LINEAR EXTENSION OPERATORS FOR ENTIRE FUNCTIONS

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Dedicated to the memory of David L. Williams

1. Introduction. Let $\{a_j\}$ be an infinite, discrete sequence of distinct points in the complex plane $\mathbf{C}$. It is well-known that given any other sequence $\{\lambda_j\}$ of complex numbers, there exists an entire function $\lambda(z)$ such that $\lambda(a_j)=\lambda_j$, $j=1,2,\ldots$ (cf. [1], ex. 1, p. 197 or [13], Th. 15.13, p. 237). It is also well-known that there can exist no one "formula" for $\lambda(z)$ in terms of the λ_j which works for *all* choices of the sequence λ_j. To say this precisely, let $A(\mathbf{C})$ denote the space of entire functions, with the usual topology of uniform convergence on compact sets, and let $A(\{a_j\})$ denote the space of all sequences of complex numbers $\{\lambda_j\}$, with the usual topology of pointwise convergence in each slot. Let $\rho: A(\mathbf{C}) \to A(\{a_j\})$ denote the restriction map, $\rho(\lambda)=\{\lambda(a_j)\}$, determined by the sequence $\{a_j\}$. Then ρ is a linear, continuous, onto map, but ρ has no linear, continuous right inverse. There is no "extension map", $E: A(\{a_j\}) \to A(\mathbf{C})$ such that E is linear, continuous, and $\rho\circ E=$ identity (cf. [12], p. 162).

On the other hand, if we formulate the corresponding problems with growth conditions, the answers may be different. For example, if we let the sequence $\{a_j\}$ be the integers, $\mathbf{Z}=\{\ldots,-2,-1,0,1,2,\ldots\}$, the analytic functions be those of exponential growth,

$$A_{\exp}(\mathbf{C}) = \{f\in A(\mathbf{C}) : |f(z)| \leqslant A\exp(B|z|),\ \text{some } A,B>0\}, \tag{1.1}$$

and the space of sequences $\{\lambda_j\}$ be those of exponential growth,

$$A_{\exp}(\mathbf{Z}) = \{(\lambda_j) : |\lambda_j| \leqslant A\exp(B|z|),\quad j=0,\pm 1,\ldots\}$$

then again the restriction map $\rho: A_{\exp}(\mathbf{C}) \to A_{\exp}(\mathbf{Z})$ is onto but this time there is a continuous, linear right inverse for ρ. For example,

$$E(\{\lambda_j\})(z) = \sum_{-\infty}^{+\infty} \lambda_j \left[\frac{(-1)^j \sin \pi z}{\pi(z-j)}\right]\left(\frac{z}{j}\right)^{|j|} \tag{1.2}$$

The results presented here grew out of an attempt to understand the difference between these two situations. The difference is not simply one of growth restrictions. If the exponential type growth condition is replaced by order 1 growth condition, then again no right inverse exists. (See Section 4.) It is also not simply that one space is a Fréchet space $(A(\mathbf{C}), A(\{a_j\}))$ and the others are duals of Fréchet spaces (see Sections 3, 5).

For some special varieties, like algebraic varieties or sub-submanifolds (of strictly pseudoconvex domains) in general position, linear extension operators are known to exist ([6], [8]). However, it may be that the existence of general "extension formulas" for all zero sets is closely related to the existence of a right inverse for the

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$\bar{\partial}$ operator. To see this, one need only recall the standard method of constructing extensions of functions defined on analytic varieties. For the simple cases discussed above, there are two main steps. As a preliminary, find an entire function $F \in A(\mathbf{C})$ (or $A_{\exp}(\mathbf{C})$) with simple zeros of the points $\{a_j\}$.

STEP 1. Construct a "good" C^∞ extension (with exponential bounds, in the second case) ϕ of $\{\lambda_j\}$. For example, let $\phi \in C(\mathbf{C})$ have the form

$$\phi = \sum_j \lambda_j \chi\left(\frac{z-a_j}{\epsilon_j}\right)$$

where $0 \leqslant \chi \leqslant 1$, $\chi = 1$ for $|z| \leqslant 1$, and $\chi = 0$ for $|z| \geqslant 2$, and the ϵ_j are so small that the discs $|z-a_j| \leqslant 2\epsilon_j$ are disjoint.

STEP 2. Construct the analytic extension $\lambda(z)$ in the form $\lambda(z) = \phi(z) - u(z)F(z)$, where $u \in C^\infty$ (with exponential bounds, in the second case). This amounts to solving $0 = \bar{\partial}\lambda = \bar{\partial}\phi - F\bar{\partial}u$, or $\bar{\partial}u = \omega$, where $\omega = \bar{\partial}\phi / F$ is a $C^\infty(0,1)$ form.

For the cases outlined above, Step 1 can clearly be done in a continuous linear way. Thus, the obstruction to finding a continuous linear right inverse for the restriction map ρ is in finding a continuous linear right inverse for the $\bar{\partial}$-operator. It is well-known that no such inverse exists for $\bar{\partial}$, as a map from C^∞ to $C^\infty(0,1)$ forms (see e.g. [8], p. 154). However, we will prove here that with exponential type bounds, there is a continuous linear right inverse for $\bar{\partial}$ (Theorem 4.2).

We know of no examples where continuous, linear extension operators exist for all "reasonable" analytic subvarieties, and where the $\bar{\partial}$-operator fails to have a continuous linear right inverse.

