PEAK SETS FOR POLYDISC ALGEBRAS

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1. Introduction. Let A_N, $N>1$, be the polydisc algebra, i.e. the algebra of all continuous functions on the closed polydisc $\bar{\Delta}^N \subset \mathbf{C}^N$, analytic on the open polydisc Δ^N, with sup norm. We call a closed set $F \subset \bar{\Delta}^N$ a peak set for A_N if there is a $\varphi \in A_N$ such that $\varphi | F = 1$, $|\varphi(z)| < 1$ $(z \in \bar{\Delta}^N - F)$; we call any such φ a peaking function for F. We call F a peak interpolation set for A_N if given any $f \in C(F)$, $f \neq 0$, there is an extension $\tilde{f} \in A_N$ of f such that $|\tilde{f}(z)| < \|f\|$ $(z \in \bar{\Delta}^N - F)$.

Peak interpolation sets for A_N have been studied extensively—it is known that they coincide with those peak sets for A_N which are contained in $(\partial\Delta)^N$, the distinguished boundary of the polydisc [3, 4].

The simplest example of a peak set for A_N is $\bar{\Delta}^N$. By the maximum modulus theorem, every other peak set for A_N is contained in $\partial(\Delta^N)$, the boundary of the polydisc. If $N=1$ then every peak set for A_1 different from $\bar{\Delta}$ is a peak interpolation set for A_1 [4]. If $N>1$ this is no longer true. To illustrate this, let $N=2$ and let $F = \{(1, \zeta) : \zeta \in \bar{\Delta}\} \subset \partial(\dot{\Delta}^2)$. It is easy to see that F is a peak set for A_2 but not a peak interpolation set for A_2. However, as we show in this paper, one can describe peak sets for A_N in terms of peak interpolation sets for A_M, $M \leqslant N$.

Let $S \subset \{1, 2, \ldots, N\}$ be a subset containing M elements. Define

$$T = \{(x_\gamma) \in \mathbf{C}^N : |x_\gamma| = 1 \ (\gamma \in S),\ x_\gamma = 0 \ (\gamma \notin S)\}$$

and $\pi : \bar{\Delta}^N \to \bar{\Delta}^N$ by $\pi((x_\gamma)) = (y_\gamma)$ where $y_\gamma = x_\gamma$ $(\gamma \in S)$, $y_\gamma = 0$ $(\gamma \notin S)$. We call any such T a torus and π the projection associated with T and we denote by A_T the restriction algebra $A \,|\, \pi(\bar{\Delta}^N)$. A_T and T can be identified in a natural way with A_M and $(\partial\Delta)^M$, respectively, and consequently the peak interpolation sets for A_T can be identified with the peak interpolation sets for A_M (see Section 2 for the precise meaning of this).

Let T be a torus, let π be the associated projection and suppose that $F \subset T$ is a peak interpolation set for A_T. There is a function $\varphi \in A_T$ such that $\varphi | F = 1$, $|\varphi(z)| < 1$ $(z \in \pi(\bar{\Delta}^N) - F)$. The function $\varphi \circ \pi \in A_N$ is a peaking function for $\pi^{-1}(F)$ so $\pi^{-1}(F)$ is a peak set for A. It follows that if T_i, $1 \leqslant i \leqslant m$, are tori, π_i the associated projections and $F_i \subset T_i$ peak interpolation sets for A_{T_i}, then

$$F = \bigcup_{i=1}^{m} \pi_i^{-1}(F_i) \tag{1}$$

is a peak set for A_N. We will show later by an example that there are other peak sets for A_N. Suppose that in (1) each of the sets F_i is the union of a sequence of peak sets for A_{T_i}. If F is closed then F is only a peak set for A_N. Our main result is that we get all peak sets for A_N in this way:

Received February 16, 1981. Revision received July 10, 1981.
This work was supported in part by the Boris Kidrič Fund, Ljubljana, Yugoslavia.
Michigan Math. J. 29 (1982).

