# LOWER BOUNDS FOR THE EIGENVALUES OF RIEMANNIAN MANIFOLDS

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**1. Introduction.** Let $M$ be a compact Riemannian manifold with boundary $\partial M$. Denote $\Delta_0$ to be the Laplacian of $M$ for functions.

Under the assumption that $M$ was negatively curved [6], we gave lower bounds for the eigenvalues of $\Delta_0$, subject to Dirichlet boundary conditions when $\partial M \neq \phi$. A main device employed was to obtain an upper bound for the trace of the heat kernel of $M$.

The present paper extends the work of [6] in several directions. First of all, we improve and simplify the elementary lemmas showing that an upper bound on the heat kernel gives lower bounds for the eigenvalues. The restriction of negative curvature is removed, giving lower bounds for the eigenvalues of $\Delta_0$ on arbitrary $M$, assuming Dirichlet boundary conditions if $\partial M \neq \phi$.

In Section 4, we consider Laplacians $\Delta$ acting on Riemannian vector bundles $V \to M$. Upper bounds are obtained for the associated heat kernels. This implies upper bounds for the number of non-positive eigenvalues and also lower bounds for the positive eigenvalues of $\Delta$. Interesting special cases include the Laplacian on Differential Forms and the Second Variation Operator of Minimal Submanifold Theory. In particular, if $M$ is minimally imbedded, upper bounds are given for the nullity and index of $M$. One also obtains upper bounds for the betti numbers of $M$.

**2. Heat kernels and eigenvalues.** Let $M$ be a compact Riemannian manifold with boundary $\partial M$. A Riemannian vector bundle $V \to M$ is a smooth vector bundle with metric and connection $\nabla$ preserving that metric. The Bochner Laplacian of $V$ is an invariantly defined second order differential operator $D: \Gamma(V) \to \Gamma(V)$, obtained by $D = \mathrm{Tr}(\nabla \circ \nabla)$. Here $\Gamma(V)$ denotes the smooth sections of $V$. More explicitly, $D$ is given by the composition:

$$\Gamma(V) \xrightarrow{\nabla} \Gamma(V \otimes T^*M) \xrightarrow{\nabla} \Gamma(V \otimes T^*M \otimes T^*M) \to \Gamma(V)$$

where the last map is a contraction.

Suppose that $E$ is a selfadjoint endomorphism of $V$ and set $\Delta = -D + E$. If $\partial M$ is empty, then $\Delta$ defines a unique selfadjoint operator in $L^2 V$. Otherwise, one must impose suitable boundary conditions. It is most typical to use either Dirichlet, $w(p) = 0$ for $p \in \partial M$, or Neumann $\nabla_\nu w(p) = 0$ for $p \in \partial M$, boundary conditions. Here $w \in \Gamma(V)$ and $\nu$ denotes a unit normal vector field along $\partial M$.

Since $M$ is compact, the operator $\Delta$ has a finite number of non-positive eigenvalues $\mu_1 \leqslant \mu_2 \leqslant \ldots \mu_l \leqslant 0$ and an infinite collection of positive eigenvalues $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots$. The $\lambda_i$ satisfy the asymptotic estimate of Minakshisundaram–Pleijel [11]:

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(2.1) $$\lambda_k \sim e_n \operatorname{vol} M \dim V k^{2/n}$$

as $k \to \infty$. Here $\operatorname{vol} M$ is the volume of $M$, $\dim V$ is the dimension of $V$, and $e_n$ is a universal constant depending only on the dimension $n$ of $M$.

The heat equation problem for $\Delta$ is:

$$\left(\frac{\partial}{\partial t} + \Delta\right) w(x,t) = 0 \qquad x \in M, \quad t > 0$$

$$w(x,0) = w_0(x)$$

with suitable boundary conditions on $w$ at $\partial M$. There is a good fundamental solution $K(t,x,y) \in \operatorname{End}(V_x, V_y)$ so that $w(x,t) = \int_M K(t,x,y) w_0(y)\, dy$.