Let us mention that in the space C^∞, it is known that many linear partial differential operators have no right inverses ([18], Appendix C). On the other hand, Palamodov has shown [10] that, while the $\bar{\partial}$-operator from functions to $(0,1)$ forms has no right inverse, it does have a continuous linear right inverse as an operator from (p,q) forms to $(p,q+1)$ forms, $q \geqslant 1$. He has also given some general criteria for existence of projections in Fréchet spaces, but they do not seem to apply in the cases studied here.

Our main results are given in Sections 2 and 3, where we discuss, in the framework of analytically uniform spaces, criterion for the existence of right inverses for overdetermined systems of partial differential operators. The point of applying this theory is to allow a reduction to a finite dimensional problem by studying the action of the right inverse on exponential functions. The question of existence then reduces to a question of the existence of entire functions satisfying certain growth and compatibility properties. In simple cases, such as for the $\bar{\partial}$-operator, it can then be decided whether the appropriate entire functions exist. We carry out the general outline in Section 2, and in Section 3 give the specific growth conditions for the special case of the $\bar{\partial}$-operator. In Section 4, the construction of the required entire function is made, and in Section 5, applications to the interpolation problem are discussed.

2. Projections in analytically uniform spaces. It is convenient to use the theory of *analytically uniform spaces* to discuss existence of right inverses. These spaces were first systematically studied by L. Ehrenpreis and are discussed extensively in [7]. A more systematic study was given by Berenstein and Dostal in [2]. We recall some of the definitions and notations. Let W be a reflexive locally convex topological vector space which contains, for some $n \geqslant 1$, the exponential functions $x \to e_z(x)$ given by

$$e_z(x) = \exp(iz \cdot x), \quad z \in \mathbf{C}^n, \quad x \in \mathbf{R}^n, \tag{2.1}$$

$(z \cdot x = z_1 x_1 + \cdots + z_n x_n)$. We further suppose that the linear span of these exponential functions is dense in W. If $L \in W'$, the space of continuous linear functionals on W, then the Fourier transform of L is defined by

$$\hat{L}(z) = L(e_z). \tag{2.2}$$

Since the span of $\{e_z : z \in \mathbf{C}^n\}$ is dense in W, $\hat{L}$ uniquely determines L. Thus, the space

$$\hat{W}' = \{\hat{L} : L \in W'\} \tag{2.3}$$

is isomorphic to W'. It is also assumed that the map $z \to e_z$ is analytic; equivalently, the functions $\hat{L}(z)$ are entire functions on $\mathbf{C}^n$. And, the main assumption is that there exists a class $\mathcal{K}$ of continuous functions, $k : \mathbf{C}^n \to [0, +\infty]$ such that for each $k \in \mathcal{K}$,

$$\|\hat{L}\|_k = \sup\{|\hat{L}(z)|/k(z) : z \in \mathbf{C}^n\} \tag{2.4}$$

defines a continuous seminorm on $\hat{W}'$ and, further, that the topology determined by these seminorms $\hat{W}'$ coincides with the strong topology on W' (carried over to $\hat{W}'$).

Let $P_j(D) = \sum a_{\alpha j} D^\alpha$ denote linear, constant coefficient partial differential operators on $\mathbf{R}^n$ and let $P_j(z) = \sum a_{\alpha j} z^\alpha$ be the corresponding polynomials, $1 \leqslant j \leqslant N$. Here we are following the usual notations, $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$, $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$, where $D_j = (1/i)\,\partial/\partial x_j$. We will denote by A, the null space of the system of equations $\{P_j(D)\}$.

$$A = \{u \in W : P_j(D)u = 0, \ \ 1 \leqslant j \leqslant N\} \tag{2.5}$$

and let V denote the algebraic variety

$$V = \{z \in \mathbf{C}^n : P_j(z) = 0, \ \ 1 \leqslant j \leqslant N\} = \{z : e_z \in A\}.$$

We will formulate our results for operators and a.u. spaces satisfying the following condition.

(*) Every $u \in A$ has a representation in the form $u(x) = \int_V e^{iz \cdot x}\, d\mu(z)/k(z)$ for some finite Borel measure μ with support in V and some $k \in \mathcal{K}$.

Actually, every a.u. space known to us has property (*) provided that derivatives of finite measures are allowed, instead of only the measures μ. Our results can be improved to cover these cases, and also the vector valued cases, when the $\{P_j(D)\}$

are replaced by a matrix of operators. However, this introduces extra technical complications so we won't discuss them here. The cases covered by (*) will suffice for our applications.

The existence of a continuous linear right inverse for a linear map is equivalent to the existence of a continuous linear projection onto its null space. We will study the latter condition.

Let $\pi: W \to A$ be a continuous linear projection onto A; i.e. $\pi^2 = \pi$ and $\pi(u) = u$ for all $u \in A$. We can then consider the *characteristic function* of the projection,

$$G(z,x) = \pi(e_z)(x). \tag{2.6}$$

This function G completely determines the projection π.

THEOREM 2.1. *Let W be an* a.u. *space and $\pi: W \to A$ a projection onto the space of solutions of $P_j(D)u = 0$, $1 \leqslant j \leqslant N$. Then*

(i) *$x \to G(z,x) \in A$ for each $z \in \mathbf{C}^n$;*

(ii) *$z \to L_x(G(z,x))$ belongs to $\hat{W}'$ for each $L \in W'$,*

(iii) *$G(z,x) = e_z(x) = \exp(iz\cdot x)$ if $z \in V$;*

(iv) *for each continuous seminorm $\| \ \|$ on W and each $k \in \mathcal{K}$, there exists a constant $C > 0$ such that*

$$\|G(z,x)\| \leqslant Ck(z), \quad z \in \mathbf{C}^n. \tag{2.7}$$

Conversely, if there exists a mapping G satisfying (i)–(iv) *and if* (*) *holds, then there exists a unique projection which induces G by the formula* (2.6).