THEOREM. *A compact set $F\subset\partial(\Delta^N)$ is a peak set for A_N if and only if*

(2) $$F=\bigcup_{i=1}^{m}\pi_i^{-1}(F_i)$$

where for each i, $1\leqslant i\leqslant m$, π_i is the projection associated with a torus T_i and $F_i\subset T_i$ is the union of a sequence of peak interpolation sets for A_{T_i}.

2. Preliminaries. Throughout, Δ is the open unit disc in $\mathbf{C}$ and $N>1$ is a fixed integer. We will write A for A_N. We denote by e_γ, $1\leqslant\gamma\leqslant N$, the standard basis in $\mathbf{C}^N$. If P is a subset of $\bar\Delta^N$ then we denote by $\bar P$ the closure of P. We denote by $\mathbf{N}$ the set of all positive integers. If S is a nonempty subset of $\{1,2,\ldots,N\}$ then we call $T=\{x=(x_\gamma)\in\bar\Delta^N: |x_\gamma|=1\ (\gamma\in S),\ x_\gamma=0\ (\gamma\notin S)\}$ the torus with support S and write $S=\operatorname{supp}T$. The map $\pi:\bar\Delta^N\to\bar\Delta^N$ defined by $\pi((x_\gamma))=(y_\gamma)$ where $y_\gamma=x_\gamma$ $(\gamma\in S)$ and $y_\gamma=0$ $(\gamma\notin S)$ is called the projection associated with T, or the projection with support S; we write $S=\operatorname{supp}\pi$. Any vector belonging to a torus is called a toroidal vector. If $x=(x_\gamma)\in\partial(\Delta^N)$ then $y=(y_\gamma)$ where $y_\gamma=x_\gamma$ if $|x_\gamma|=1$ and $y_\gamma=0$ if $|x_\gamma|<1$, is called the toroidal part of x and the set $\{\gamma:|x_\gamma|=1\}$ is called the toroidal support of x. As usual, the support of $x=(x_\gamma)\in\bar\Delta^N$ is the set $\operatorname{supp}x=\{\gamma:x_\gamma\neq 0\}$. If $x=(x_\gamma)\in\partial(\Delta^N)$ we define the face of x, $\mathfrak{F}(x)$, as $\mathfrak{F}(x)=\{y=(y_\gamma)\in\bar\Delta^N: y_\gamma=x_\gamma$ for all those γ for which $|x_\gamma|=1\}$.

Let T be a torus with support containing M elements and let π be the associated projection. We denote by A_T the restriction algebra $A\,|\,\pi(\bar\Delta^N)$. Let $b:S\to\{1,2,\ldots,M\}$ be a bijection and define the map $\varphi:\pi(\bar\Delta^N)\to\bar\Delta^M$ by

(3) $$\varphi\Big(\sum_{i\in S}x_ie_i\Big)=\sum_{i\in S}x_ie_{b(i)}.$$

By the map $f\mapsto f\circ\varphi$ $(f\in A_M)$ the polydisc algebra A_M can be identified with A_T. Call a closed set $F\subset T$ a peak interpolation set for A_T if given any $f\in C(F)$, $f\neq 0$, there is an extension $\tilde f\in A_T$ of f such that $|\tilde f(z)|<\|f\|$ $(z\in\pi(\bar\Delta^N)-F)$; equivalently, $F\subset T$ is a peak interpolation set for A_T if $\varphi(F)$ is a peak interpolation set for A_M. By the properties of peak interpolation sets for A_M [3] a closed set $F\subset T$ is a peak interpolation set for A_T if and only if it is an interpolation set for A_T, i.e. if given any $f\in C(F)$ there is an $\tilde f\in A_T$ which extends f.

Call a set $F\subset\bar\Delta^N$ a zero set for A_N if there is a $\varphi\in A_N$ such that $\varphi\,|\,F=0$, $\varphi(z)\neq 0$ $(z\in\bar\Delta^N-F)$. If $F\subset\partial(\Delta^N)$ then $\bar\Delta^N-F$ is simply connected. Consequently a set $F\subset\partial(\Delta^N)$ is a peak set for A_N if and only if it is a zero set for A_N [3, pp. 132–133]. Using an argument of Stout [5, p. 8] it is easy to see that any compact subset of $\bar\Delta^N$ which is the union of a sequence of peak sets for A_N is itself a peak set for A_N.

