QUASI-INVARIANT MEASURES AND MAXIMAL ALGEBRAS ON MINIMAL FLOWS

Jun-Ichi Tanaka

Dedicated to Professor Takizo Minagawa on his 70th birthday.

1. Introduction. Let Γ be a dense subgroup of the real line $\mathbf{R}$, endowed with the discrete topology, and let K be the dual group of Γ. For each t in $\mathbf{R}$, e_t denotes the element of K defined by $e_t(\lambda) = e^{i\lambda t}$ for any λ in Γ. The mapping from t to e_t embeds $\mathbf{R}$ continuously into K, and it is well known that the translation by e_t defines a strictly ergodic flow. Fix a positive γ in Γ, and let K_γ be the compact subgroup consisting of all x such that $x(\gamma) = 1$. If we put $\beta(x,t) = x + e_t$, then β carries $K_\gamma x[0, 2\pi/\gamma)$ continuously onto K. Furthermore, β is one to one and its inverse, β^{-1}, is continuous except at points on K_γ. It follows that Borel sets are taken to Borel sets in both directions. Thus K is represented measure theoretically, and almost topologically, as a product space $K_\gamma x[0, 2\pi/\gamma)$. Also it can be easily seen that the above flow, $x + e_t$, on K can be characterized by the homeomorphism S on K_γ defined by $S(y) = y + e_{2\pi/\gamma}$. This local product decomposition is very useful for understanding the structure of K, and is also highly important in the study of analyticity on compact abelian groups (cf. [6; Chapter II], and [5, Chapter VII, Section 6]). Especially, we notice that, by using this decomposition, a representation of quasi-invariant measures on K was shown by deLeeuw and Glicksberg [2].

Our principal objective in this article is to extend the local product decomposition in quotients of the Bohr group to minimal flows, and particular attention is given to representing quasi-invariant measures on minimal flows. Moreover, as an application of this representation, we investigate the maximality of algebras of analytic functions associated with a minimal flow. Conceivably, our proof enables us to make clearer the relation between Forelli's generalization [4] of Wermer's maximality theorem and Muhly's result [7; Corollary 3.1] concerning maximal weak-* Dirichlet algebras.

On the other hand, a famous theorem of Ambrose [1] showed that any measurable ergodic flow can be represented as a flow built under a function. Our main result may be regarded as a refinement of this theorem concerning continuous flows.

In the next section, we present some preliminary material which we shall need. In Section 3, our representation of a minimal flow, Theorem 3.3, is obtained, and we also give a representation of quasi-invariant measures. We deal with analytic measures and provide simpler proofs of two known theorems concerning maximal algebras in Section 4. We close with some remarks in Section 5.

The author would like to express his sincere gratitude to Professor Yuji Ito for his useful advice. He is also grateful to the referee for his valuable suggestions in the presentation of the first version of this paper.

Received January 14, 1981. Revision received May 12, 1981.

This research was partially supported by a Grant-in-Aid for Scientific Research, (No. 574078), from the Japanese Ministry of Education.

Michigan Math. J. 29 (1982).

2. Preliminaries. Let Ω be a locally compact Hausdorff space, and let $\mathcal{B}_\Omega$ be the Borel field on Ω, i.e., the smallest σ-algebra of subsets of Ω which contains every compact set. A one to one mapping S of Ω onto itself is called a *Borel isomorphism* if S carries $\mathcal{B}_\Omega$ onto itself. Let $\{S_t\}_{t\in\mathbf{R}}$ be a one-parameter group of one to one mappings of Ω onto itself. The pair $(\Omega, \{S_t\}_{t\in\mathbf{R}})$ is called a *continuous flow* (resp., *Borel flow*) if each S_t is a homeomorphism (resp., Borel isomorphism) and the mapping of (w,t), $S_t(w)$, is a continuous mapping on $\Omega\times\mathbf{R}$ (resp., Borel mapping with respect to $\mathcal{B}_{\Omega\times\mathbf{R}}$). A continuous flow $(\Omega, \{S_t\}_{t\in\mathbf{R}})$ is said to be *minimal* if, for each w in Ω, the orbit of w is dense in Ω. Let $C_0(\Omega)$ denote the space of all continuous complex-valued functions which vanish at infinity. The dual of $C_0(\Omega)$ is the space of all bounded regular Borel measures on Ω and it will be denoted by $M(\Omega)$. When Ω is compact $C(\Omega)$ denotes the space of all continuous complex-valued functions. A measure is called *quasi-invariant* on $(\Omega, \{S_t\}_{t\in\mathbf{R}})$ if its null sets are preserved under the translation by $\{S_t\}_{t\in\mathbf{R}}$. A quasi-invariant measure is said to be *ergodic* in case the only invariant sets in $\mathcal{B}_\Omega$ are null or have null complement.

Throughout this paper, we shall always assume that the flow $(X, \{T_t\}_{t\in\mathbf{R}})$ is a minimal continuous flow defined on a fixed compact Hausdorff space X. The assumption that the flow is minimal will not be necessary in some of our arguments, however there is no essential loss of generality from assuming it throughout. When it is convenient, we will often write $x+t$ for the translate of x in X by T_t. Similarly, if E is a subset of X and if J is a subset of $\mathbf{R}$, then $E+J$ denotes the set of all $x+t$ for any x in E and t in J. Using $\{T_t\}_{t\in\mathbf{R}}$, one may convolve a function ϕ in $C(X)$ or a measure μ in $M(X)$ with a function f in $L^1(\mathbf{R})$ in the following way. The convolution $\phi * f$ is defined by setting

$$\phi * f(x) = \int_{-\infty}^{\infty} \phi(x+t) f(t)\, dt,$$

and the convolution $\mu * f$ is defined to be the measure such that, for all ψ in $C(X)$,

$$\int_X \psi(x)\, d(\mu * f) = \int_X \psi * \tilde{f}(x)\, d\mu$$

where $\tilde{f}(t) = f(-t)$. The *spectrum* of a function ϕ in $C(X)$ (resp., a measure μ in $M(X)$), in the sense of spectral synthesis, is then defined to be the hull of its annihilator and will be denoted by $\mathrm{sp}(\phi)$ (resp., $\mathrm{sp}(\mu)$). A function ϕ in $C(X)$ (resp., a measure μ in $M(X)$) is said to be *analytic* if $\mathrm{sp}(\phi)$ (resp., $\mathrm{sp}(\mu)$) is nonnegative. Let $\mathfrak{A}$ be the space of all analytic functions in $C(X)$. Then $\mathfrak{A}$ is a uniform algebra on X. Let m be a representing measure for $\mathfrak{A}$, and let $1 \leqslant p \leqslant \infty$. Then we shall denote the abstract Hardy space associated with $\mathfrak{A}$, m, and p by $H^p(m)$. Recall that if m is not a point mass then m is quasi-invariant on $(X, \{T_t\}_{t\in\mathbf{R}})$ ([7; Theorem III]), and also recall that if μ is an invariant ergodic probability measure in $M(X)$, then μ is a representing measure for $\mathfrak{A}$ and $\mathfrak{A}$ is a weak-$*$ Dirichlet algebra in $L^\infty(\mu)$ ([7; Theorem I]). Our references for the basic facts about spectra are [3] and [7].