Proof. First suppose $\pi: W \to A$ exists. Then assertion (i) follows from the definition of G. If $L \in W'$, then $L \circ \pi \in W'$ so (ii) follows from the analyticity of the map $z \to e_z$ of $\mathbf{C}^n$ into W. Property (iii) is exactly the condition $\pi(f) = f$ for $f = e_z \in A$. And (iv) is a consequence of the continuity of π. For, if $k \in \mathcal{K}$, then the set $B = \{e_z / k(z) : z \in \mathbf{C}^n\} \subset W$ is a bounded subset of W, since for each $L \in W'$,

$$\sup\{|L(f)| : f \in B\} = \sup\{|\hat{L}(z)|/k(z) : z \in \mathbf{C}^n\} < +\infty.$$

Thus, $\pi(B)$ is also a bounded subset of W. Consequently, given any continuous seminorm $\| \ \|$ on W, there is a constant $C > 0$ such that $\|f\| \leqslant C$ for all $f \in \pi(B)$, and this is exactly assertion (iv).

To prove the converse part of the Theorem, recall the representation theorem for a.u. spaces ([7], Theorem 1.5). For each $f \in W$, there exists a finite Borel measure μ on $\mathbf{C}^n$ and $k \in \mathcal{K}$ such that $f(x) = \int e^{iz\cdot x} d\mu(z)/k(z)$. Convergence of the integral is in the topology of W; μ and k are not unique. Define $\pi: W \to A$ by $\pi(f)(x) = \int G(z\cdot x)\, d\mu(z)/k(z)$. Because of (iv), the integral converges (in W). Also, π is well-defined, because the exponentials $z \to \exp(iz\cdot x)$, $x \in \mathbf{R}^n$, have dense linear span in W' (cf. [2], Chapter 1). Thus, if μ, k are such that $\int e_z\, d\mu(z)/k(z) \equiv 0$, then

$$0 = \int L(G(z,x))\, d\pi(z)/k(z) = L\left(\int G(z,x)\, d\mu(z)/k(z)\right)$$

for all $L \in W'$, so $\int G(z,x)\, d(z)/k(z) \equiv 0$. Because of (iii) and (*), π is the identity on A. And, again by (iv), π is a bounded linear operator, hence continuous, because W is reflexive. This completes the proof. □

3. Case of the $\bar{\partial}$-operator. We want to specialize Theorem 2.1 to the cases of interest for our problem. First, for the a.u. space W it will be convenient to choose a space of infinitely differentiable functions on $\mathbf{R}^n$ determined as follows. Let Φ denote a family of nonnegative functions $\phi : \mathbf{R}^n \to [0, +\infty]$. For each $\phi \in \Phi$ and each integer $m \geqslant 0$, let

$$\|f\|_{m,\phi} = \sup\{|D^j f(x)| \exp(-\phi(x)) : x \in \mathbf{R}^n,\ |j| \leqslant m\}$$

(where the $D^j = \partial^{|j|}/\partial x_1^{j_1}, \ldots, \partial x_n^{j_n}$ are the usual partial derivatives), let $\mathcal{E}(\phi, m) = \{f \in C^\infty(\mathbf{R}^n) : \|f\|_{m,\phi} < +\infty\}$ and let

$$\mathcal{E}(\Phi) = \bigcap_{m=0}^{+\infty} \left(\bigcup_{\phi \in \Phi} \mathcal{E}(\phi, m) \right). \tag{3.1}$$

Under suitable conditions on Φ (such as Φ1–Φ3, listed later), the space $\mathcal{E}(\Phi)$ is an analytically uniform space. (See [7], Chapter 5, [2], or [16] for the proofs of such results.) We note four interesting examples, which are the only ones we will treat explicitly. The last example, (3.5), is somewhat different from the first three.

(3.2) $\Phi =$ all functions which are $+\infty$ outside a compact set,
$\mathcal{E}(\Phi) = C^\infty(\mathbf{R}^n)$

(3.3) $\Phi = \{j|x| : j = 1, 2, \ldots\}$
$\mathcal{E}(\Phi) = C^\infty(\mathbf{R}^n, \exp) =$ all C^∞ functions f on $\mathbf{R}^n$, each of whose partial derivatives is of exponential type. That is, for each $j = (j_1, \ldots, j_n)$ there exist constants A_j, B_j such that $|D^j f(x)| \leqslant A_j \exp(B_j |x|)$, $x \in \mathbf{R}^n$,

(3.4) $\Phi = \{\phi$ convex on $\mathbf{R}^n : \phi(x) = O(|x|^{1+\epsilon})$, $|x| \to +\infty$, for each $\epsilon > 0\}$.
$\mathcal{E}(\Phi) =$ all C^∞ functions on $\mathbf{R}^n$, each of whose partial derivatives is of order $\leqslant 1$.

(3.5) $\mathbf{R}^n = \mathbf{R}^{2p} = \mathbf{C}^p = \{w = u + iv : u, v \in \mathbf{R}^p\}$.
$\Phi = \{j(|v| + \log(1 + |w|)) : j = 1, 2, \ldots\}$
$\mathcal{E}(\Phi) =$ all C^∞ functions on $\mathbf{R}^{2p}$, each of whose derivatives is of exponential type and of polynomial growth in the real directions.

Some technical conditions we impose on Φ which are sufficient that $\mathcal{E}(\Phi)$ be an a.u. space are the following (see e.g. [16]).