3. Peak sets and faces.

LEMMA 1. *Let $F\subset\partial(\Delta^N)$ be a peak set for A. Then $\mathfrak{F}(x)\subset F$ for every $x\in F$.*

Proof. Let $f\in A$ be a peaking function for F. Let $x\in F$, let x_t be the toroidal part of x and let π be the projection with support $\{\gamma:|x_\gamma|<1\}$. The function $y\mapsto\varphi(y)=f(x_t+y)$ is continuous on $\pi(\bar\Delta^N)$. Since f is analytic on Δ^N it follows that for each α, $0<\alpha<1$, the function $y\mapsto\varphi_\alpha(y)=f(\alpha x_t+y)$ is analytic on $\pi(\Delta^N)$. By the uniform continuity of f the functions φ_α converge to φ uniformly on $\pi(\Delta^N)$ as $\alpha \to 1$ and consequently φ is analytic on $\pi(\Delta^N)$. Since φ assumes its maximum modulus at the point $x - x_t \in \pi(\Delta^N)$ it follows by the maximum modulus theorem that $\varphi(y) =$ const $(y \in \pi(\bar{\Delta}^N))$. It follows that $f(x_t + y) = 1$ $(y \in \pi(\bar{\Delta}^N))$ and the assertion follows. □

Let $F \subset \partial(\Delta^N)$ be a peak set for A. Denote by x_t the toroidal part of $x \in \partial(\Delta^N)$. Since $x \in \mathfrak{F}(x) = \mathfrak{F}(x_t)$ for every $x \in \partial(\bar{\Delta}^N)$ it follows by Lemma 1 that

$$F = \bigcup_{x \in F} \mathfrak{F}(x_t). \tag{4}$$

Call a face $\mathfrak{F}(x_t)$, $x \in F$, maximal if there is no $y \in F$ such that $\mathfrak{F}(x_t)$ is properly contained in $\mathfrak{F}(y_t)$. Further, if $x = (x_\gamma) \in \bar{\Delta}^N$, $y = (y_\gamma) \in \bar{\Delta}^N$ then write $x \prec y$ if

(a) supp x is properly contained in supp y

(b) $x_\gamma = y_\gamma$ for each $\gamma \in \operatorname{supp} x$.

It is easy to see that if $x, y \in \partial(\Delta^N)$ then $\mathfrak{F}(x)$ is properly contained in $\mathfrak{F}(y)$ if and only if $y_t \prec x_t$.

LEMMA 2. *Let $F \subset \partial(\Delta^N)$ be a peak set for A. Every face $\mathfrak{F}(x_t)$, $x \in F$, is contained in a maximal face.*

Proof. Let $x \in F$. If there is no $y \in F$ such that $y_t \prec x_t$ then $\mathfrak{F}(x_t)$ is already maximal. If there is some $y \in F$ such that $y_t \prec x_t$ then $\mathfrak{F}(x_t)$ is properly contained in $\mathfrak{F}(y_t)$—in this case the number of indices in supp y_t is strictly smaller than the number of indices in supp x_t. It is easy to see that there are $m \in \mathbf{N}$ and $y_1, y_2, \ldots, y_m \in F$ such that $y_{m,t} \prec y_{m-1,t} \prec \cdots \prec y_{1,t} \prec x_t$ and such that either $\mathfrak{F}(y_{m,t})$ is maximal or supp $y_{m,t}$ contains only one element which again implies that $\mathfrak{F}(y_{m,t})$ is maximal. In either case we have $\mathfrak{F}(x_t) \subset \mathfrak{F}(y_{m,t})$. This completes the proof. □

4. Proof of the theorem. Suppose that $F \subset \partial(\Delta^N)$ is a compact set of the form (3) where for each i, $1 \leqslant i \leqslant m$, π_i is the projection associated with a torus T_i and $F_i = \bigcup_{j=1}^{\infty} F_{ij}$ where F_{ij}, $j \in \mathbf{N}$, are peak interpolation sets for A_{T_i}. Since for each i, j, $1 \leqslant i \leqslant m$, $j \in \mathbf{N}$, the set $\pi_i^{-1}(F_{ij})$ is a peak set for A it follows that for each $j \in \mathbf{N}$ the set $G_j = \bigcup_{i=1}^{m} \pi_i^{-1}(F_{ij})$ is a peak set for A. Since F is compact and since $F = \bigcup_{j=1}^{\infty} G_j$ it follows that F is a peak set for A. This proves the if part of the theorem.