The following lemma is a direct consequence from the definition of continuous flows, so the proof is omitted.

LEMMA 2.1. *Let U be an open subset of X, and let I be a compact subset of* $\mathbf{R}$. *If $x+I$ is contained in U for some x in X, then there is a neighborhood $V(x)$ of x such that U contains $V(x)+I$.*

For a given E in $\mathcal{B}_X$, we define

$$\tilde{E}=\left\{x \in X;\ \int_{-\infty}^{\infty} \chi_E(x-t)\,\frac{dt}{1+t^2}>0\right\}, \tag{2.1}$$

where $\chi_E(x)$ denotes the characteristic function of E. Then $\tilde{E}$ is invariant, and it follows from Fubini's theorem that $\tilde{E}$ is measurable with respect to each μ in $M(X)$ (cf. [11; Chapter 7]). It is useful in the next section to characterize the null sets of quasi-invariant measures.

LEMMA 2.2. *Let μ be a positive quasi-invariant measure in $(X, \{T_t\}_{t\in\mathbf{R}})$. Then $\mu(E)=0$ if and only if $\mu(\tilde{E})=0$ for an E in $\mathcal{B}_X$.*

Proof. By Fubini's theorem, we have

$$\int_X\left[\int_{-\infty}^{\infty} \chi_E(x-t)\,\frac{dt}{1+t^2}\right] d\mu(x) = \int_{-\infty}^{\infty}\left[\int_X \chi_E(x-t)\,d\mu(x)\right]\frac{dt}{1+t^2}. \tag{2.2}$$

If we assume $\mu(E)=0$, then $\int \chi_E(x-t)\,d\mu(x)=0$ for each t in $\mathbf{R}$, since μ is quasi-invariant. We thus have $\mu(\tilde{E})=0$ by (2.2). Conversely, assume that $\mu(\tilde{E})=0$. Then, from (2.2), we may choose a t in $\mathbf{R}$ such that $\mu(E+t)=0$. This implies that $\mu(E)=0$. □

3. Representation of minimal flows. In this section, we represent a minimal flow by an analogue of [1], and provide a representation of quasi-invariant measures on it. This representation will be useful for studying the uniform algebras of analytic functions. We begin with some lemmas.

LEMMA 3.1. *There exist a positive function $\tilde{\phi}$ in $C(X)$, a neighborhood $V(x_0)$ of some point x_0 in X, and a positive t_0 in* $\mathbf{R}$ *which have the following properties*:

(i) *$\tilde{\phi}(x) \leqslant 1/4$ for each x in $V(x_0)$,*

(ii) *$\tilde{\phi}(x) \geqslant 3/4$ for each x in $V(x_0)+t_0$, and*

(iii) *if y belongs to $V(x_0)$, then the function of t, $\tilde{\phi}(y+t)$, is strictly increasing on $[0,t_0]$.*

Proof. Let ψ be any real-valued function in $C(X)$ that is not constant. Since $(X, \{T_t\}_{t\in\mathbf{R}})$ is minimal, we may choose an x_1 in X and a positive t_1 in $\mathbf{R}$ such that $\psi(x_1)<\psi(x_1+t_1)$. Let $P_u(s)=\dfrac{u}{\pi(u^2+s^2)}$ for any positive u in $\mathbf{R}$, and consider the convolution

$$\psi * P_u(x) = \int_{-\infty}^{\infty} \psi(x+s)\,\frac{u}{\pi(u^2+s^2)}\,ds.$$

Then it is easy to see that, for each x in X, the function of t, $\psi * P_u(x+t)$, is continuously differentiable in t. Moreover we obtain $\lim_{u\to 0} \psi * P_u(x)=\psi(x)$ for each x in X, since $P_u(s)$ is the Poisson kernel for the upper half-plane. This implies that $\psi * P_r(x_1) < \psi * P_r(x_1+t_1)$ for some positive r in $\mathbf{R}$. It therefore follows from the mean value theorem that there exist an x_0 in $x_1+[0,t_1]$, and two positive numbers t_0 and a in $\mathbf{R}$ such that $\frac{d}{dt}\psi*P_r(x_0+t)>a$ for each t in $[0,t_0]$. On the other hand, for any x in X, we see that

$$\frac{d}{dt}\psi*P_r(x) = \lim_{t\to 0}\frac{\psi*P_r(x+t)-\psi*P_r(x)}{t}$$
$$= \int_{-\infty}^{\infty}\psi(x+s)\,\frac{-2rs}{\pi(r^2+s^2)^2}\,ds.$$

Since the function of s, $\frac{-2rs}{\pi(r^2+s^2)^2}$, belongs to $L^1(\mathbf{R})$, $\frac{d}{dt}\psi*P_r(x)$ is continuous on X. So, by Lemma 2.1, we may find a neighborhood $V'(x_0)$ of x_0 which satisfies $\frac{d}{dt}\psi*P_r(x+t)>a$ for each t in $[0,t_0]$ and x in $V'(x_0)$. Since there exist a positive b in $\mathbf{R}$, and a c in $\mathbf{R}$ such that $b\cdot\psi*P_r(x)+c$ is positive on X, $b\cdot\psi*P_r(x_0)+c<1$, and $b\cdot\psi*P_r(x_0+t_0)+c\geqslant 1$, we may choose a positive integer n and a neighborhood $V(x_0)$ of x_0 with $V(x_0)\subseteq V'(x_0)$ for which the function $\tilde\phi(x)=\{b\cdot\psi*P_r(x)+c\}^n$ has the desired properties. □

Let $\tilde\phi(x)$, x_0, $V(x_0)$, and t_0 be as in Lemma 3.1, and let $W(x_0)$ be a compact neighborhood of x_0 with $W(x_0)\subseteq V(x_0)$. Then we put

$$\text{(3.1)}\qquad \begin{cases} H=\{x\in X;\ \ \tilde\phi(x)=\frac{1}{2}\}, & \text{and}\\ Y=H\cap(W(x_0)+[0,t_0]). \end{cases}$$

