficient $k|\phi|/\phi$ on $F(E)$ and zero on $R_1-F(E)$. It also follows that such differentials are extremal. To a certain extent these results can be viewed as analogous to those presented by Reich in [6]. In Reich's notation we are treating the case where $b_1=0$, which Reich says has special interest and is not included in his own treatment.

The second problem we consider concerns mappings F which are Teichmüller trivial on R and which are also conformal on $R-E$. The space of Beltrami differentials of such mappings is denoted by $M_0(R,E)$. We are unable to show that $M_0(R,E)$ is connected but can show that given μ in $M_0(R,E)$ corresponding to F there is a mapping μ_1 in $M_0(R,E)$ corresponding to F_1 such that if μ_2 is the Beltrami coefficient for $F\circ F_1^{-1}$ then $\|\mu_i\|_\infty < \|\mu\|_\infty$ for $i=1$ and $i=2$. The result also gives real analytic parametric path of Beltrami differentials in $M_0(R,E)$ connecting μ to μ_1.

1. Preliminaries. Let R be a Riemann surface, C be the union of the ideal boundary curves of R and let σ be a closed subset of C. It is possible for either C or σ to be empty. When C is nonempty we consider it to be part of R so in this case R is a bordered Riemann surface.

Let $A(R,\sigma)$ be the space of integrable, holomorphic, quadratic differentials on R which are real with respect to real boundary parameters at points of $C-\sigma$. If R is of finite type with genus g, n boundary contours and k interior punctures and σ is a finite subset of C, then from the Riemann-Roch theorem one can show that the real dimension of $A(R,\sigma)$ is

$$6(g-1)+3n+2k+\operatorname{card}(\sigma)+\rho \tag{1}$$

where ρ is the real dimension of the continuous group of holomorphic homeomorphisms of R. ρ can be positive only in special cases when $g=1$ or 0.

Let $L(R)$ be the set of all Beltrami differentials on R. An element μ of $L(R)$ is an assignment of a measurable function μ^z to each local parameter z such that

(a) $\mu^z(z)\dfrac{d\bar{z}}{dz}=\mu^\zeta(\zeta)\dfrac{d\bar{\zeta}}{d\zeta}$ for any two parameters z and ζ with overlapping domains and

(b) $\|\mu\|_\infty=\sup\{\|\mu^z(z)\|_\infty \text{ for all } z\}<\infty$.

Received March 25, 1981. Revision received November 30, 1981.
Partially supported by NSF.
Michigan Math. J. 29 (1982).

Let $M(R)$ be the open unit ball in the Banach space $L(R)$. From the theory of the Beltrami equation [1] for any μ in $M(R)$ there will be a quasiconformal homeomorphism $w=w^{\mu}$ from R onto another Riemann surface R^{μ} satisfying

$$w_{\bar{z}} = \mu w_z. \tag{2}$$

Let $D_0(R,\sigma)$ be the group of quasiconformal, homeomorphic self-mappings h of R which are homotopic to the identity in the sense that there exists a continuous curve of continuous self-mappings g_t of R for which

(i) $g_0(z)=z$ and $g_1(z)=h(z)$ for z in R and

(ii) $g_t(p)=h(p)=p$ for p in σ and $0\leqslant t\leqslant 1$.

The group $D_0(R,\sigma)$ induces an equivalence relation on $M(R)$ by defining $\mu\sim\nu$ if $\nu=(w^{\mu}\circ h)_{\bar{z}}/(w^{\mu}\circ h)_z$ for some h in $D_0(R,\sigma)$.

The Teichmüller space $T(R,\sigma)$ is the space of equivalence classes under this equivalence relation.