Let $\phi_i$ be the normalized eigenfunctions corresponding to the $\mu_i$ and $\psi_j$ the normalized eigenfunctions corresponding to the $\lambda_j$. Then one has

$$K(t,x,y) = \sum_{i=1}^{l} e^{-t\mu_i} \phi_i(x) \phi_i(y) + \sum_{j=1}^{\infty} e^{-t\lambda_j} \psi_j(x) \psi_j(y).$$

Consequently,

(2.2) $$\operatorname{Tr} K(t) = \sum_i e^{-t\mu_i} + \sum_j e^{-t\lambda_j}.$$

The formula (2.2) provides a most useful device for studying the eigenvalues of $\Delta$. We show in this section how upper bounds for $\operatorname{Tr} K(t)$ give lower bounds for the $\lambda_i$. The remaining sections are devoted to obtaining upper bounds for $\operatorname{Tr} K(t)$ in some interesting geometric situations. These bounds for $\operatorname{Tr} K(t)$ will be of the form

(2.3) $$\operatorname{Tr} K(t) \leqslant e^{C_3 t}(C_1 t^{-n/2} + C_2 t)$$

with $C_1 > 0$; $C_2, C_3 \geqslant 0$ and the inequality holding for all $t > 0$. Even though the right-hand side of (2.3) need not decay properly as $t \to \infty$, the inequality (2.3) still gives interesting information about the eigenvalues of $\Delta$.

Clearly one has

LEMMA 2.4. *If $K(t)$ satisfies* (2.3), *then the number $l$ of non-positive eigenvalues for $\Delta$ is bounded above by $l \leqslant e^{C_3}(C_1 + C_2)$.*

*Proof.* Using (2.2), we see that for any $t > 0$, $l \leqslant \operatorname{Tr} K(t)$. Now set $t = 1$.

A similar argument yields

LEMMA 2.5. *Suppose that* (2.3) *is satisfied. Then the k'th positive eigenvalue $\lambda_k$ is bounded below by $\lambda_k \geqslant \log k - \log(C_1 + C_2) - C_3$.*

*Proof.* For any $t > 0$, $ke^{-\lambda_k t} \leqslant \operatorname{Tr} K(t)$. Now substitute $t = 1$.

The lower bound of Lemma 2.5 is deficient in two respects. Since the right-hand side may be less than zero for small $k$, the inequality is trivial until $k$ is sufficiently large. Secondly, the quantity $\log k$ grows too slowly as $k$ increases. A good lower bound should be compatible with the asymptotic formula (2.1) of Minakshisundaram–Pleijel when $k \to \infty$.

Proceeding more carefully, one obtains a lower bound with the correct asymptotic growth as $k \to \infty$:

LEMMA 2.6. *Suppose that* $\lambda_m \geqslant C_4 > 0$, *for some* $m > 0$. *If* (2.3) *is satisfied, then one has for* $k \geqslant m$, $\lambda_k \geqslant C_5 k^{2/n}$, *where* $C_5$ *depends only on* $C_1$, $C_2$, $C_3$, $C_4$, *and* $n$.

*Proof.* It follows from (2.3) that for all $t > 0$:

$$ke^{-\lambda_k t} \leqslant \operatorname{Tr} K(t) \leqslant e^{C_3 t}(C_1 t^{-n/2} + C_2 t).$$

Set $t = A/\lambda_k$, where $A$ is a constant to be chosen later:

$$ke^{-A} \leqslant e^{AC_3/\lambda_k}(C_1 \lambda_k^{n/2} A^{-n/2} + C_2 A/\lambda_k).$$

Since $\lambda_k \geqslant \lambda_m > 0$, $ke^{-A} \leqslant e^{AC_3/\lambda_m}(C_1 \lambda_k^{n/2} A^{-n/2} + C_2 A/\lambda_m)$.
Using $\lambda_m \geqslant C_4 > 0$, $ke^{-A} \leqslant e^{AC_3/C_4}(C_1 \lambda_k^{n/2} A^{-n/2} + C_2 A/C_4)$.
Choose $A > 0$ sufficiently small to satisfy $(1/2)e^{-A} \geqslant e^{AC_3/C_4}(C_2 A/C_4)$.
Since $k/2 \geqslant 1/2$, we may combine the last two inequalities, yielding $(k/2)e^{-A} \leqslant e^{AC_3/C_4}(C_1 \lambda_k^{n/2} A^{-n/2})$ from which Lemma 2.6 follows immediately.
If $C_2 = C_3 = 0$ in (2.3), then a stronger conclusion holds:

LEMMA 2.7. *If* $\operatorname{Tr} K(t) \leqslant C_1 t^{-n/2}$ *then* $\lambda_k \geqslant C_1^{-2/n} k^{2/n}$ *for all* $k > 0$.