Since Y is closed in the compact subspace H of X, we denote by bd Y the topological boundary of Y with respect to H. Notice that bd Y may be empty in some cases. We also denote by int Y the interior of Y with respect to H. It will be helpful to note that an orbit starting at y in Y must return to Y in finite time. In fact, by construction of $W(x_0)$ and Lemma 3.1, it can be seen that, for each x in $W(x_0)$, there exists unique u in $(0,t_0)$ such that $x+u$ belongs to Y. On the other hand, since X is compact, the minimality of $(X,\{T_t\}_{t\in\mathbf{R}})$ implies that $W(x_0)-[0,a]=X$ for some positive a in $\mathbf{R}$. Then we see that if y belongs to Y, then $y+t$ must return to Y for some t in $(0,a+t_0)$ (see Figure). A bounded function F on Y and a transformation S on Y are defined as follows:

$$\text{(3.2)}\qquad \begin{cases} F(y)=\min\{t;\ y+t\in Y, \text{ and } t>0\}, & \text{and}\\ S(y)=y+F(y), \end{cases}$$

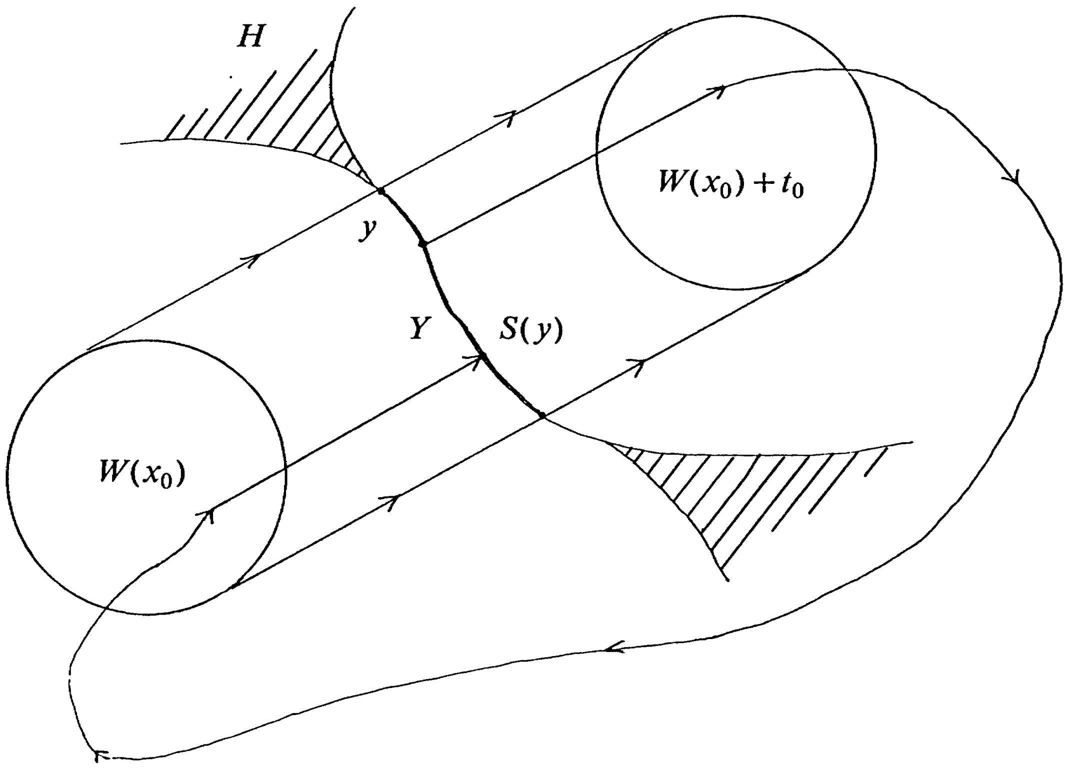
for each y in Y. It follows easily from (iii) of Lemma 3.1 and the above remark that F is well defined.

LEMMA 3.2. *Let Y, F, and S be as above. The we have*:

(i) *F is lower semi-continuous on Y. In particular, F is continuous on* $Y\backslash S^{-1}(\text{bd } Y)$,

(ii) *F is bounded and bounded away from zero, i.e., there exist two positive numbers m and M such that $m<F(y)<M$ for each y in Y, and*

(iii) *S is a one to one Borel isomorphism of Y onto itself.*



Figure

Proof. (i) Suppose that $S(y_0)=y_0+F(y_0)$ belongs to int Y. Then there is an open neighborhood $U(S(y_0))$ of $S(y_0)$ in X such that $U(S(y_0))\cap H\subseteq \text{int } Y$. Let $\tilde{\phi}$ be the function defined in Lemma 3.1. Then, for a given positive ϵ in $\mathbf{R}$, we may find a positive δ with $\delta<\epsilon$ and an open neighborhood $Q(S(y_0)-\delta)$ of $S(y_0)-\delta$ in X such that the function of t, $\tilde{\phi}(S(y_0)-\delta+t)$, is strictly increasing on $[0,2\delta]$, $\tilde{\phi}<\frac{1}{2}$ on $Q(S(y_0)-\delta)$, $\tilde{\phi}>\frac{1}{2}$ on $Q(S(y_0)-\delta)+2\delta$, and $Q(S(y_0)-\delta)+[0,2\delta]\subseteq U(S(y_0))$ by Lemma 2.1 and Lemma 3.1. Since $y_0+F(y_0)-\delta$ belongs to $Q(S(y_0)-\delta)$ and $y_0+(0,F(y_0)-\delta]\cap Y$ is empty, we may easily choose a neighborhood $V(y_0)$ of y_0 in Y such that, for any y in $V(y_0)$, $y+F(y_0)-\delta$ belongs to $Q(S(y_0)-\delta)$, and $y+[0,F(y_0)-\delta]\cap Y=\{y\}$. So it follows from (3.2) $F(y_0)-\delta<F(y)<F(y_0)+\delta$. We thus have $|F(y)-F(y_0)|<\epsilon$ for each y in $V(y_0)$. By an argument similar to one above, it may be seen that if $S(y_0)$ belongs to bd Y, then, for each positive ϵ in $\mathbf{R}$, there is a neighborhood $V(y_0)$ of y_0 in Y such that $F(y_0)-\epsilon<F(y)$ for any y in $V(y_0)$. Thus we have the property (i).

(ii) We have seen that F is bounded, so it suffices to show that F has a positive lower bound m. If y belongs to Y, then there is a positive δ such that the function of t, $\tilde{\phi}(y+t)$, is strictly increasing on $[0,\delta]$. This shows that $F(y)$ is positive for each y in Y. Since F is lower semicontinuous on compact space Y, we may choose a positive m such that $F(y)>m$ for any y in Y, so the property (ii) holds.