(Φ1) if $\phi_1, \phi_2 \in \Phi$, there exists $\phi_3 \in \Phi$, ϕ_3 convex on the set $\{\phi_3 < +\infty\}$ and such that $\phi_3 \geqslant \max(\phi_1, \phi_2)$.

(Φ2) for each $A > 0$, there exists $\phi \in \Phi$ such that $A|x| \leqslant \phi(x)$.

(Φ3) If $\phi \in \Phi$, then there exists $\eta > 0$ and $\phi_1 \in \Phi$ such that $\Phi(x+y) + \eta|x| \leqslant \phi_1(x)$ for all $x, y \in \mathbf{R}^n$, $|y| \leqslant 1$.

Note that the last example (3.5) does not satisfy properties Φ2 or Φ3. Strictly speaking, it is not an a.u. space, since it contains the exponentials $e^{iz \cdot w}$ only for $z = (z_1, z_2)$ pure real (i.e. $z_1, z_2 \in \mathbf{R}^p$). Its analysis, therefore, will have to be carried out outside the general framework.

In case Φ satisfies (Φ1), (Φ2), (Φ3) above it is easy to write down the class $\mathcal{K}$ which gives the a.u. structure for $\mathcal{E}(\Phi)$. Namely, recall that if ϕ is a convex function on $\mathbf{R}^n$, then the *conjugate* convex-function of ϕ is

$$\phi^*(y) = \sup\{x \cdot y - \phi(x) : x \in \mathbf{R}^n\}. \tag{3.6}$$

The class $\mathcal{K}$ consists of all functions $k(z) = (1+|z|)^{\omega(|z|)} \exp(\phi^*(\operatorname{Im} z))$ where $\operatorname{Im} z \in \mathbf{R}^n$ denotes the imaginary part of z, and $\omega(|z|)$ is any function which increases to $+\infty$ as $|z| \to +\infty$.

Thus, we have determined the space $W = \mathcal{E}(\Phi)$ in which Theorem 2.1 will be applied. To consider the case of analytic functions we must restrict the underlying $\mathbf{R}^n$ on which the functions $f \in \mathcal{E}(\Phi)$ are defined to be even dimensional, $\mathbf{R}^n = \mathbf{R}^{2p} = \mathbf{C}^p = \{w = u + iv : u, v \in \mathbf{R}^p\}$, and the differential operators to be the $\bar{\partial}$-operator, $\partial/\partial \bar{w}_j = \frac{1}{2}(\partial/\partial u_j + \partial/\partial v_j)$. Then the space A of Section 2 is

$$\begin{aligned} A = A(\Phi) &= \{f \in \mathcal{E}(\Phi) : \bar{\partial} f = 0\} \\ &= \{f \in \mathcal{E}(\Phi) : f \text{ is analytic on } \mathbf{C}^p\}. \end{aligned}$$

The cases corresponding to (3.2)–(3.5) are

(3.2′) A: all entire functions

(3.3′) A: all entire functions of exponential type

(3.4′) A: all entire functions of order $\leqslant 1$

(3.5′) A: all entire functions of exponential type with polynomial growth on the real subspace of $\mathbf{C}^p$.

The analytic variety $V = \{z \in \mathbf{C}^{2p} : e_z \in A\}$ is the same in all cases:

$$V = \{z = (z_1, z_2) \in \mathbf{C}^{2p} : z_2 = iz_1\}$$

i.e. $e^{iz \cdot w} = e^{i(z_1 u + z_2 v)}$ is an analytic function of $w = u + iv$ if and only if $z_2 = iz_1$.

Thus, the characteristic function of the projection π, if it exists, is

$$G(z_1, z_2, w) = \pi_w(e^{i(z_1 u + z_2 v)}), \quad w = u + iv \in \mathbf{C}^p. \tag{3.8}$$

The direct translation of Theorem 2.1 to this case is as follows.

THEOREM 3.1. *Suppose* $W = \mathcal{E}(\Phi)$, $P_j = \partial/\partial \bar{w}_j$, *and* $A = A(\Phi)$, *where* Φ *satisfies the conditions* $\Phi 1$, $\Phi 2$, $\Phi 3$. *Then there exists a continuous linear projection of* $\mathcal{E}(\Phi)$ *onto* $A(\Phi)$ *if and only if there exists an entire function* $G : \mathbf{C}^{3p} \to \mathbf{C}$ *such that*

(i) $G(z_1, z_2, w) = e^{iz_1 w}$ *for* $z_2 = iz_1$; *and*

(ii) *given* $\psi \in \Phi$, *there exists* $\phi \in \Phi$, $C > 0$, $m > 0$ *such that* $|G(z_1, z_2, w)| \leqslant C(1+|z|)^m \exp(\phi(w) + \psi^*(\operatorname{Im} z_1, \operatorname{Im} z_2))$.

Proof. The conditions of Theorem 2.1 have only to be written in this context. We omit the direct verification. □

To apply Theorem 3.1 we then have to analyze the growth condition to find out when G exists. The simplest case is Example (3.2) where, as we have already remarked, it is known that no such projection exists. We won't give the details of the analysis for this case, but they proceed as follows. First, the growth condition of Theorem 3.1 is equivalent to

$$|G(z_1, z_2, w)| \leqslant (1+|z|)^m \exp[\alpha(|w|)(|\operatorname{Im} z_1| + |\operatorname{Im} z_2|) + \beta(|w|)].$$

This then implies a stronger estimate

$$|G(z_1, z_2, w)| \leqslant C(1+|z|)^m \exp[C(|\operatorname{Im} z_1| + |\operatorname{Im} z_2|) + \beta(|w|)]$$

for some $C > 0$ and some function $\beta(|w|)$. However, the latter estimate is incompatible with $G(z, iz, w) = \exp(iz \cdot w)$, so G cannot exist. □

4. We will consider two special cases in this section, the cases (3.3) and (3.4) of Section 3. After treating these cases we will discuss some more general examples in which similar arguments can be given. The only differences between the special cases and the general cases is technical, so we won't give the proofs in the general cases.