2. The extremal problem. Let $F:R\to R_1$ and $M(F,E,R,\sigma)=M$ be the set of elements μ in $M(R)$ which are identically zero on $R-E$ and which are equivalent to the Beltrami coefficient of F under the action of $D_0(R,\sigma)$. Let

$$k^*(M) = \inf\{\|\mu\|_\infty : \mu \in M(F,E,R,\sigma)\}$$

and $K^*(M)=(1+k^*(M))/(1-k^*(M))$. If μ is the Beltrami coefficient of G and μ is in $M=M(F,E,R,\sigma)$ and if $\|\mu\|_\infty=k^*(M)$, we say G is extremal for the class M. The existence of at least one extremal mapping G for the class M is assured by the compactness of a family of quasiconformal mappings whose dilatations have a uniform bound. We will write $z=F(w)$, $f=F^{-1}$, $\kappa(w)=F_{\bar{w}}/F_w$, $\kappa_1(z)=f_{\bar{z}}/f_z$. For a mapping G whose Beltrami coefficient is in $M(F,E,R,\sigma)$, we may as well assume G and F map onto the same Riemann surface. Suppose $F:R\to R_1$ and $G:R\to R_2$. The equivalence relation implies there is a conformal map $c:R_2\to R_1$ such that $C\circ G$ is homotopic to F by a homotopy fixing points of σ. But $C\circ G$ and G have identical Beltrami coefficients. Of course, we cannot assume $F(E)=G(E)$.

The following theorem is a form of Hamilton's necessary condition for extremality, [4].

THEOREM 1. *Suppose $T(R,\sigma)$ is finite dimensional. Suppose F is extremal in the class $M(F,E,R,\sigma)$. Then*

$$\sup\left|\operatorname{Re}\iint_{F(E)} \kappa_1(z)\phi(z)\,dx\,dy\right| = \|\kappa_1\|_\infty$$

where the supremum is taken over all ϕ in $A(R_1,\phi_1)$ for which $\|\phi\|_{F(E)}=1$.

Proof. Hamilton's proof [4] goes through verbatim except for the requirement that one must invoke the existence of trivial curves of Beltrami coefficients on R_1 with given tangent vector and with support contained in the set $F(E)$. The existence of such curves is proved in [3] and depends on the finite dimensionality of $T(R,\sigma)$. We begin the proof by applying the Hahn–Banach theorem to obtain τ in $L(E)$ such that

$$\operatorname{Re}\iint_{F(E)} \tau\phi\,dx\,dy = \operatorname{Re}\iint_{F(E)} \kappa_1\phi\,dx\,dy$$

for all ϕ in $A(R_1,\sigma_1)$ with the property that

$$\|\tau\|_\infty = \sup\left\{\left|\operatorname{Re}\iint_{F(E)} \kappa_1\phi\, dx\, dy\right| \Big/ \iint_{F(E)} |\phi|\, dx\, dy\right\}.$$

If the conclusion of the theorem were not true, then $\|\tau\|_\infty < \|\kappa_1\|_\infty$. Let $\nu = \kappa_1 - \tau$. Then ν is infinitesimally trivial and so there exists a curve $\nu(t,z)$ of trivial Beltrami coefficients on R_1 with support in $F(E)$ such that $\nu(t,z) = t\nu(z) + o(t)$ uniformly in z. Now, one forms the mapping $H = G\circ F = G\circ f^{-1}$ where G has Beltrami coefficient $\nu(t,z)$. By calculating the Beltrami coefficient of H just as in [2, page 42] one finds that for sufficiently small t the Beltrami coefficient of H has norm less than $\|\kappa\|$ and H is in the same class as F. This contradicts the fact that κ is extremal. □

COROLLARY 1. *If $T(R,\sigma)$ is finite dimensional and if F is extremal in the class $M(F,E,R,\sigma)$ then $\kappa_1(z) = k|\phi(z)|/\phi(z)$ for z in $F(E)$ and $\kappa_1(z) = 0$ for z in $R_1 - F(E)$ where $0 \leqslant k < 1$ and ϕ is a nonzero element of $A(R_1,\sigma_1)$.*