(iii) It follows easily from (3.2) that S is one to one. As above, we see that $X=W(x_0)+[0,a]$ for some positive a. Hence if y belongs to Y, then there is a positive u for which $y-u$ belongs to Y. This implies that $S^n(y-u)=y$ for some positive integer n, so S is an onto mapping. We next show that S^{-1} is a Borel mapping. It suffices to show that $S(E)$ belongs to $\mathcal{B}_Y$ for every compact subset E of Y. Let m and M be as in (ii), and let k be the positive integer satisfying $km<M\leqslant (k+1)m$. We notice that $E+[m,(j+1)m]$ is compact in X for $j=1,2,\ldots,k$, since Y is a compact subset of X. We set

$$E_j = (E+[m,(j+1)m])\cap Y.$$

Then it is easy to see that

$$S(E) = E_1\cup(E_2\backslash E_1)\cup(E_3\backslash E_2)\cup\cdots\cup(E_k\backslash E_{k-1}).$$

This shows that $S(E)$ is a Borel set in Y. On the other hand, since S is the composition of Borel functions, it is a Borel mapping. So the proof is complete. □

Let Y, F, and S be as above. Then by (iii) of Lemma 3.2, the pair (Y,S) defines a (Borel) dynamical system, so we say that a measure in $M(Y)$ is *quasi-invariant* on (Y,S) if its null sets are preserved under the translation by $\{S^n\}_{n\in Z}$. In order to state our result, we need some notation. Define a subset Ω of $Y\times\mathbf{R}$ as follows:

$$\Omega = \{(y,u);\ y\in Y,\ \text{ and }\ 0\leqslant u<F(y)\}.$$

Since F is lower semi-continuous, we observe that Ω is locally compact as a subspace of $Y\times\mathbf{R}$ and that Ω belongs to $\mathcal{B}_{Y\times\mathbf{R}}$. Let τ be the function on $Y\times Z$ defined by the formula

$$\tau(y,n) = \begin{cases} \sum_{k=0}^{n-1} F(S^k(y)) & \text{if } n>0, \\ 0 & \text{if } n=0, \\ -\tau(S^n(y),-n) & \text{if } n<0. \end{cases} \tag{3.3}$$

By using (3.3), a one-parameter group $\{S_t\}_{t\in\mathbf{R}}$ of mappings of Ω onto itself is defined by the formula

$$S_t(y,u) = (S^n(y),\ u+t-\tau(y,n)) \tag{3.4}$$

if $\tau(y,n)\leqslant u+t<\tau(y,n+1)$ (cf. [10; Section 2]). This is an analogue of the one which was introduced in [1]. We also define a mapping β of Ω onto X by the formula

$$\beta(y,u) = y+u \tag{3.5}$$

for each (y,u) in Ω. Then β is a Borel isomorphism of Ω onto X, i.e., β carries $\mathcal{B}_\Omega$ onto $\mathcal{B}_X$. In fact, we see easily that β is a one to one continuous mapping of Ω onto X, so β is a Borel mapping. On the other hand, for any compact subset K of Ω, $\beta(K)$ is also compact in X. This implies that β^{-1} is also a Borel mapping. If μ is a measure in $M(X)$, then $\mu\circ\beta$ denotes the measure in $M(\Omega)$ defined by $\mu\circ\beta(E)=\mu(\beta(E))$ for E in $\mathcal{B}_\Omega$.

We may now give the statement of our theorem which provides a representation of quasi-invariant measures on minimal flows.

THEOREM 3.3. *Let $(X,\{T_t\}_{t\in\mathbf{R}})$ be a minimal flow, and let S, Ω, $\{S_t\}_{t\in\mathbf{R}}$, and β be as above. Then we have:*

(i) $(\Omega, \{S_t\}_{t\in\mathbf{R}})$ *is a Borel flow, and it is (Borel) isomorphic to* $(X, \{T_t\}_{t\in\mathbf{R}})$ *via the Borel isomorphism* β *of* Ω *onto* X, *i.e.*,

$$S_t(\omega) = \beta^{-1}\circ T_t\circ\beta(\omega) \tag{3.6}$$

for each ω *in* Ω *and* t *in* $\mathbf{R}$, *and*

(ii) *suppose that a measure* μ *in* $M(X)$ *is quasi-invariant on* $(X, \{T_t\}_{t\in\mathbf{R}})$. *Then there exists a measure* $\tilde\mu$ *in* $M(Y)$ *which is quasi-invariant on* (Y, S) *so that* $\mu\circ\beta$ *and* $(\tilde\mu\times m_{\mathbf{R}})_\Omega$ *are mutually absolutely continuous, where* $(\tilde\mu\times m_{\mathbf{R}})_\Omega$ *denotes the restriction to* Ω *of the product* $\tilde\mu$ *with Lebesgue measure* $m_{\mathbf{R}}$ *on* $\mathbf{R}$. *In particular, if* μ *is invariant on* $(X, \{T_t\}_{t\in\mathbf{R}})$, *then* μ *may be represented as* $\mu\circ\beta = (\tilde\nu\times m_{\mathbf{R}})$ *for some invariant measure* $\tilde\nu$ *on* (Y, S).

Proof. (i) It follows easily from (3.4) and (3.5) that the equation (3.6) holds. So $\{S_t\}_{t\in\mathbf{R}}$ is a one-parameter group of Borel isomorphisms of Ω onto itself, since β is a Borel isomorphism. We have to show that the mapping of (ω, t), $S_t(\omega)$, is a Borel mapping with respect to $\mathcal{B}_{\Omega\times\mathbf{R}}$. Recall that β is continuous on Ω. So if we set $\Phi(\omega, t) = (\beta(\omega), t)$, then Φ is a Borel mapping of $\Omega\times\mathbf{R}$ onto $X\times\mathbf{R}$. For any E in $\mathcal{B}_\Omega$, it follows from (3.6) that

$$\Phi^{-1}\{(x,t);\ T_t(x)\in\beta E\} = \{(\omega,t);\ S_t(\omega)\in E\}.$$

Therefore, since βE belongs to $\mathcal{B}_X$, we see that the mapping of (ω, t), $S_t(\omega)$, is a Borel mapping. Thus $(\Omega, \{S_t\}_{t\in\mathbf{R}})$ is a Borel flow.