*Proof.* As above, $ke^{-\lambda_k t} \leqslant \operatorname{Tr} K(t) \leqslant C_1 t^{-n/2}$, for all $t$. Now set $t = 1/\lambda_k$.
Lemma 2.7 was given in the authors' earlier paper [6]. However, the proof presented there is unnecessarily complicated.

**3. Laplacian for functions.** Let $\mathfrak{D}$ be a relatively compact domain in a complete Riemannian manifold $X$. Of course, any compact Riemannian manifold with boundary can be realized as a domain in some complete $X$. We will derive an upper bound for the heat kernel of $\mathfrak{D}$ with Dirichlet boundary conditions.

Denote $K_\sigma(t,x,y)$ to be the heat kernel for the sphere $S^n$ with constant curvature $\sigma$. Here $n$ is the dimension of $X$. Since $S^n$ is a two point homogeneous space and $K_\sigma$ is invariant under isometries, one has $K_\sigma(t,x,y) = K_\sigma(t, r(x,y))$, where $r$ is the geodesic distance from $x$ to $y$.

The following lemma is standard [3]:

LEMMA 3.1. *For each fixed* $t > 0$, $K_\sigma(t,r)$ *is a decreasing function of its argument* $r$.

Let $I_\mathfrak{D}$ be a finite positive number so that, for each $p \in \mathfrak{D}$, the exponential map $\exp B(p, I_\mathfrak{D}) \to X$ is a diffeomorphism. Here $B(p, I_\mathfrak{D})$ is the ball of radius $I_\mathfrak{D}$ in the tangent space to $X$ at $p$. Denote $\mathfrak{D}' = \bigcup_{p \in \mathfrak{D}} \exp B(p, I_\mathfrak{D})$. Then $\mathfrak{D}'$ is a relatively compact open set containing $\mathfrak{D}$.

Choose $\sigma$ to be an upper bound for the sectional curvature of $X$ at points in $\mathfrak{D}'$. Without loss of generality, we may assume that $\sigma > 0$. For sharper estimates on negatively curved manifolds, the reader may refer to [6]. Set $I = \min(I_\mathfrak{D}, 2\pi/\sqrt{\sigma})$.

If $\rho : \mathbf{R} \to \mathbf{R}$ is a smooth function, we may require $\rho$ to satisfy: $\rho(\alpha) = 1$ for $|\alpha| < 1/2$, $\rho(\alpha) = 0$ for $|\alpha| \geqslant 1$, and $\rho$ is a decreasing function of $|\alpha|$. Define a cut-off function $\rho : X \times X \to \mathbf{R}$ by $\rho(x,y) = \rho(r(x,y)/I)$.

We will compare the heat kernel $K$ of $\mathfrak{D}$, for Dirichlet boundary conditions, with a transplanted kernel $L(t,x,y)=K_\sigma(t,r(x,y))\rho(x,y)+tB$. Here $B>0$ is as yet an undetermined constant. The basic technical lemma required is:

LEMMA 3.2. *If $\Delta$ denotes the Laplacian of $X$ acting on functions, then*

$$\left(\frac{\partial}{\partial t}+\Delta_x\right)L(t,x,y)\geqslant 0$$

*for $B$ sufficiently large. The choice of $B$ depends only on $\sigma$, $I$, and the dimension $n$ of $X$.*

*Proof.* Fix $y\in X$ and let $\theta(x,y)$ be the volume element in spherical coordinates centered at $y$. Denote $\theta_\sigma$ to be the analogous volume element on the sphere of constant sectional curvature $\sigma$. Since $S^n$ is two point homogeneous, $\theta_\sigma$ is a function of the geodesic distance alone. This allows one to transplant $\theta_\sigma$ to a function on a neighborhood of the diagonal in $X\times X$, $\theta_\sigma(x,y)=\theta_\sigma(r(x,y))$, for $(x,y)$ near the diagonal.

Using the standard formula for the Laplacian applied to a function of the geodesic distance [6, p. 32]:

$$\left(\frac{\partial}{\partial t}+\Delta\right)L(t,x,y)=\rho\left(\frac{\theta_\sigma'}{\theta_\sigma}-\frac{\theta'}{\theta}\right)\frac{\partial K_\sigma}{\partial r}-2\frac{\partial K_\sigma}{\partial r}\frac{\partial\rho}{\partial r}+K_\sigma\Delta\rho+B.$$

Here the prime denotes a radial derivative.