Proof. Obviously $\kappa_1(z) = 0$ on $R_1 - F(E)$ since $\kappa(w) = 0$ on $R - E$. Choose ϕ such that $\|\phi\|_{F(E)} = 1$ and $\iint_{F(E)} \kappa_1\phi\, dx\, dy = \|\kappa_1\|_\infty$. But then

$$\|\kappa_1\|_\infty = \iint_{F(E)} \kappa_1\phi\, dx\, dy \leqslant \iint_{F(E)} \|\kappa_1\|_\infty |\phi|\, dx\, dy \leqslant \|\kappa_1\|_\infty$$

and we conclude that $\kappa_1\phi = \|\kappa_1\|_\infty |\phi|$ almost everywhere in $F(E)$ and this proves the corollary. □

REMARK. There is an initial holomorphic quadratic differential ψ on E with property that F has Beltrami coefficient $\kappa = -k|\psi(w)|/\psi(w)$ for w in E and $\kappa(w) = 0$ for w in $R - E$. But ψ is not necessarily in $A(R,\sigma)$ and it does not necessarily extend into the domain $R - E$. This is a reflection of the asymmetry of the extremal problem. If one poses the problem of solving the extremal problem for $M(f, F(E), R_1, \sigma_1)$, the extremal value will in general be smaller than $k(F)$. The fact that F is extremal in its class does not imply that $f = F^{-1}$ is extremal in its class.

DEFINITION. If a Beltrami differential $\kappa(z) = k|\phi(z)|/\phi(z)$ for z in E_1 and $\kappa(z) = 0$ for z in $R_1 - E_1$ for some element $\phi \in A(R_1,\sigma_1)$, we will say $\kappa(z)$ has partially Teichmüller form on the set E_1.

One of the results of the next section will show that if $F = f^{-1}$ and κ and κ_1 are the Beltrami coefficients of F and f and if κ_1 has partially Teichmüller form on $F(E)$, then F is extremal in its class. In the analogous case treated by Reich in [6], he says he can give conditions under which F is unique extremal. These conditions involve uniqueness of the Hahn-Banach extension of the linear functional

$$\iint_{F(E)} \kappa(z)\varphi(z)\, dx\, dy.$$

We do not know whether his method works in this case.

3. The main inequality of Reich and Strebel. Let the Beltrami coefficient of G be an element of $M(F,E,R,\sigma)$ and $E_1 = F(E)$. Let $f = F^{-1}$, $g = G^{-1}$ and μ, μ_1, κ, κ_1 be the Beltrami coefficients of G, g, F, f respectively. The main inequality of Reich and Strebel, [7, p. 464] applied to $G\circ F$ on the surface $R_1 = F(R)$ can be stated as follows: for any quadratic differential ϕ in $A(R_1,\sigma_1)$,

(3) $$\iint_{E_1}|\phi| \leqslant \iint_{E_1}|\phi| \frac{\left|1-\kappa_1 \dfrac{\phi}{|\phi|}\right|^2}{1-|\kappa_1|^2} \cdot \frac{\left|1-\mu \dfrac{\bar{p}}{p}\dfrac{\phi}{|\phi|} \cdot \dfrac{1-\bar{\kappa}_1\bar{\phi}/|\phi|}{1-\kappa_1\phi/|\phi|}\right|^2}{1-|\mu|^2} dx\,dy$$

where $p=f_z$. This inequality involves integrals over the subset E_1 and it follows from the inequality in [7] because the integrals over R_1-E_1 of both sides of (3) are identical since $\kappa_1(z)=\mu(f(z))\equiv 0$ on R_1-E_1.

In the case where κ_1 has partially Teichmüller form

(4) $$\kappa_1(z)=\begin{cases} k|\phi(z)|/\phi(z) & \text{on } E_1 \\ 0 & \text{on } R-E_1 \end{cases}$$

where ϕ is in $A(R_1,\sigma_1)$, one obtains

(5) $$K \leqslant \iint_{E_1}|\phi(z)| \frac{1+|\mu(f(z))|}{1-|\mu(f(z))|} dx\,dy$$

where ϕ is normalized so that $\|\phi\|_{E_1}=1$ and $K=(1+k)/(1-k)$. From this one sees that $K \leqslant (1+\|\mu\|_\infty)/(1-\|\mu\|_\infty)$ for any μ in $M(F,E,R,\sigma)$ and hence κ is extremal. We summarize this as a theorem.