(ii) Let μ be a quasi-invariant measure on $(X, \{T_t\}_{t\in\mathbf{R}})$. By considering the total variation measure of μ, we may assume that μ is positive. Let m be the positive number in (ii) of Lemma 3.2, and let $\tilde\mu$ be the measure defined by the equation

$$\tilde\mu(A) = \mu\circ\beta(A\times[0,m]) \tag{3.7}$$

for each A in $\mathcal{B}_Y$. Notice that β is a homeomorphism of $Y\times[0,m]$ onto $\beta(Y\times[0,m])$. This implies that $\tilde\mu$ is a regular Borel measure. For a given B in $\mathcal{B}_Y$, suppose that $\tilde\mu(B) = 0$. If we set $E = B\times[0,m]$, then $\mu(\beta E) = 0$ by (3.7). Let $\widetilde{\beta E}$ be the invariant set defined by (2.1). Then it follows from Lemma 2.2 that $\mu(\widetilde{\beta E}) = 0$. Since it is easy to see that $\widetilde{\beta E} = \widetilde{\beta(S(B)\times[0,m])}$, we obtain that

$$\mu\circ\beta(S(B)\times[0,m]) = 0,$$

so $\tilde\mu(S(B)) = 0$. Similarly it may be shown that $\tilde\mu(S^{-1}(B)) = 0$, thus $\tilde\mu$ is quasi-invariant on (Y, S). Next we show that $\mu\circ\beta$ is mutually absolutely continuous with respect to $(\tilde\mu\times m_{\mathbf{R}})_\Omega$. We first note that

$$\begin{cases} (\tilde\mu\times m_{\mathbf{R}})_\Omega(E) = \displaystyle\int_Y\left[\int_0^{F(y)} \chi_E(y,u)\,du\right]d\tilde\mu(y) \\ \qquad\qquad = \displaystyle\int_Y\left[\int_0^{F(y)} \chi_{\beta E}(y+u)\,du\right]d\tilde\mu(y), \end{cases} \tag{3.8}$$

for an E in $\mathcal{B}_\Omega$. Suppose that $\mu\circ\beta(E) = 0$. Then it follows from Lemma 2.2 that $\mu(\widetilde{\beta E}) = 0$. Since $(\widetilde{\beta E}\cap Y)\times[0,m]$ is contained in $\beta^{-1}(\widetilde{\beta E})$, we see that $\tilde\mu(\widetilde{\beta E}\cap Y) = 0$. Together with the equation (3.8), this shows that $(\tilde\mu\times m_{\mathbf{R}})_\Omega(E) = 0$.

On the other hand, suppose that $(\tilde{\mu}\times m_{\mathbf{R}})_{\Omega}(E)=0$. Then there exists a $\tilde{\mu}$-null set N such that the inner integral of (3.8) equals to 0 for each y in $Y\backslash N$. If we set $\hat{N}=\bigcup_{n\in Z}S^n(N)$, then $\hat{N}$ is also a $\tilde{\mu}$-null set, since $\tilde{\mu}$ is quasi-invariant on (Y,S). Let $\tau(y,n)$ be the function defined by (3.3). Then since $S^n(y)$ does not belong to N for any y in $Y\backslash\hat{N}$, it is easy to verify that

$$\int_0^{F(S^n(y))}\chi_{\beta E}(S^n(y)+u)\,du=\int_{\tau(y,n)}^{\tau(y,n+1)}\chi_{\beta E}(y+u)\,du=0.$$

Since $\bigcup_{n\in Z}[\tau(y,n),\ \tau(y,n+1)]=\mathbf{R}$, we have

$$\int_{-\infty}^{\infty}\chi_{\beta E}(y+u)\,\frac{du}{1+u^2}=0,$$

for each y in $Y\backslash\hat{N}$. Since $\mu\circ\beta(\hat{N}\times[0,m])=0$, it follows that $\mu(\widetilde{\beta E})=0$, so by Lemma 2.2 we obtain $\mu\circ\beta(E)=0$. When μ is invariant on $(X,\{T_t\}_{t\in\mathbf{R}})$, we may see that $\tilde{\mu}$ is also invariant on (Y,S). In fact, let $\tilde{F}(y,u)=F(y)$ for any (y,u) in Ω. Then there is an increasing sequence $\tilde{F}_n(\omega)$ of step functions which converges to $F(\omega)$ for $\mu\circ\beta-$a.e. ω in Ω. If A belongs to $\mathfrak{B}_Y$, then it can be easily seen that

$$\chi_{A+[0,m]}(x-\tilde{F}\circ\beta^{-1}(x))=\lim_{n\to\infty}\chi_{A+[0,m]}(x-\tilde{F}_n\circ\beta^{-1}(x))$$

for $\mu-$a.e. x in X. Since μ is invariant, we see that

$$\mu\circ\beta(A\times[0,m])=\int_X\chi_{A+[0,m]}(x-\tilde{F}_n\circ\beta^{-1}(x))\,d\mu(x)$$

for $n=1,2,\ldots$. It follows from (3.2) and the bounded convergence theorem that $\mu\circ\beta(A\times[0,m])=\mu\circ\beta(S(A)\times[0,m])$. This shows that $\tilde{\mu}$ is invariant on (Y,S), so we can find a positive a in $\mathbf{R}$ for which $\mu\circ\beta=((a\tilde{\mu})\times m_{\mathbf{R}})_{\Omega}$. This completes the proof. □

There are, of course, many possibilities to choose such a compact subspace Y and a function F for a fixed minimal flow $(X,\{T_t\}_{t\in\mathbf{R}})$. So it is natural to raise the following question connected with Theorem 3.3:

Is it possible to make F to be continuous on Y?

It follows from (3.2) that if F is continuous, then S is a homeomorphism of Y onto itself. Hence, in these cases, (Y,S) defines a topological dynamical system. So many results and concepts associated with $(X,\{T_t\}_{t\in\mathbf{R}})$ can be reduced to those of (Y,S). We however content ourselves in treating the case where F is constant.

For ψ in $C(X)$, we say that ψ is a *continuous eigenfunction* if there is a λ in $\mathbf{R}$ such that $\psi(x+t)=e^{i\lambda t}\psi(x)$ for each x in X and t in $\mathbf{R}$. This λ is called an *eigenvalue* for the flow.