Since the sectional curvatures in $\mathfrak{D}'$ are bounded above by $\sigma$, a basic comparison theorem [2, p. 253] gives $\theta'/\theta-\theta_\sigma'/\theta_\sigma\geqslant 0$. Using Lemma 3.1, we see that:

$$\left(\frac{\partial}{\partial t}+\Delta\right)L\geqslant -2\frac{\partial K_\sigma}{\partial r}\frac{\partial\rho}{\partial r}+K_\sigma\Delta\rho+B.$$

Now $\Delta\rho=\dfrac{-\partial^2\rho}{\partial r^2}-\dfrac{\theta'}{\theta}\dfrac{\partial\rho}{\partial r}$. Since $\partial\rho/\partial r\leqslant 0$, we may apply the basic comparison theorem again to write:

$$\left(\frac{\partial}{\partial t}+\Delta\right)L\geqslant -2\frac{\partial K_\sigma}{\partial r}\frac{\partial\rho}{\partial r}-K_\sigma\frac{\partial^2\rho}{\partial r^2}-\frac{\theta_\sigma'}{\theta_\sigma}\frac{\partial\rho}{\partial r}K_\sigma+B.$$

Since $\partial\rho/\partial r$, $\partial^2\rho/\partial r^2$ are supported in $[I/2,I]$:

$$\left(\frac{\partial}{\partial t}+\Delta\right)L\geqslant B_1\sup_{\substack{I/2<r<I\\ t>0}}\max\left(K_\sigma,\left|\frac{\partial K_\sigma}{\partial r}\right|\right)+B.$$

The quantity $\max(K_\sigma,|\partial K_\sigma/\partial r|)$ is bounded above if $r>\sigma/2$ since one stays uniformly away from the diagonal. Choosing $B$ sufficiently large, we obtain

$$\left(\frac{\partial}{\partial t}+\Delta\right)L\geqslant 0.$$

It is now easy to deduce:

THEOREM 3.3. *Let $\mathfrak{D}$ be a relatively compact domain in a complete Riemannian manifold X. The heat kernel $K(t,x,y)$ of $\mathfrak{D}$ with Dirichlet boundary conditions satisfies $K(t,x,y) \leqslant C_1 t^{-n/2} + C_2 t$ where $C_1$, $C_2$ depend only on I, $\sigma$, and n.*

*In particular, if $\mathfrak{D} = X$ is a compact manifold without boundary then $C_1$, $C_2$ depend only on a lower bound for the injectivity radius of X, an upper bound for the sectional curvature of X, and n.*

*Proof.* One employs Lemma 3.2 and Duhamel's principle to write:

$$\begin{aligned} L(t,x,y) - K(t,x,y) &= \int_0^t \frac{d}{ds} \int_{\mathfrak{D}} L(s,x,z) K(t-s,z,y)\, dz\, ds \\ &= \int_0^t \int_{\mathfrak{D}} \frac{\partial L}{\partial s}(s,x,z) K(t-s,z,y)\, dz\, ds \\ &\quad + \int_0^t \int_{\mathfrak{D}} L(s,x,z)\, \Delta_z K(t-s,z,y)\, dz\, ds. \end{aligned}$$

Since Dirichlet boundary conditions are imposed for $K$, integration by parts gives

$$L(t,x,y) - K(t,x,y) \geqslant \int_0^t \int_{\mathfrak{D}} \left( \frac{\partial}{\partial s} + \Delta \right) L(s,x,z) K(t-s,z,y)\, dz\, ds.$$

From Lemma 3.2: $K(t,x,y) \leqslant L(t,x,y)$ which proves Theorem 3.3. □

The trace of the heat kernel $K$, on $L^2\mathfrak{D}$, is given by $\operatorname{Tr} K(t) = \int_{\mathfrak{D}} K(t,x,x)$. So

$$\operatorname{Tr} K(t) \leqslant (C_1 t^{-n/2} + C_2 t) \operatorname{vol}(\mathfrak{D}) \tag{3.4}$$

where $\operatorname{vol}(\mathfrak{D})$ is the volume of $\mathfrak{D}$ and $C_1$, $C_2$ are from Theorem 3.3.