We first take up the case (3.3), of functions of exponential type. To interpret the growth condition (ii) in this case, recall that $\Phi = \{j|w| : j = 1, 2, \dots\}$ so the functions ψ^*, $\psi \in \Phi$, are equal to $+\infty$ outside some large compact set. Therefore, the condition (ii) is equivalent to:

(4.1) for each compact set K in $\mathbf{R}^{2p}$, there exists $C > 0$, $m > 0$ such that $|G(z_1, z_2, w)| \leqslant C(1+|z|)^m \exp(c|w|)$ for all $w \in \mathbf{C}^p$ and all z with $\operatorname{Im} z \in K$.

THEOREM 4.1. *There exists an entire function G on $\mathbf{C}^{3p}$ such that $G(z_1, z_2, w) = e^{iz_1 \cdot w}$ for $z_2 = iz_1$ and such that for some $m \geqslant 0$, $c > 0$*

$$|G(z_1, z_2, w)| \leqslant C(1+|z|+|w|))^m \exp(C|w| \exp(C|\operatorname{Im} z|)).$$

Such a function clearly satisfies (4.1). *Thus, Theorem* 3.1 *implies the existence of a right inverse for* $\bar{\partial}$.

THEOREM 4.2. *Let $\Phi = \{j|w| : j = 1, 2, \dots\}$. Then there is a continuous linear projection $\pi : \mathcal{E}(\Phi) \to A(\Phi)$.*

Proof of Theorem 4.1. We have the function $G(z_1, z_2, w)$ defined on the variety $\{(z_1, iz_1, w)\} \subset \mathbf{C}^{3p}$ and wish to extend it to all of $\mathbf{C}^p$ so that the estimate of the Theorem holds. To this end consider the plurisubharmonic function on $\mathbf{C}^{3p}$ $\beta(z, w) = (1+|w|) \exp(|\operatorname{Im} z_1| + |\operatorname{Im} z_2|)$. Note that if $z_2 = iz_1$ then $|e^{iz_1 \cdot w}| \leqslant \exp(|w||z_1|) \leqslant \exp(\beta(z, w))$. Hence, by Theorem 2.3 of [5], or by following directly the extension technique of [9], Theorem 4.4.3, such an extension exists.

Consider next the case (3.2) of functions of order 1. To interpret the growth condition (ii) in this case, recall that Φ is the set of all convex increasing functions on $\mathbf{C}^p$ which satisfy $\phi(w) = o(|w|^{1+\epsilon})$ for each $\epsilon > 0$. Thus, the functions ψ^*, $\psi \in \Phi$ range over the class of all convex functions on $\mathbf{C}^p$ which satisfy for each $N > 0$. $|w|^N = O(\psi^*(w))$ (see e.g. [15]). Consequently, the growth condition (ii) of Theorem (3.1) implies:

(4.2) Given any $\epsilon > 0$ and any convex function α on $t \geqslant 0$ such that $t^N = O(\alpha(t))$ for all $N > 0$, there exists $C, m > 0$ such that

$$|G(z, w)| \leqslant C(1+|z|)^m \exp(|w|^{1+\epsilon} + \alpha(|\operatorname{Im} z|)).$$

However, this implies an apparently stronger inequality. □

LEMMA 4.3. *Given any* $\epsilon > 0$, *there exists* C, m, D, N *such that* $|G(z, w)| \leqslant C(1+|z|)^m \exp(|w|^{1+\epsilon} + D|\operatorname{Im} z|^N)$.

Proof. This is a standard argument (cf. [15] or [17]). Assume not. Then there exist points $(z_j, w_j) \in \mathbf{C}^{3p}$ and $\eta > 0$ such that

$$(4.3) \qquad |G(z_j, w_j)| \geqslant j(1+|z_j|)^j \exp(|w_j|^{1+\eta} + j|\operatorname{Im} z_j|^j).$$

Now, the set $\{\operatorname{Im} z_j\}$ cannot be bounded. Otherwise, the set of functions

$$e_{z_j}(w)/(1+|z_j|)^j$$

is bounded in $\mathcal{E}(\Phi)$ and therefore so is the set of functions $G(z_j, w)/(1+|z_j|)^j = \pi(e_{z_j}/(1+|z_j|)^j)$ which forces $|G(z_j, w)| \leqslant C(1+|z_j|)^j \exp(\phi(w))$ for some $\phi \in \Phi$, all $w \in \mathbf{C}^p$, and thus violates (4.3). Similarly, we must have $|w_j| \to +\infty$, because otherwise, the functions $z \to \exp(iz \cdot w_j)/(1+|z_j|)^j$ form a bounded set in $\hat{W}' = \hat{\mathcal{E}}(\Phi)'$. Thus, $|G(z, w_j)| \leqslant C(1+|z_j|)^j \exp(D|\operatorname{Im} z|^N)$ for some $C, D, N > 0$, which violates (4.3) for $z = z_j$ and large j. Thus, we may assume $|w_j| \to +\infty$, $|\operatorname{Im} z_j| \to +\infty$.