The proof of the only if part of the theorem is more difficult. We divide it into several parts.

PART 1. Let $F \subset \partial(\Delta^N)$ be a peak set for A. Put

$$G = \{x_t : x \in F \text{ and } \mathfrak{F}(x_t) \text{ is maximal}\}.$$

By Lemma 2

(5) for each $x \in F$ there is a $y \in G$ such that $\mathfrak{F}(x) \subset \mathfrak{F}(y)$.

Since $x \in \mathfrak{F}(x)$ $(x \in F)$ (5) implies that $F \subset \bigcup_{x \in G} \mathfrak{F}(y)$. On the other hand, by (4) $\bigcup_{y \in G} \mathfrak{F}(y) \subset \bigcup_{x \in F} \mathfrak{F}(x_t) = F$ which proves that

$$(6)\qquad \bigcup_{y\in G} \mathfrak{F}(y) = F.$$

In particular, $G \subset F$.

For each i, $1 \leqslant i \leqslant N$, let

$$G_i = \{x \in G : \operatorname{supp} x \text{ contains } i \text{ elements}\}$$

$$H_i = \{x \in G : \operatorname{supp} x \text{ contains at most } i \text{ elements}\}.$$

Clearly $H_i = \bigcup_{j=1}^{i} G_j$ $(1 \leqslant i \leqslant N)$. Further, for each i, $1 \leqslant i \leqslant N$, let $L_i = \bigcup_{x\in H_i} \mathfrak{F}(x)$.

We prove first that the set L_i is compact for each i, $1 \leqslant i \leqslant N$. Fix i, $1 \leqslant i \leqslant N$, and let $x_n \in L_i$, $x_n \to x_0$. We will prove that $x_0 \in L_i$. By the definition of L_i there is a sequence $y_n \in H_i$ such that $x_n \in \mathfrak{F}(y_n)$ $(n \in \mathbf{N})$. Since the class of all subsets of $\{1,2,\ldots,N\}$ containing at most i elements is finite it follows that there are m, $1 \leqslant m \leqslant i$, and a set $P \subset \{1,2,\ldots,N\}$ containing m elements such that $\operatorname{supp} y_n = P$ for infinitely many n. Passing to a subsequence if necessary we may assume that $\operatorname{supp} y_n = P$ $(n \in \mathbf{N})$ and by compactness we may assume that there is y_0 such that $y_n \to y_0$. Clearly $\operatorname{supp} y_0 = P$. Since $y_n \in F$ $(n \in \mathbf{N})$ and since F is closed it follows that $y_0 \in F$. Further, since $y_n \to y_0$, $x_n \to x_0$ and since $x_n \in \mathfrak{F}(y_n)$ $(n \in \mathbf{N})$ it is easy to see that $x_0 \in \mathfrak{F}(y_0)$. By (5) there is $z_0 \in G$ such that $\mathfrak{F}(y_0) \subset \mathfrak{F}(z_0)$. Since both y_0 and z_0 are toroidal vectors it follows that $\operatorname{supp} z_0 \subset \operatorname{supp} y_0$ which implies that $\operatorname{supp} z_0$ contains at most m elements. Since $m \leqslant i$ it follows that $\mathfrak{F}(z_0) \subset L_i$ and consequently $x_0 \in L_i$. This proves that L_i is compact.

PART 2. For each i, $1 \leqslant i \leqslant N$, let I_i be the class of all subets of $\{1,2,\ldots,N\}$ containing i elements. If $1 \leqslant i \leqslant N$ and $J \in I_i$ denote

$$G_{i,J} = \{x \in G : \operatorname{supp} x = J\}.$$

In what follows we assume that all sets $G_{i,J}$, $1 \leqslant i \leqslant N$, $J \in I_i$, are nonempty. It is easy to modify the proof in the case when some of these sets are empty.