THEOREM 2. *Let F have Beltrami coefficient in $M(F,E,R,\sigma)$ and $f=F^{-1}$. Suppose the Beltrami coefficient κ_1 of f is of partially Teichmüller form as in* (4) *where $\iint_{R_1}|\phi|<\infty$. Then F is extremal in its class.*

Another consequence of (3) concerns elements of $M_0(E,R,\sigma)$. $M_0(E,R,\sigma)$ is, by definition, the set $M(F,E,R,\sigma)$ where $F=$the identity. In this case, we put $\kappa\equiv\kappa_1\equiv 0$ and $p\equiv 1$ and (3) becomes

(6) $$\iint_E|\phi| \leqslant \iint_E|\phi| \frac{\left|1-\mu \dfrac{\phi}{|\phi|}\right|^2}{1-|\mu|^2} dx\,dy$$

for any ϕ in $A(R,\sigma)$ and any μ in $M_0(E,R,\sigma)$. Expanding out the numerator in the right side of (6), subtracting $\iint_E|\phi|\,dx\,dy$ from both sides and dividing by 2, (6) is seen to be equivalent to

(7) $$\operatorname{Re}\iint_E \frac{\mu\phi}{1-|\mu|^2}\,dx\,dy \leqslant \iint_E \frac{|\mu|^2}{1-|\mu|^2}|\phi|\,dx\,dy$$

for any μ in $M_0(E,R,\sigma)$ and any ϕ in $A(R,\sigma)$.

The inequalities (3) and (7) differ from ones given by Reich and Strebel in [5] and [7] only insofar as they involve integrals over a subset of the given Riemann surface.

4. Paths in $M_0(E,R,\sigma)$. Using the notations introduced in the previous sections, we prove the following theorem. It is similar to the result of Reich in [5, Theorem 1, p. 18].

THEOREM 3. *Suppose F has Beltrami coefficient μ in $M_0(E,R,\sigma)$ and $\|\mu\|_\infty=k=(K-1)/(K+1)$. Then there exist positive numbers t_0, δ and C and an analytic curve ν_t in $M_0(F(E),R,\sigma)$ such that if h_t has Beltrami coefficient ν_t then*

(i) *$K(h_t)\leqslant 1+Ct$ and*

(ii) *$K(h_t\circ F)\leqslant K-\delta t$ for $0\leqslant t\leqslant t_0$.*

REMARK. In this theorem the numbers t_0, δ, and C depend on μ. It would be desirable to know if the same theorem were true with the numbers t_0, δ and C depending only on $\|\mu\|_\infty = k$.

Proof. Here we modify the method used by Reich, [5]. Consider the linear functional L_μ defined on $A(R,\sigma)$ by

$$L_\mu(\phi) = \operatorname{Re} \iint_{E_1} \frac{\mu_1 \phi}{1-|\mu_1|^2}\, dx\, dy,$$

where μ_1 is the Beltrami coefficient of a quasiconformal mapping which is inverse to a quasiconformal mapping with Beltrami coefficient μ. Let

$$\|L_\mu\| = \sup\{|L_\mu(\phi)|;\ \|\phi\|_{E_1} = 1\}.$$

By the Hahn–Banach theorem and the Riesz representation theorem there is a Beltrami differential τ on R_1 with support in $E_1 = F(E)$ such that $\|\tau\|_\infty = \|L\|$

$$\operatorname{Re} \iint_{E_1} \frac{\mu_1 \phi}{1-|\mu_1|^2}\, dx\, dy = \operatorname{Re} \iint_{E_1} \tau \phi\, dx\, dy$$

for all ϕ in $A(R_1, \sigma_1)$.