PROPOSITION 3.4. *The function F for a minimal flow $(X,\{T_t\}_{t\in\mathbf{R}})$ can be chosen to be constant if and only if there exists a continuous eigenfunction which is not constant.*

Proof. If ψ is a non-constant continuous eigenfunction with eigenvalue λ, then we may assume that λ is positive and $|\psi(x)|=1$ for any x in X. Let $Y=\{x\in X;\psi(x)=1\}$, and let $F(y) \equiv 2\pi/\lambda$. We notice that Y is a compact subspace of X, and that $\Omega = Y \times [0, 2\pi/\lambda)$. Let β and $\{S_t\}_{t\in\mathbf{R}}$ be as in (3.5) and (3.4), respectively. Then it follows easily that $(\Omega, \{S_t\}_{t\in\mathbf{R}})$ is isomorphic to $(X, \{T_t\}_{t\in\mathbf{R}})$ via the Borel isomorphism β. Conversely, suppose that $F(y) = a$ for some positive a in $\mathbf{R}$. We define $v(y,u) = e^{i(2\pi/a)u}$ for any (y,u) in $Y\times\mathbf{R}$. Then it is easy to see that $v\circ S_t(y,u) = e^{i(2\pi/a)t}v(y,u)$ for each (y,u) in Ω. Therefore if we put $\psi(x) = v\circ\beta^{-1}(x)$, then

$$\begin{aligned}\psi(x+t) &= v\circ\beta^{-1}\circ T_t(x) = v\circ\beta^{-1}\circ T_t\circ\beta(\beta^{-1}(x))\\ &= v\circ S_t(\beta^{-1}(x)) = e^{i(2\pi/a)t}v\circ\beta^{-1}(x) = e^{i(2\pi/a)t}\psi(x)\end{aligned}$$

for each x in X and t in $\mathbf{R}$. We next show that ψ is continuous on X. Let $\bar{\Omega}$ be the quotient topological space of $Y\times[0,a]$ by identifying (y,a) and $(S(y),0)$ for each y in Y. Then β is a homeomorphism of $\bar{\Omega}$ onto X. Since we see that $v(y,a) = v(S(y),0)$ for any y in Y, v may be considered as a continuous function on $\bar{\Omega}$. Thus $\psi(x) = v\circ\beta^{-1}(x)$ is a continuous eigenfunction with eigenvalue $2\pi/a$ which is not constant, so the proof is complete. □

By considering $\bar{\Omega}$ as above, we can always construct a new continuous flow from a given topological dynamical system (Y, S).

4. Maximality of uniform algebras. In this section, by using Theorem 3.3, we provide elementary proofs to some fundamental theorems concerning the maximality of algebras of analytic functions. Our proofs rest on some techniques which were used in [2] and [10]. We begin to study some properties of quasi-invariant measures.

Let Y, F, S, Ω, and β be as in Section 3, and define

$$(4.1)\qquad \begin{cases}\sigma(y,u) = (S(y), u - F(y)), & \text{and}\\ U_t(y,u) = (y, u+t)\end{cases}$$

for each (y,u) in $Y\times\mathbf{R}$ and t in $\mathbf{R}$. Then it is easy to see that the hypotheses of F imply that $Y\times\mathbf{R}$ is the disjoint union $\bigcup_{n\in Z}\sigma^n(\Omega)$. We also notice that $(Y\times\mathbf{R}, \{U_t\}_{t\in\mathbf{R}})$ defines a continuous flow. Let π be the mapping of $Y\times\mathbf{R}$ onto Ω defined by the formula

$$(4.2)\qquad \pi(y,u) = (S^n(y), u - \tau(y,n))$$

if $\tau(y,n) \leqslant u < \tau(y,n+1)$, where $\tau(y,n)$ denotes the function defined in (3.3). For a μ in $M(X)$, we define a σ-finite regular Borel measure on $Y\times\mathbf{R}$ by the formula

$$(4.3)\qquad \mu'(E) = \sum_{n\in Z}\mu\circ\beta\circ\pi(E\cap\sigma^n(\Omega))$$

for each E in $\mathcal{B}_{Y\times\mathbf{R}}$. Let μ be a quasi-invariant measure on $(X, \{T_t\}_{t\in\mathbf{R}})$. Then there exists a quasi-invariant measure $\tilde{\mu}$ on (Y,S) such that $\mu\circ\beta$ is mutually absolutely continuous with respect to $(\tilde{\mu}\times m_{\mathbf{R}})_\Omega$ by (ii) of Theorem 3.3. We denote by $|\tilde{\mu}|$ the total variation measure of $\tilde{\mu}$, and a function γ in $L^1(|\tilde{\mu}|\times m_{\mathbf{R}})$ is said to be *nonvanishing* if $|\gamma(y,u)| > 0$ for $|\tilde{\mu}|\times m_{\mathbf{R}}$−a.e. (y,u) in $Y\times\mathbf{R}$. We now point out that

$$(4.4)\qquad d\mu'(y,u) = \gamma(y,u)\,d(\tilde{\mu}\times m_{\mathbf{R}})(y,u)$$

for some nonvanishing Borel function $\gamma(y,u)$ in $L^1(|\tilde{\mu}|\times m_{\mathbf{R}})$. In fact, since there is a nonvanishing $v(y,u)$ in $L^1((|\tilde{\mu}|\times m_{\mathbf{R}})_\Omega)$ such that $d\mu\circ\beta = v(y,u)\,d(\tilde{\mu}\times m_{\mathbf{R}})_\Omega$, we have

$$\begin{aligned}\mu'(E) &= \sum_{n\in Z}\int_{\pi(E\cap\sigma^n(\Omega))} v(y,u)\,d(\tilde{\mu}\times m_{\mathbf{R}})_\Omega(y,u)\\ &= \sum_{n\in Z}\int_{E\cap\sigma^n(\Omega)} v\circ\pi(y,u)\,d(\tilde{\mu}\times m_{\mathbf{R}})_\Omega\circ\pi(y,u)\\ &= \sum_{n\in Z}\int_{E\cap\sigma^n(\Omega)} v\circ\pi(y,u)\,p_n(y)\,d(\tilde{\mu}\times m_{\mathbf{R}})(y,u)\end{aligned}$$

for some nonvanishing $p_n(y)$ in $L^1(|\tilde{\mu}|)$. By setting $\gamma(y,u)=v\circ\pi(y,u)p_n(y)$ for any (y,u) in $\sigma^n(\Omega)$, we thus obtain (4.4).

Next we let $h(u)$ be a nonvanishing continuous function on $\mathbf{R}$ which has the following properties:

$$\text{(4.5)}\qquad \begin{cases} |h(u)| \leqslant 1/(1+u^2) \\ \hat{h}(s) = 0 \quad \text{if} \quad s\leqslant 0, \quad \text{and} \\ |\hat{h}(s)| > 0 \quad \text{if} \quad s>0, \end{cases}$$

where $\hat{h}(s)$ denotes the Fourier transform of h in $L^1(\mathbf{R})$. We may observe that if μ is analytic on $(X,\{T_t\}_{t\in\mathbf{R}})$, then $hd\mu'$ is orthogonal to any bounded analytic function $G(y,u)$ on $(Y\times\mathbf{R},\{U_t\}_{t\in\mathbf{R}})$, so especially $hd\mu'$ is also an analytic measure. In fact, let $G'(y,u)=G(y,u)h(u)$. Then $G'(y,u)$ is analytic and satisfies that $\sup_{y\in Y}|G'(y,u)| = O(u^2)$ as $|u|\to\infty$. So it can be seen that

$$\begin{aligned}\int_{Y\times\mathbf{R}} G'(y,u)\,d\mu'(y,u) &= \sum_{n\in Z}\int_{\sigma^n(\Omega)} G'(y,u)\,d\mu\circ\beta\circ\pi(y,u)\\ &= \sum_{n\in Z}\int_{\Omega} G'\sigma^{-n}(y,u)\,d\mu\circ\beta(y,u)\\ &= \int_{\Omega}\sum_{n\in Z} G'\sigma^{-n}(y,u)\,d\mu\circ\beta(y,u)\\ &= \int_X\sum_{n\in Z} G'\sigma^{-n}\circ\beta^{-1}(x)\,d\mu(x).\end{aligned}$$