We first prove that

$$(7)\qquad \overline{G_{i,J}} - G_{i,J} \subset L_{i-1} \quad (1 < i \leqslant N,\ J \in I_i).$$

Let $1 < i \leqslant N$, $J \in I_i$ and let $x_0 \in \overline{G_{i,J}} - G_{i,J}$. There is a sequence $x_n \in G_{i,J}$, $x_n \to x_0$. Since $G_{i,J} \subset F$ and since F is closed it follows that $x_0 \in F$. Note that x_0 is a toroidal vector, $\operatorname{supp} x_0 = J$. By the definition of G this implies that the face $\mathfrak{F}(x_0)$ is not maximal (otherwise we would have $x_0 \in G$ which, together with $\operatorname{supp} x_0 = J$, would imply that $x_0 \in G_{i,J}$, contradicting the assumption). By (5) it follows that there is $y_0 \in G$ such that $\mathfrak{F}(x_0)$ is properly contained in $\mathfrak{F}(y_0)$. Since x_0 and y_0 are both toroidal vectors this implies that $\operatorname{supp} y_0$ is properly contained in $\operatorname{supp} x_0$. Consequently $\operatorname{supp} y_0$ contains at most $i-1$ elements which implies that $\mathfrak{F}(y_0) \subset L_{i-1}$. Since $\mathfrak{F}(x_0) \subset \mathfrak{F}(y_0)$ it follows that $x_0 \in L_{i-1}$. This proves (7).

Further, observe that

$$(8)\qquad \mathfrak{F}(x) \subset L_{i-1} \quad (x \in L_{i-1},\ 1 < i \leqslant N).$$

Namely, if $x \in L_{i-1}$ then $x \in \mathfrak{F}(y)$ for some $y \in H_{i-1}$ which implies that $\mathfrak{F}(x) \subset \mathfrak{F}(y)$. Since $\mathfrak{F}(y) \subset L_{i-1}$ (8) follows.

PART 3. Let $f \in A$ be a peaking function for F. If $1 \leq i \leq N$ and $J \in I_i$ denote by T_J, π_J the torus and the projection with support J and write $D_J = \pi_J(\bar{\Delta}^N)$. Observe that $G_{i,J} \subset T_J \subset D_J$. We prove that

$$(9) \qquad \begin{cases} \text{if} \quad 1 < i \leq N \quad \text{and if} \quad J \in I_i \quad \text{then} \\ |f(z)| < 1 \quad (z \in D_J : z \notin L_{i-1},\ z \notin G_{i,J}) \end{cases}$$

and

$$(10) \qquad \text{if} \quad J \in I_1 \quad \text{then} \quad |f(z)| < 1 \quad (z \in D_J,\ z \notin G_{1,J}).$$

Fix i, $1 < i \leq N$ and let $J \in I_i$. Suppose that $K \in I_j$ for some $j \geq i$, $K \neq J$. Then $K - J$ is not empty. Let $\gamma \in K - J$. If $y \in G_{j,K}$ and $z \in \mathfrak{F}(y)$ then $|z_\gamma| = 1$. Since $\gamma \notin J$ we have $z_\gamma = 0$ $(z \in D_J)$. It follows that $D_J \cap \mathfrak{F}(y) = \varnothing$. Thus we have proved that

$$(11) \qquad D_J \cap \mathfrak{F}(y) = \varnothing \quad \text{if} \quad y \in G_{i,K},\ K \neq J, \quad \text{or if} \quad y \in G_{j,K},\ j > i.$$

By (6) we have

$$F = L_{i-1} \cup \left[\bigcup_{K \in I_i} \bigcup_{y \in G_{i,K}} \mathfrak{F}(y) \right] \cup \left[\bigcup_{j > i} \bigcup_{K \in I_j} \bigcup_{y \in G_{j,K}} \mathfrak{F}(y) \right]$$

and it follows by (11) that $(D_J - L_{i-1}) \cap F = (D_J - L_{i-1}) \cap \bigcup_{y \in G_{i,J}} \mathfrak{F}(y)$. Further, since $G_{i,J} \subset T_J$ it follows that $\mathfrak{F}(y) \cap D_J = \{y\}$ $(y \in G_{i,J})$. Consequently

$$(D_J - L_{i-1}) \cap F = (D_J - L_{i-1}) \cap G_{i,J}$$

which implies (9). In the same way we prove (10).