Choose a convex increasing function $\alpha(t)$, $t \geqslant 0$, such that $\alpha(|\operatorname{Im} z_j|) \leqslant |\operatorname{Im} z_j|^{j/2}$ for infinitely many j, but $t^N = O(\alpha(t))$, $t \to +\infty$, for every $N > 0$. Then let $\epsilon = \eta/2$ and choose $C, m > 0$ as in (4.2). This contradicts (4.3) for infinitely many large j. Hence, the Proposition must hold. □

We can now make a significant improvement of the estimate.

LEMMA 4.4. *Let* $G(z, w)$ *be an entire function which satisfies the estimates of Lemma* 4.3. *Then there exists a constant* $N > 0$ *such that for each* $\epsilon > 0$, *there exist constants* C, D, m *such that*

$$|G(z, w)| \leqslant C(1+|z|)^m \exp(|w|^{1+\epsilon}) \exp(D|\operatorname{Im} z|^N).$$

That is, the constant N *can be chosen independent of* $\epsilon > 0$.

Proof. The factors $(1+|z|)^m$ present an annoying technical problem which must be circumvented. Otherwise, the argument is straightforward. To eliminate the factors $(1+|z|)^m$, note that $G(z, w)$ can be written as a sum of terms of the form $P(z)\tilde{G}(z, w)$ where $P(z)$ is a polynomial of degree $\leqslant q$, and $\tilde{G}$ satisfies the estimates of Lemma 4.3 with m replaced by $\max(m-q, 0)$. For example (in case $p = 1$, $q = 1$), the decomposition $G(z_1, z_2, w) = z_1[(G(z_1, z_2, w) - G(0, z_2, w))/z_1] + G(0, z_2, w)$ reduces the power of $(1+|z_1|)$ in the estimate by at least 1. A tedious induction argument, which we omit, shows the procedure can always be carried out. Thus, if it were not for the possibility that $m = m(\epsilon) \to +\infty$ as $\epsilon \to 0$, we could directly reduce to the case $m = 0$.

Choose constants C, m, D, N so that the inequality of Lemma 4.3 holds for $\epsilon=1$. Then choose C_1, m_1, D_1, N so that the estimate of Lemma 4.3 holds with another, fixed (small) value of $\epsilon>0$. Make the decomposition outlined in the preceding paragraph with $q=\max(m,m_1)$. Thus, we have $G(z,w)=\sum P_i(z,w)G_i(z,w)$ where (possibly different constants)

$$|G_i(z,w)| \leqslant C\exp(|w|^2)\exp(D|\operatorname{Im} z|^N) \tag{4.4}$$

$$|G_i(z,w)| \leqslant C_1\exp(|w|^{1+\epsilon})\exp(D_1|\operatorname{Im} z|^{N_1}) \tag{4.5}$$

Dropping the index i, set

$$\begin{aligned} u(z,w) &= \sup\{\log|G(z+x,Tw)| : x\in\mathbf{R}^{2p},\ T \text{ an orthogonal map of } \mathbf{C}^p\} \\ &= u(\operatorname{Im} z, w). \end{aligned}$$

The function u is plurisubharmonic, so if $|w|=e^x$, $y=\operatorname{Im} z$, $h(y,x)=u(z,w)$ then the function h is convex on $\mathbf{R}^{2n}\times\mathbf{R}$. Therefore, if $\eta>1$ is given, then from $(y,x)=(1-1/\eta)(\eta y/(\eta-1),0)+1/\eta(0,\eta y)$, we have

$$h(y,x) \leqslant (1-1/\eta)h(\eta y/(\eta-1),0)+h(0,\eta y)/\eta$$

But, by (4.4), $h(ay,0)\leqslant a^N D|y|^N+\log(eC)$ and, by (4.5),

$$h(0,\eta x) \leqslant |w|^{\eta(1+\epsilon)}+\log C_1.$$

We conclude that

$$|G_i(z,w)| \leqslant C'\exp(|w|^{\eta(1+\epsilon)})\exp(D'|\operatorname{Im} z|^N).$$

Since $\eta>1$, $\epsilon>0$ are arbitrary, we see that the original function G must satisfy

$$|G(z,w)| \leqslant C(\epsilon)(1+|z|)^{m(\epsilon)}\exp(|w|^{1+\epsilon})\exp(D(\epsilon)|\operatorname{Im} z|^N).$$

In other words, N can be chosen to be independent of $\epsilon>0$. This completes the proof. □

THEOREM 4.5. *There exists no entire function $G(z,w)$ satisfying the conditions of Lemma 4.3 and $G(z_1,iz_1,w)=e^{iz_1\cdot w}$. Consequently, there exists no continuous linear projection $\pi:\mathcal{E}(\Phi)\to A(\Phi)$ where Φ is given in (3.4).*

Proof. Suppose G exists. Choose N as in Lemma 4.4 and then take $\epsilon>0$ so small that

$$1+\epsilon<N/(N-1). \tag{4.6}$$

The function $f(y)=Dy^N$, where $D=D(\epsilon)$, is convex, and for $x>0$, there is a unique $y=y(x)>0$ such that $f(y)-xy$ is minimized at $y(x)$. Namely, the solution of $f'(y)=x$ or $NDy^{N-1}=x$. And, for this value of y, we have $f(y)=xy-Bx^q$ where B is a constant and $q=N/N-1$ is the conjugate index to N. Into the equation $G(z_1,iz_1,w)=e^{iz_1\cdot w}$, substitute $w=(x,0,\ldots,0)$, $z_1=(iy(x),0,\ldots,0)$ so that G takes the value

$$e^{xy}=\exp(Dy^N+Bx^q) \tag{4.7}$$

by construction of x and y. If we now apply the estimate of Lemma 4.4 we obtain $\exp(Dy^N+Bx^q) \leqslant C(1+|y|)^m \exp(Dy^N+x^{1+\epsilon})$ or $\exp(Bx^q-x^{1+\epsilon}) \leqslant C(1+|x|)^{m'}$. But $q=N/N-1>1+\epsilon$, so we obtain a contradiction by letting $x \to +\infty$. This completes the proof. □