We set $\phi(x)=\sum_{n\in Z}G'\sigma^{-n}\beta^{-1}(x)$. Then it is easy to see that $\phi(x)$ is a bounded analytic Borel function on X, whose spectrum, $\mathrm{Sp}(\phi)$, is positive. Since μ is analytic, we conclude that $\int_X\phi(x)\,d\mu(x)=0$ (cf. [3; Section 4. (24) and (25)]).

We now set $\rho(u)=1/\pi(1+u^2)$, and denote by $H^p(\rho du)$, $1\leqslant p\leqslant\infty$, the Hardy space on $\mathbf{R}$ associated with the representing measure ρdu.

PROPOSITION 4.1. *Let μ be a quasi-invariant measure on $(X,\{T_t\}_{t\in\mathbf{R}})$, and let ψ be a nonvanishing function in $L^\infty(|\mu|)$. Suppose that $\psi^n d\mu$ is analytic for $n=1,2,\ldots$. Then the function of u, $\psi(x+u)$, belongs to $H^\infty(\rho du)$ for $\mu-$a.e. x in X.*

Proof. Let $d\mu' = \gamma(y,u)\,d(\tilde{\mu}\times m_{\mathbf{R}})$ be as in (4.4), and let $h(u)$ be a continuous function on $\mathbf{R}$ with the properties in (4.5). If we define $U(y,u) = \psi\circ\beta\circ\pi(y,u)$ for any (y,u) in $Y\times\mathbf{R}$, then $(\psi^n d\mu)' = U^n(y,u)\gamma(y,u)\,d(\tilde{\mu}\times m_{\mathbf{R}})$. For any $p(y)$ in $C(Y)$ and any positive rational r, it follows from the above remark that

$$\int_Y p(y)\left[\int_{-\infty}^{\infty} e^{iru}h(u)\,U^n(y,u)\gamma(y,u)\,du\right]d\tilde{\mu}(y) = 0,$$

because the function of (y,u), $p(y)e^{iru}$, is bounded analytic on $(Y\times\mathbf{R}, \{U_t\}_{t\in\mathbf{R}})$. Since $p(y)$ is arbitrary, there is a $\tilde{\mu}$-null set $N(r,n)$ such that

$$\int_{-\infty}^{\infty} e^{iru}h(u)\,U^n(y,u)\gamma(y,u)(1+u^2)\rho(u)\,du = 0 \tag{4.6}$$

for each y in $Y\backslash N(r,n)$. Let $N = \bigcup_{r,n} N(r,n)$. Then it follows from (4.6) that the function of u, $U^n(y,u)\gamma(y,u)h(u)(1+u^2)$, belongs to $H^1(\rho du)$ for each y in $Y\backslash N$ and for $n = 1,2,\ldots$. Therefore, by [5; Chapter VII, Lemma 8.1], we see that the function of u, $U(y,u)$, belongs to $H^\infty(\rho du)$. Notice that $U(y,u) = \psi(y+u)$ for any (y,u) in $Y\times\mathbf{R}$. For the definition (3.7) of $\tilde{\mu}$, it can be easily seen that the function of u, $\psi(x+u)$, belongs to $H^\infty(\rho du)$ for $\mu -$a.e. x in X. □

The following lemma is a weak version of [9; Theorem I].

LEMMA 4.2. *Let ν be a ergodic representing measure for $\mathfrak{A}$. Then $\mathfrak{A}$ is a weak-$*$ Dirichlet algebra in $L^\infty(\nu)$, and the following assertions are equivalent for a function ϕ in $L^\infty(\nu)$:*

(i) *ϕ belongs to $H^\infty(\nu)$, and*

(ii) *for $\nu -$a.e. x in X, the function of u, $\phi(x+u)$, belongs to $H^\infty(\rho du)$.*

Proof. It was shown in [13; Theorem 3] that if ν is an ergodic representing measure for $\mathfrak{A}$, then $\mathfrak{A}$ is a weak- $*$ Dirichlet algebra in $L^\infty(\nu)$. So it suffices to show the equivalence of (i) and (ii). Since ν and $\nu*\rho$ are mutually absolutely continuous and $\nu*\rho$ is a representing measure for $\mathfrak{A}$, ν and $\nu*\rho$ lie in the same Gleason part ([5; Chapter VI, Section 2]). Hence there is a positive b with $b>1$ such that $b^{-1} < \dfrac{d\nu*\rho}{d\nu} < b$, where $\dfrac{d\nu*\rho}{d\nu}$ denotes the Radon-Nikodym derivative. We denote by $\mathcal{H}$ the family of all ϕ *in* $L^\infty(\nu)$ which satisfy the property (ii). We now claim that $H^2(\nu)$ is the closure $\mathcal{H}^2$ of $\mathcal{H}$ in $L^2(\nu)$. It is easy to see that $\mathcal{H}^2$ contains $H^2(\nu)$. On the other hand, suppose that ϕ in $\mathcal{H}^2$ is orthogonal to $H^2(\nu)$. Since there is a sequence ϕ_n in $\mathcal{H}$ such that $\|\phi_n - \phi\|_2 \to 0$, we have

$$\int_X\left[\int_{-\infty}^{\infty} |\phi_n(x+u)-\phi(x+u)|^2\rho(u)\,du\right]d\nu(x) = \int_X |\phi_n(x)-\phi(x)|^2\,d\nu*\rho(x)$$
$$\leqslant b\|\phi_n-\phi\|_2^2 \to 0$$