Consequently, from (3.4) and Lemmas 2.5, 2.6, one has

COROLLARY 3.5. *Let $0 < \lambda_1 \leqslant \lambda_2 \ldots$ denote the positive eigenvalues of $\Delta$ acting on $L^2\mathfrak{D}$ with Dirichlet boundary conditions. Then for some $m > 0$ and all $k \geqslant m$, $\lambda_k \geqslant C_3 k^{2/n}$ where n is the dimension of $\mathfrak{D}$. Here m, $C_3$ depend upon n, I, $\sigma$, and an upper bound for the volume of $\mathfrak{D}$.*

*In particular, if $\mathfrak{D} = X$ is compact without boundary, the m, $C_3$ depend upon upper bounds for the sectional curvature and volume of X, a lower bound for the injectivity radius of X, and n.*

In the case that the sectional curvatures of $X$ are non-positive, Theorem 3.3 was obtained in the authors' earlier paper [6]. The proof was by comparison with the Euclidean heat kernel, using an argument similar to the one employed above. For negative sectional curvatures, a better upper bound is found by comparison with the heat kernel of a suitable hyperbolic space [6].

The second author and S. Y. Cheng [4] used a quite different approach to find an upper bound for $K(t,x,y)$:

THEOREM 3.6 (Cheng-Li). *Let $K(t,x,y)$ denote the heat kernel of a compact Riemannian manifold with boundary. If $\partial M \neq \phi$ and one imposes Dirichlet boundary conditions, then $K(t,x,x) \leqslant C_4 t^{-n/2}$.*

*When $\partial M=\phi$ or for Neumann boundary conditions,*

$$K(t,x,x) \leqslant (\operatorname{vol} M)^{-1}+C_5 t^{-n/2}.$$

*Here $C_4$, $C_5$ depend on certain Sobolev constants for $M$.*

If $\partial M=\phi$, then C. Croke [5] estimated the relevant Sobolev constant in terms of an upper bound for the diameter and lower bounds for the Ricci curvature and volume of $M$. Thus one obtains an upper bound for the heat kernel using these quantities.

The eigenvalue estimates following from Theorem 3.6 are detailed in [4]. Cheng and Li employ the observation which appears as Lemma 2.7 in the present paper.

**4. Semigroup domination.** We return to the setting of Section 2. Let $M$ be a compact Riemannian manifold with boundary and $V$ a Riemannian vector bundle over $M$. For a given endomorphism $E$, set $\Delta=-D+E$, where $D$ is the Bochner Laplacian. Suppose that $-b$ is a lower bound for $E$ in the sense that $\langle Ev,v\rangle \geqslant -b\langle v,v\rangle$, where $b$ is a real number and $v\in V$.

One may study $\Delta$ with either Dirichlet or Neumann boundary conditions. A section $w$ of $V$ is said to satisfy Dirichlet boundary conditions if $w(p)=0$ for $p\in\partial M$, and $w$ is said to satisfy Neumann boundary conditions if $\nabla_\nu w(p)=0$ for $p\in\partial M$. Here $\nu$ is a unit normal vector along $\partial M$. Denote $K_d$, $K_n$ to be the heat kernels associated to $\Delta$ with Dirichlet, Neumann boundary conditions. Of course,

$$K_d(t,x,y), K_n(t,x,y) \in \operatorname{End}(V_y, V_x),$$

for each fixed $t$. In the special case where $V$ is the trivial line bundle and $E=0$, $\Delta_0$ is the standard Laplacian acting on functions. Let $K_{d,0}$, $K_{n,0}$ be the heat kernels associated to the Laplacian for functions.

Set $|K_n|$, $|K_d|$ to be the pointwise norm of the endomorphisms $K_n$, $K_d$. A basic technical lemma is:

LEMMA 4.1. *Suppose that $E=0$. Considered as distributions, one has both*

(i) $$\left(\frac{\partial}{\partial t}+\Delta_0\right)|K_n| \leqslant 0$$

*and*

(ii) $$\left(\frac{\partial}{\partial t}+\Delta_0\right)|K_d| \leqslant 0.$$

*Proof.* Let $K$ be either $K_n$ or $K_d$. To finesse the singularity of derivatives when $K$ vanishes, set $|K|_\epsilon=(|K|^2+\epsilon)^{1/2}$. Later, we will let $\epsilon\to 0$.