Then $\nu_1(z) = \dfrac{\mu_1}{1-|\mu_1|^2} - \tau$ is orthogonal to $A(R_1,\sigma_1)$ and has support in E_1. By (7)

$$\|\tau\|_\infty = \|L_\mu\| \leqslant \frac{k^2}{1-k^2} \tag{8}$$

since μ_1 is in $M_0(E_1, R_1, \sigma_1)$.

By Theorem 1 of [3, p. 1099] applied to the surface R with subset E_1 there is a curve of Beltrami coefficients $\nu_t(z)$ in $M_0(E_1, R, \sigma)$ such that

$$\nu_t(z) = t\nu_1(z) + o(t) \tag{9}$$

uniformly in z as $t \to 0$.

Let $h^t : R \to R$ be a quasiconformal mapping with Beltrami coefficient ν_t. We claim that (i) and (ii) hold for this choice of h_t. The complex dilatation of ρ_t of $h^t \circ F$ satisfies

$$|\rho_t| = \left| \frac{\mu_1 - \nu_t}{1 - \bar{\nu}_t \mu_1} \right| \tag{10}$$

We will show there are constants δ' and t_0' such that

$$|\rho_t(z)| \leqslant k - \delta' t \quad \text{for} \quad 0 \leqslant t \leqslant t_0' \quad \text{and} \quad w \text{ in } R.$$

As in [5, p. 20], let α be the positive solution of the equation

$$\frac{\alpha}{1-\alpha^2} = \frac{1}{2}\left(\frac{k^2}{1-k^2} + \frac{k}{1-k^2} \right) = \frac{k}{2(1-k)}.$$

Let

$$S_1 = \{z \in R : |\mu_1(z)| \leqslant \alpha\}$$

$$S_2 = \{z \in E_1 : \alpha < |\mu_1(z)| \leqslant k\}$$

Since $|\mu(z)| \leqslant \alpha < k$ for z in S_1, it is obvious from (10) that there are constants δ_1 and t_1 such that

(11) $$|\rho_t| \leqslant k-\delta_1 t \quad \text{for} \quad 0 \leqslant t \leqslant t_1 \quad \text{and} \quad z \in S_1.$$

From (10) we have

(12) $$|\rho_t|^2 = \frac{|\mu_1|^2 - 2\,\mathrm{Re}(\nu_t \bar{\mu}_1) + |\nu_t|^2}{1 - 2t\,\mathrm{Re}(\nu_t \bar{\mu}_1) + |\nu_t|^2 |\mu_1|^2}.$$

From (9) and (12) one obtains

(13) $$|\rho_t(w)| = |\mu_1(z)| - t\,\frac{1-|\mu_1(z)|^2}{|\mu_1(z)|}\,\mathrm{Re}[\nu_1(z)\overline{\mu_1(z)}] + o(t).$$

By definition of $\nu_1(z)$,

(14) $$\begin{aligned}\mathrm{Re}[\nu_1(z)\overline{\mu_1(z)}] &= \mathrm{Re}\left[\frac{|\mu_1|^2}{1-|\mu_1|^2} - \tau\bar{\mu}_1\right] \geqslant \frac{|\mu_1|^2}{1-|\mu_1|^2} - |\tau|\,|\mu| \\ &= |\mu_1|\left[\frac{|\mu_1|}{1-|\mu_1|^2} - |\tau|\right].\end{aligned}$$

By (8) and (14) and the definition of α, the coefficient of $-t$ in (13) is bounded below by

$$|\mu_1|\left[1 - \frac{1-|\mu_1|^2}{|\mu_1|}\,|\tau|\right] \geqslant \alpha\left[1 - \frac{1-\alpha^2}{\alpha}\,\frac{k^2}{1-k^2}\right] = \frac{1-k}{1+k}\,\alpha$$

for z in S_2. Hence, there exists a $\delta_2 > 0$ and $t_2 > 0$ such that $|\rho_t| \leqslant k - \delta_2 t$. Taking $\delta' = \min(\delta_1, \delta_2)$ and $t_0' = \min(t_1, t_2)$, we get the desired result on $R = S_1 \cup S_2$. □

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