Therefore there is a subsequence $\{\phi_j\}$ of $\{\phi_n\}$ such that

$$\int_{-\infty}^{\infty} |\phi_j(x+u)-\phi(x+u)|^2\rho(u)\,du \to 0, \quad \text{as} \quad j\to\infty,$$

for $\nu-$a.e. x in X. So the function of u, $\phi(x+u)$, belongs to $H^2(\rho du)$. Recall that $L^2(\nu)=H^2(\nu)\oplus\bar{H}_0^2(\nu)$ where $H_0^2(\nu)=\{\phi\in H^2(\nu);\ \int\phi\,d\nu=0\}$. Since ϕ belongs to $\bar{H}_0^2(\nu)$, and $\mathcal{H}^2$ contains $H_0^2(\nu)$, the function of u, $\bar{\phi}(x+u)$, also belongs to $H^2(\rho du)$. Hence ϕ is invariant, because every real-valued function in $H^2(\rho du)$ is constant. Since ν is ergodic, this implies that $\phi\equiv 0$, thus we have $H^2(\nu)=\mathcal{H}^2$. By the same argument as above, we may see easily that $\mathcal{H}$ contains $\mathcal{H}^2\cap L^\infty(\nu)$. We therefore obtain $\mathcal{H}=H^\infty(\nu)$, so the equivalence of (i) and (ii) is established. □

THEOREM 4.3. *Let $\mathfrak{A}$ be the uniform algebra of all continuous analytic functions induced by a minimal flow $(X,\{T_t\}_{t\in\mathbf{R}})$. Then we have:*

(i) *$\mathfrak{A}$ is a maximal closed subalgebra of $C(X)$, and*

(ii) *if ν is an ergodic representing measure for $\mathfrak{A}$, then $H^\infty(\nu)$ is a weak- $*$ maximal closed subalgebra of $L^\infty(\nu)$.*

Proof. (i) Let $\mathfrak{C}$ be any subalgebra of $C(X)$ that contains $\mathfrak{A}$. Suppose that $\mathfrak{C}$ is not uniformly dense in $C(X)$. Then there is a non-zero measure μ in $M(X)$ such that μ is orthogonal to $\mathfrak{C}$. For any non-zero ϕ in $\mathfrak{C}$ and $n=0,1,2,\ldots$, $\phi^n d\mu$ is orthogonal to $\mathfrak{A}$. This shows that $\phi^n d\mu$ is analytic, so it is quasi-invariant on $(X,\{T_t\}_{t\in\mathbf{R}})$ by [3; Theorem 3]. It can be observed that ϕ is non-vanishing as a function in $L^\infty(|\mu|)$. Therefore we may find an x in X such that the function of u, $\phi(x+u)$, belongs to $H^\infty(\rho du)$ by Proposition 4.1. Since ϕ is continuous, it follows from the minimality that $\phi*f=0$ for any f in $L^1(\mathbf{R})$ with $\hat{f}(s)=0$ on $[0,\infty)$. This shows that ϕ belongs to $\mathfrak{A}$, thus we have $\mathfrak{A}=\mathfrak{C}$.

(ii) Let $\mathfrak{D}$ be any subalgebra of $L^\infty(\nu)$ that contains $H^\infty(\nu)$. Suppose that $\mathfrak{D}$ is not weak- $*$ dense in $L^\infty(\nu)$. Then we can find a nonzero function ψ in $L^1(\nu)$ such that $\psi d\nu$ is orthogonal to $\mathfrak{D}$. In the same fashion as above, it can be seen that $\phi^n\psi d\nu$ is analytic for any nonzero ϕ in $\mathfrak{D}$ and for $n=0,1,2,\ldots.$ Since ν is ergodic, we observe that $\phi^n\psi d\nu$ and ν are mutually absolutely continuous. So it follows from Proposition 4.1 that the function of u, $\phi(x+u)$, belongs to $H^\infty(\rho du)$ for $\nu-$a.e. x in X. Hence ϕ belongs to $H^\infty(\nu)$ by Lemma 4.2, and we have $H^\infty(\nu)=\mathfrak{D}$. This completes the proof. □

5. Remarks. (a) Recall that the main theorem in [3] is an extension of F. and M. Riesz theorem, which states that every analytic measure is quasi-invariant. We remark that Theorem 3.3 enables us to provide another proof of this result. In fact, suppose that μ is analytic on $(X,\{T_t\}_{t\in\mathbf{R}})$. Let μ' and h be as in (4.3) and (4.5) respectively. Then we have already seen that $hd\mu'$ is analytic on $(Y\times\mathbf{R},\{U_t\}_{t\in\mathbf{R}})$. By the argument in [2], we may verify without difficulty that $hd\mu'$ is quasi-invariant on $(Y\times\mathbf{R},\{U_t\}_{t\in\mathbf{R}})$. However it is easy to see that this fact implies that μ is quasi-invariant on $(X,\{T_t\}_{t\in\mathbf{R}})$.

(b) We do not know whether $\mathfrak{A}$ is a Dirichlet algebra without the assumption that $(X,\{T_t\}_{t\in\mathbf{R}})$ is strictly ergodic (cf. [7] and [8; Section 6]). However it is easy to verify that this problem may be reduced to the following one:

Does $\mathfrak{A}$ separate the probability invariant measures on $(X,\{T_t\}_{t\in\mathbf{R}})$?

Since Theorem 3.3 provides a representation of invariant measures, it may be useful for dealing with this problem.

REFERENCES

1. W. Ambrose, *Representation of ergodic flows.* Ann. of Math. (2) 42 (1941), 723–739.
2. K. deLeeuw and I. Glicksberg, *Quasi-invariance and analyticity of measures on compact groups.* Acta Math. 109 (1963), 179–205.
3. F. Forelli, *Analytic and quasi-invariant measures.* Acta Math. 118 (1967), 33–59.
4. ———, *A maximal algebra.* Math. Scand. 30 (1972), 152–158.
5. T. Gamelin, *Uniform algebras*, Prentice-Hall, Englewood Cliffs, N.J. 1969.
6. H. Helson, *Analyticity on compact abelian groups.* Algebras in Analysis (Proc. Instructional Conf. and NATO Advanced Study Inst., Birmingham, 1973), pp. 1–62, Academic Press, London, 1975.
7. P. S. Muhly, *Function algebras and flows.* Acta Sci. Math. (Szeged) 35 (1973), 111–121.
8. ———, *Function algebras and flows. II.* Ark. Mat. 11 (1973), 203–213.
9. ———, *Function algebras and flows. III.* Math. Z. 136 (1974), 253–260.
10. ———, *Ergodic Hardy spaces and duality.* Michigan Math. J. 25 (1978), no. 3, 317–323.
11. W. Rudin, *Real and complex analysis.* MacGraw-Hill, New York, 1966.
12. T. P. Srinivasan and J.-K. Wang, *Weak- * Dirichlet algebras.* Function Algebras (Proc. Internat. Sympos. on Function Algebras, Tulane Univ., 1965), pp. 216–249, Scott-Foresman, Chicago, Ill., 1966.
13. J.-I. Tanaka, *A note on Helson's existence theorem.* Proc. Amer. Math. Soc. 69 (1978), no. 1, 87–90.

Department of Mathematics
Tsuru University
Tsuru City, Yamanashi-ken
402 Japan