The Bochner formulas read

$$\text{(4.2)}\qquad \begin{aligned} \tfrac12\Delta_0|K|^2 &= \langle\Delta K,K\rangle-|\nabla K|^2 \\ \tfrac12\Delta_0|K|^2 &= \tfrac12\Delta_0|K|_\epsilon^2 = |K|_\epsilon\Delta|K|_\epsilon-|\nabla|K|_\epsilon|^2 \end{aligned}$$

where the differential operators act on the variable $x$ in $K(t,x,y)$.

Since $K_t=-\Delta K$, one has for each $\epsilon>0$,

$$|K|_\epsilon|K|_{\epsilon,t}=\frac12\frac{d}{dt}(|K|_\epsilon)^2=\frac12\frac{d}{dt}|K|^2=\langle K_t,K\rangle=\langle-\Delta K,K\rangle.$$

Using (4.2) yields,

$$|K|_\epsilon |K|_{\epsilon,t} = -|\nabla K|^2 - \tfrac{1}{2}\Delta_0 |K|^2$$

and so for each $\epsilon > 0$,

$$|K|_\epsilon |K|_{\epsilon,t} = -|\nabla K|^2 - \tfrac{1}{2}\Delta_0 |K|_\epsilon^2.$$

Employing (4.2) again, $|K|_\epsilon |K|_{\epsilon,t} = |\nabla |K|_\epsilon|^2 - |\nabla K|^2 - |K|_\epsilon \Delta_0 |K|_\epsilon$. By definition of $|K|_\epsilon$, it follows that $|\nabla |K|_\epsilon| \leqslant |\nabla K|$ and thus $\left(\frac{\partial}{\partial t} + \Delta_0\right)|K|_\epsilon \leqslant 0$.

Now let $\epsilon \to 0$ to prove Lemma 4.1.

The main result of the present section is

THEOREM 4.3. *Let $V \to M$ be a Riemannian vector bundle over the compact Riemannian manifold $M$. Suppose that $\Delta = -D + E$ acts on $\Gamma(V)$, smooth sections of $V$. Let $E$ be bounded below by $-b$ in the sense that $(Ev, v) \geqslant -b(v, v)$. Denote $K_n$, $K_d$ to be the heat kernels associated to $\Delta$ with Neumann, Dirichlet boundary conditions.*

*The following inequalities hold for pointwise norms of heat kernels:*

(i) $|K_n(t,x,y)| \leqslant e^{bt} K_{n,0}(t,x,y)$

(ii) $|K_d(t,x,y)| \leqslant e^{bt} K_{d,0}(t,x,y)$.

*Here $K_{n,0}$, $K_{d,0}$ are the analogous heat kernels for the Laplacian acting on functions.*

*If $\partial M = \phi$, then it is not necessary to specify boundary conditions.*

*Proof.* (i) First suppose that $E = 0$, so that $\Delta = -D$. For either type of boundary conditions one may apply Duhamel's principle as follows:

$$|K|(t,x,y) - K_0(t,x,y) = \int_0^t \frac{d}{ds} \int_M |K|(s,x,z) K_0(t-s,z,y)\, dz\, ds$$

$$|K|(t,x,y) - K_0(t,x,y) = \int_0^t \int_M \frac{\partial}{\partial s} |K|(s,x,z) K_0(t-s,z,y)\, dz\, ds$$

$$+ \int_0^t \int_M |K|(s,x,z) \Delta_0 K_0(t-s,z,y)\, dz\, ds.$$

So that integrating by parts

$$|K|(t,x,y) - K_0(t,x,y) = \int_0^t \int_M \left(\frac{\partial}{\partial s} + \Delta_0\right) |K|(s,x,z) K_0(t-s,z,y)\, dz\, ds$$

where one uses that either Dirichlet or Neumann boundary conditions are imposed on both $K$ and $K_0$.

Lemma 4.1 completes the proof in case $E = 0$.

(ii) The general case follows from Part (i) and the Trotter product formula [14, p. 297]: $e^{-t\Delta} = \lim_{k\to\infty} (e^{tD/k} e^{-tE/k})^k$ where $D$ is Bochner's Laplacian.

One just observes that for the pointwise norm of heat kernels:

$$|e^{tD/k} e^{-tE/k}| \leqslant |e^{tD/k}|\,|e^{-tE/k}| \leqslant e^{tb/k} e^{-t\Delta_0/k}$$

via Part (i) and the definition of $b$.