Since $G_{1,J} \subset F$ $(J \in I_1)$, (10) implies that for each $J \in I_1$, $G_{1,J}$ is a peak set for A_{T_J} and since $G_{1,J} \subset T_J$ it follows that $G_{1,J}$ is a peak interpolation set for A_{T_J}.

PART 4. Let $1 < i \leq N$ and let $J \in I_i$. We will prove that each compact subset of $G_{i,J}$ disjoint from L_{i-1} is a peak interpolation set for A_{T_J}.

Let $x \in G_{i,J}$, $x \notin L_{i-1}$. Since L_{i-1} is compact there is a compact convex neighbourhood $U \subset D_J$ of x disjoint from L_{i-1}. We will prove that $G_{i,J} \cap U$ is a peak interpolation set for the algebra $P(U)$, the uniform closure on U of polynomials in z_γ, $\gamma \in J$. Since U is a compact convex subset of the subspace $E \subset C^N$ spanned by e_γ, $\gamma \in J$, U is polynomially convex and it follows that $U \subset E$ is the maximal ideal space of $P(U)$ [1, p. 67]. Note that $f \mid D_J$ belongs to A_{T_J} which implies that $f \mid D_J$ is a uniform limit of polynomials in z_γ, $\gamma \in J$. It follows that $f \mid U \in P(U)$.

Write $H = G_{i,J} \cap U$ and $\varphi = f \mid U$. Since $\varphi \in P(U)$ it follows by (9) that H is a peak set for $P(U)$ and that φ is a peaking function for H. Now we use the proof of Stout [5, p. 7; 3, p. 133] to prove that H is an interpolation set for $P(U)$. In the same way as in [3, p. 133–134] we prove that $P(U) \mid H$ is closed in $C(H)$ and that H is the maximal ideal space of $P(U) \mid H$. Since $H \subset T_J$ the coordinate functions z_γ, $\gamma \in J$, have no zeros on H so they have inverses in $P(U) \mid H$. On H we have $z_\gamma^{-1} = \bar{z}_\gamma$ $(\gamma \in J)$ and it follows that $P(U) \mid H$ contains the algebra generated by $z_\gamma \mid H$, $\bar{z}_\gamma \mid H$ $(\gamma \in J)$. Since $P(U) \mid H$ is closed in $C(H)$ it follows by the Stone–Weierstrass theorem that $P(U) \mid H = C(H)$ [3, p. 134]. This proves that H is an interpolation set for $P(U)$. Since H is a peak set for $P(U)$ it follows that H is a peak interpolation set for $P(U)$ [5].

Now, the localization result of Stout [4, p. 224–225] implies that every $x \in G_{i,J} - L_{i-1}$ has a closed neighbourhood $U(x) \subset D_J$ disjoint from L_{i-1} such that $G_{i,J} \cap U(x)$ is a zero set for A_{T_J}. It follows that every compact subset of $G_{i,J}$ disjoint from L_{i-1} is a subset of a zero set for A_{T_J} contained in T_J, hence a subset of a peak interpolation set for A_{T_J}. Consequently every compact subset of $G_{i,J}$ disjoint from L_{i-1} is a peak interpolation set for A_{T_J}.

PART 5. Put $F_J = G_{1,J}$ $(J \in I_1)$ and $F_J = G_{i,J} - L_{i-1}$ $(J \in I_i,\ 1 < i \leq N)$. If $1 < i \leq N$ then $L_i - L_{i-1} = \bigcup_{J \in I_i} \bigcup_{x \in G_{i,J}} \mathfrak{F}(x) - L_{i-1}$. By (8) it follows that $L_i - L_{i-1} = \bigcup_{J \in I_i} \bigcup_{x \in G_{i,J} - L_{i-1}} \mathfrak{F}(x) - L_{i-1}$, which, by (6) gives

$$L_i - L_{i-1} \subset \bigcup_{J \in I_i} \bigcup_{x \in F_J} \mathfrak{F}(x) \subset F.$$