Let us now mention some other examples where these methods apply, with almost no change. The two cases we discussed involved entire functions of order 1. An arbitrary order $\rho \geqslant 1$ can be considered with the same results. In the case of functions of order $\rho \geqslant 1$, finite type, i.e. $|(D^j f)(w)| \leqslant C\exp(D|w|^\rho)$, there does exist a continuous projection onto the analytic functions. And, for the case of functions of order ρ, i.e. $|D^j f(w)| = O(\exp(|w|^{\rho+\epsilon}))$ for each $\epsilon>0$, there is no projection. It is quite likely true that similar results hold for more general growth conditions which are functions of $|w|$.

In general, we know of no natural *Fréchet spaces* of smooth functions on $\mathbf{C}^p$ where there exist projections onto the entire functions in the space.

As the referee has kindly pointed out, the Fréchet space condition by itself cannot rule out the existence of projections—a finite dimensional subspace of C^∞ can always be added to a space of analytic functions. However, it would be interesting to know if some general easily recognizable structural property of the space is enough to rule out the existence of projections.

5. Applications to the interpolation problem. Let $\mathcal{E}(\Phi)$, $A(\Phi)$ be the spaces of functions of exponential type given in (3.3). Let $\mathcal{I}$ be a closed ideal in $A(\Phi)$ and let

$$\rho : A(\Phi) \to A(\Phi)/\mathcal{I} \tag{5.1}$$

be the quotient map. The space $A(\Phi)/\mathcal{I}$ can be interpreted as a space of analytic functions analytic on the zero set of $\mathcal{I}$. For functions of one variable, the range of the maps can be explicitly described as a sequence space (see e.g. [4], Theorem 8).

THEOREM 5.1. *The map* ρ *of* (5.1) *has a continuous linear right inverse when* $n=1$.

Proof. For functions of one variable, the method of proof outlined in the introduction carries through directly. See, for example, Theorem 8 and Theorem 7 of [4]. Further, an explicit formula shows that the local problem (Step 1 of the introduction) always has a right inverse (see pp. 133, 134 of [4]). Hence a continuous linear right inverse exists if the $\bar{\partial}$-operator has a continuous linear right inverse, which it does by Theorem 4.2. □

REMARK 5.1. The same argument applies to all spaces $A_p(\mathbf{C})$ and ideals $\mathcal{I}$ which are generated by slowly decreasing functions, and for which $\bar{\partial} : \mathcal{E}(\Phi) \to \mathcal{E}_{(0,1)}(\Phi)$ has a right inverse. In particular, to the spaces of functions of order $\leqslant \rho$, finite type in $\mathbf{C}$. See [5] for details, including the definition of slowly decreasing.

REMARK 5.2. The "local" part of the problem (Step 1 in the introduction) also has a linear solution for discrete, slowly decreasing varieties in $\mathbf{C}^n$. For such ideals, and for the spaces of functions of order $\leqslant \rho$, finite type, the analogue of Theorem 5.1 also holds. We refer to [5] for details.

Next, consider the Fréchet space example $\mathcal{E}(\Phi)$, $A(\Phi)$ given by (3.4). We already saw in Section 4 that there is no projection onto $A(\Phi)$; hence the $\bar{\partial}$ operator on $\mathcal{E}(\Phi)$ has no continuous, linear right inverse. It is also the case that the restriction map $f \to f(z_k)$ of $A(\Phi)$ onto any discrete infinite sequence $\{z_k\}$ has no continuous linear right inverse. For, suppose it does have such a right inverse, E, and let $f_k \in A(\Phi)$ be the image under E of the sequence equal to 1 at z_k and 0 at the other points of the sequence. We will derive two bounds for the f_k which show such f_k cannot exist. First, because E is continuous, the f_k form a bounded set in $A(\Phi)$. Thus, there exists $\omega(|z|) \to +\infty$ such that

$$|f_k(z)| \leqslant e^{\omega(|z|)} \tag{5.2}$$

and

$$\omega(|z|) = O(|z|^{1+\epsilon}), \quad \text{each} \quad \epsilon > 0. \tag{5.3}$$

Next, because E is continuous for each $\epsilon > 0$, there exists $\eta > 0$ such that $|\Sigma c_k f_k(z)| \leqslant \exp(|z_k|^{1+\epsilon})$ whenever $|c_k| \leqslant \eta \exp(|z_k|^{1+\eta})$. In particular, if $|z| \leqslant 1$, then there exists $\eta > 0$, $C > 0$ such that

$$|f_k(z)| \leqslant C \exp(-|z_k|^{1+\epsilon}). \tag{5.4}$$

From the 3-circles theorem applied on the circles $r_1 = 1$, $r_2 = |z|$, $r_3 = |z|^{1+\tau}$, and the bounds (5.2), (5.4), we have

$$|f_k(z)| \leqslant \exp\left[-\frac{\tau}{1+\tau}|z_k|^{1+\eta} + \frac{1}{1+\tau}\omega(|z|^{1+\tau}) + \frac{\tau}{1+\tau}\log C\right]. \tag{5.5}$$

If we choose $\tau < \eta$, set $z = z_k$, and apply (5.3) with ϵ chosen so small that

$$(1+\tau)(1+\epsilon) < 1+\eta,$$

then we see that $f_k(z_k) \to 0$ as $k \to +\infty$. But this is a contradiction since $f_k(z_k) = 1$. Thus, no such map E exists.