Therefore, for all $k$, $|(e^{tD/k}e^{-tE/k})^k| \leqslant (e^{tb/k}e^{-t\Delta_0/k})^k \leqslant e^{tb}e^{-t\Delta_0}$ from the semigroup property of $e^{-t\Delta_0}$. Now let $k \to \infty$ and use the Trotter product formula.

This completes the proof of Theorem 4.3. □

In the case $E=0$, Theorem 4.3 was proved in [7]. The method there relied on general semigroup domination theory and proceeded by establishing Kato's inequality. The alternative approach here using Duhamel's principle is somewhat more direct.

An immediate consequence of the above results is:

COROLLARY 4.4. *In the notation of Theorem* 4.3:

(i) $\operatorname{Tr} K_n(t) \leqslant (\dim V)e^{bt} \operatorname{Tr} K_{n,0}(t)$

*and*

(ii) $\operatorname{Tr} K_d(t) \leqslant (\dim V)e^{bt} \operatorname{Tr} K_{d,0}(t)$

*where* Tr *denotes the global trace on* $L^2$ *sections. Here* $\dim V$ *is the dimension of the vector bundle* $V$.

*Proof.* For either case,

$$\begin{aligned}\operatorname{Tr} K(t) &= \int_M \operatorname{Tr} K(t,x,x) \leqslant \dim V \int_M |K(t,x,x)| \\ &\leqslant \dim V \int_M e^{bt} K_0(t,x,x) = \dim V e^{bt} \operatorname{Tr} K_0(t)\end{aligned}$$

where $\operatorname{Tr} K(t,x,x)$ denotes the pointwise trace.

Combining Corollary 4.4 with Theorems 3.3 and 3.6 gives

COROLLARY 4.5. (i) *One has for Neumann boundary conditions or if* $\partial M = \phi$:

$$\operatorname{Tr} K_n(t) \leqslant (\dim V)e^{bt}(1 + C_5 t^{-n/2} \operatorname{vol} M)$$

(ii) *For Dirichlet boundary conditions or if* $\partial M = \phi$:

$$\operatorname{Tr} K_d(t) \leqslant (\dim V)e^{bt} \operatorname{vol} M(C_1 t^{-n/2} + C_2 t)$$

(iii) *Finally, for Dirichlet boundary conditions requiring* $\partial M \neq \phi$:

$$\operatorname{Tr} K_d(t) \leqslant (\dim V)e^{bt} \operatorname{vol} M C_4 t^{-n/2}.$$

*Here* $C_1$, $C_2$, $C_4$, $C_5$ *have the geometric dependence described in Section* 3.

Using the elementary Lemmas 2.5, 2.6, and 2.7 one has eigenvalue estimates corresponding to each case in Corollary 4.5:

COROLLARY 4.6. (i) *For Neumann boundary conditions or if* $\partial M = \phi$, *then for some* $m$ *and all* $k \geqslant m$, $\lambda_k \geqslant C_6 k^{2/n}$. *Here* $m, C_6$ *depend only on* $\operatorname{vol} M$, $b$, $n$, $\dim V$, *and* $C_5$ *of Corollary* 4.5.

(ii) *For Dirichlet boundary conditions or if* $\partial M = \phi$, *then for some* $m$ *and* $k \geq m$, $\lambda_k \geq C_7 k^{2/n}$. *Here* $m, C_7$ *depend only on* $\operatorname{vol} M$, $b$, $n$, $\dim V$ *and* $C_1, C_2$ *of Corollary* 4.5.

(iii) *For Dirichlet boundary conditions and* $\partial M \neq \phi$, *one has for some* $m$ *and all* $k \geq m$, $\lambda_k \geq C_8 k^{2/n}$. *Here* $m, C_8$ *depend only on* $\operatorname{vol} M$, $b$, $n$, $\dim V$, *and* $C_4$ *of Corollary* 4.5.

In this Corollary, $0 < \lambda_1 \leq \lambda_2 \leq \dots$ are the positive eigenvalues of $\Delta$.

If $b < 0$, then $m = 1$ in all parts of Corollary 4.6. If $b \leq 0$, then $m = 1$ in Part (iii) of Corollary 4.6.