Consequently

$$\begin{aligned} F &= \bigcup_{i=1}^{N} L_i = L_1 \cup \left[\bigcup_{i=2}^{N} (L_i - L_{i-1}) \right] \\ &= \left[\bigcup_{J \in I_1} \bigcup_{x \in F_J} \mathfrak{F}(x) \right] \cup \left[\bigcup_{i=2}^{N} \bigcup_{J \in I_i} \bigcup_{x \in F_J} \mathfrak{F}(x) \right] \\ &= \bigcup_{i=1}^{N} \bigcup_{J \in I_i} \pi_J^{-1}(F_J). \end{aligned}$$

The proof of the theorem will be completed once we have shown that each F_J is the union of a sequence of peak interpolation sets for A_{T_J}. This is clear for the sets $F_J = G_{1,J}$ $(J \in I_1)$ since they are peak interpolation sets for A_{T_J}, respectively. Let $1 < i \leq N$ and let $J \in I_i$. Recall that $G_{i,J} \subset T_J$. If $D_J \cap L_{i-1} = \varnothing$ then by (7) $G_{i,J}$ is compact and disjoint from L_{i-1} so by Part 4 $F_J = G_{i,J}$ is a peak interpolation set for A_{T_J}. Suppose that $D_J \cap L_{i-1} \neq \varnothing$. For $n \in \mathbf{N}$, let

$$V_n = \{ z \in D_J : \operatorname{dist}(z, D_J \cap L_{i-1}) \geq 1/n \}.$$

Observe that for every $n \in \mathbf{N}$ V_n is a compact subset of D_J disjoint from L_{i-1}. By the compactness of L_{i-1} we have $D_J - L_{i-1} = \bigcup_{n=1}^{\infty} V_n$ which implies that

$$F_J = G_{i,J} - L_{i-1} = \bigcup_{n=1}^{\infty} (V_n \cap G_{i,J}).$$

Let $n \in \mathbf{N}$. By (7) the set $V_n \cap G_{i,J}$ is compact. Since it is disjoint from L_{i-1} it follows by Part 4 that $V_n \cap G_{i,J}$ is a peak interpolation set for A_{T_J}. Consequently F_J is the union of a sequence of peak interpolation sets for A_{T_J}. This completes the proof of the theorem. □

5. An example. In this section we present an example which shows that not every peak set for A has the form

$$F = \bigcup_{i=1}^{m} \pi_i^{-1}(F_i) \tag{12}$$

where T_i, $1 \leq i \leq m$, are tori, π_i the associated projections and $F_i \subset T_i$ peak interpolation sets for A_{T_i}.

Let $N=2$. Let $\varphi_n, \varphi_n < \pi/2$, be a decreasing sequence of positive numbers converging to 0 and let ψ_n, $0 \leqslant \psi_n \leqslant 2\pi$, be a sequence dense in $[0, 2\pi]$. Let T_1, T_2 be the tori with supports supp $T_1 = \{1\}$, supp $T_2 = \{1, 2\}$ and let π_1, π_2 be the associated projections. Let $F_1 = \{(1, 0)\}$, $F_2 = \{(e^{i\varphi_n}, e^{i\psi_n}) : n \in \mathbf{N}\}$ and put

$$F = \pi_1^{-1}(F_1) \cup \pi_2^{-1}(F_2).$$

Since F is compact and since it is the union of a sequence of peak sets for A it follows that F is a peak set for A. F_1 is a peak interpolation set for A_{T_1} but F_2 is not a peak interpolation set for A_{T_2} since it is not closed. By the properties of φ_n and ψ_n we have $\overline{F_2} = F_2 \cup \{(1, \zeta) : \zeta \in \partial\Delta\}$ and consequently $F = \pi_1^{-1}(F_1) \cup \pi_2^{-1}(\overline{F_2})$. It is easy to see that $\overline{F_2}$ is not a peak interpolation set for $A_{T_2} = A$. It follows that the set F above cannot be written in the form (12) with F_i being peak interpolation sets for A_{T_i}.

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