The same argument applies with only trivial modifications to the spaces of analytic functions of order $\leqslant \rho$, $\rho \neq 1$.

The argument sketched in the introduction relating interpolation and right inverses for $\bar{\partial}$ then gives another proof of the nonexistence of continuous linear projections onto the analytic functions in these spaces.

Finally, we discuss the example given in (3.5). Let $\{z_k\}$ be a discrete sequence of points in $\mathbf{C}^n$. We will say that $\{z_k\}$ is an *interpolating sequence* if the restriction map $f \to f(z_k)$ maps $A(\Phi)$ onto the space Γ of all sequences $\{\gamma_k\}$ such that $|\gamma_k| \leqslant B(1+|z_k|)^C \exp(D|\text{Im}\, z_k|)$ for some $B, C, D > 0$.

THEOREM 5.2. *Let $\mathcal{E}(\Phi)$, $A(\Phi)$ be as in* (3.5) *Let $\rho : A(\Phi) \to \Gamma$ be the restriction map for an interpolating sequence $\{z_k\}$. Then there exists a continuous linear right inverse for ρ if and only if*

$$\limsup_{k\to\infty} \frac{|\text{Im}\, z_k|}{\log(1+|z_k|)} < +\infty.$$

There is no continuous linear projection of $\mathcal{E}(\Phi)$ onto $A(\Phi)$.

Proof. Suppose such a right inverse E exists. As usual, set $f_k = E(\delta_k)$, where δ_k is the sequence equal to 1 at z_k and zero at all other z_i. Then $\{f_k\}$ is a bounded sequence in $A(\Phi)$, so

$$|f_k(z)| \leq B(1+|z|)^C \exp(D|\operatorname{Im} z|) \tag{5.6}$$

for some B, C, D independent of k. Also, $\Sigma c_k f_k(z)$ converges for all $C=(c_k) \in \Gamma$. Therefore, given $C_1, D_1 > 0$, there exists $B_2, C_2, D_2 > 0$ such that

$$|f_k(z)| \leq \frac{B_2(1+|z|)^{C_2}}{(1+|z_k|)^{C_1}} \exp(D_2|\operatorname{Im} z| - D_1|\operatorname{Im} z_k|) \tag{5.7}$$

(compare with (5.4)). Thus, from (5.7) we obtain a bound on the real axis, while (5.6) gives a bound on the imaginary axis. From the Phragmen-Lindelöf theorem, we conclude:

(5.8) there exists $D>0$ such that, for all $C_1, D_1 > 0$, constants B_2, C_2 exist such that

$$|f_k(z)| \leq \frac{B_2(1+|z|)^{C_2}}{(1+|z_k|)^{C_1}} \exp(D|\operatorname{Im} z| - D_1|\operatorname{Im} z_k|).$$

Thus, we can choose $D_1 = 2D$ and deduce that $f_k(z_k) \to 0$ provided that

$$\limsup \frac{|\operatorname{Im} z_k|}{\log(1+|z_k|)} = +\infty. \tag{5.9}$$

Thus, since $f_k(z_k) = 1$, this proves E cannot exist.

On the other hand, if the upper limit in (5.9) is finite, then we can write down the $f_k(z)$. The estimate of (5.8) becomes simply

$$|f_k(z)| \leq \frac{B_2(1+|z|)^{C_2}}{(1+|z_k|)^{C_1}} \exp(D|\operatorname{Im} z|) \tag{5.10}$$

because the term $D_1|\operatorname{Im} z_k|$ can be absorbed into the term $(1+|z_k|)^{C_1}$. Since we have assumed the sequence $\{z_k\}$ is an interpolating sequence, there exists $\tilde{f}_k$ with $\tilde{f}_k(z_k)=1$, $\tilde{f}_k(z_i) = 0$ for $i \neq k$, and $|\tilde{f}_k(z)| \leq B(1+|z|)^C \exp(D|\operatorname{Im} z|)$. See e.g. [4] or [14]. Choose a function $\phi \in A(\Phi)$ such that $\phi(0)=1$ and

$$|\phi(z)| \leq \frac{B \exp(D|\operatorname{Im} z|)}{\omega(|z|)}$$

where $(1+|z|)^C = O(\omega(|z|))$ for each $C>0$. (See e.g. [11], Ch. 1.) Then set $f_k(z) = \tilde{f}_k(z)\phi(z-z_k)$. It is routine to verify that f_k satisfies the required estimates so that $E: \Gamma \to A(\Phi)$ given by $E(\{c_k\}) = \Sigma c_k f_k(z)$ is the right inverse.

To complete the proof of the Theorem, we sketch how the argument of the introduction can be carried out. Let the interpolating sequence be of the form

$$\{\pm i y_k\} \cup \{x_l\}$$

where $y_k = 2^k$ and x_l is a subset of the integers which is the zero set of a function ϕ in $A(\Phi)$ with the property that $F(z) = \phi(z)\prod_{k=1}^{\infty}(1-(z/iy_k)^2) \in A(\Phi)$ (in one variable). Such a ϕ always exists. Then the sequence is an interpolating sequence in $A(\Phi)$ (see e.g. [14]). For this sequence, Step 1 of the introduction is easily carried out, so the only obstruction to the existence of a continuous, linear right inverse for ρ is the existence of a continuous linear right inverse for $\bar{\partial}$. By part (1), no such inverse exists for ρ. Consequently, no such inverse exists for $\bar{\partial}$. □

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