Similarly, one may employ Lemma 2.4 to give an upper bound on the number of non-positive eigenvalues of $\Delta$:

COROLLARY 4.7. *Let* $l$ *denote the number of non-positive eigenvalues for* $\Delta$. *Then*

(i) *For Neumann boundary conditions or if* $\partial M = \phi$, $l \leq (\dim V) e^b (1 + C_5 \operatorname{vol} M)$.

(ii) *For Dirichlet boundary conditions or if* $\partial M = \phi$, $l \leq (\dim V) e^b \operatorname{vol} M (C_1 + C_2)$.

(iii) *For Dirichlet boundary conditions and* $\partial M \neq \phi$, $l \leq (\dim V) e^b \operatorname{vol} M C_4$.

*Here we use the notation of Corollary* 4.5.

**5. Second variation operator.** The second variation operator, which arises in the study of minimal submanifolds, provides an interesting example for application of the results in Section 4. Let $M$ be a compact Riemannian manifold with boundary which is minimally imbedded in the ambient space $\bar{M}$. That is, the mean curvature vector vanishes identically on the interior of $M$. The normal bundle $T^\perp M \to M$ of $M$ in $\bar{M}$ is naturally a Riemannian vector bundle with metric and connection induced by the Riemannian metric and Levi-Civita connection of $\bar{M}$. We take $V = T^\perp M$, in the notation of Section 2.

Let $D$ denote the Bochner Laplacian of $T^\perp M$, i.e. $D = \operatorname{Tr}(\nabla^2)$. For the definition of the second variation operator, we follow [15]. Denote $SM$ to be the space of symmetric linear transformations $TM \to TM$. The second fundamental form of $M$ in $\bar{M}$ may be regarded as an endomorphism $A \in \operatorname{End}(T^\perp M, SM)$. Set $\tilde{A} = {}^t A \circ A$, where the superscript $t$ denotes the transpose with respect to the induced metrics. Then $\tilde{A} : T^\perp M \to T^\perp M$. If $\bar{R}$ denotes the curvature tensor of $\bar{M}$, then a partial Ricci transformation $\bar{\mathrm{R}}\mathrm{ic}$ is defined for $v \in T^\perp M$ by:

$$\bar{\mathrm{R}}\mathrm{ic}(v) = \sum_{i=1}^{n} (\bar{R}_{e_i, v} e_i)^\perp .$$

Here the sum runs over an orthonormal basis $e_i$ of $TM$ and the perpendicular sign $\perp$ means to take the normal component. Of course, $\bar{\mathrm{R}}\mathrm{ic}$ is an endomorphism $\bar{\mathrm{R}}\mathrm{ic} : T^\perp M \to T^\perp M$. Given these preliminaries, one may define the second variation operator $\Delta : \Gamma(T^\perp M) \to \Gamma(T^\perp M)$ by

$$\Delta v = -Dv + \bar{\mathrm{R}}\mathrm{ic}(v) - \tilde{A} v \tag{5.1}$$

for $v \in T^\perp M$.

The operator $\Delta$ has an interesting geometrical meaning for Dirichlet boundary conditions. In fact, if $v \in \Gamma(T^\perp M)$ vanishes on $\partial M$ then $\langle \Delta v, v \rangle$, global inner product, represents the second derivative of area for a deformation of $M$ along direction $v$.

The index of $M$ is the number of negative eigenvalues of $\Delta$, counted to multiplicity. The nullity of $M$ is the number of zero eigenvalues of $\Delta$.

Suppose that $b$ is a real number satisfying $\langle \bar{\mathrm{R}}\mathrm{ic}(v) - \tilde{A}v, v\rangle \geq -b\langle v, v\rangle$. Then one may apply the results of Section 4, with $E = \bar{\mathrm{R}}\mathrm{ic} - \tilde{A}$. It is most interesting to give an upper bound for the index and nullity of $M$:

PROPOSITION 5.2. *Let $l$ denote the number of non-positive eigenvalues of $\Delta : \Gamma(T^{\perp}M) \to \Gamma(T^{\perp}M)$ acting with Dirichlet boundary conditions. Here $\Delta$ is the second variation operator given by* (5.1). *Denote $n'$ to be the codimension of $M$. Then*

(i) *If $\partial M = \phi$, $l \leq n'e^{b}(1 + C_5 \operatorname{vol} M)$, where $C_5$ has the geometrical dependence described in Section* 3. *In particular, $C_5$ may be bounded above given an upper bound for the diameter and lower bounds for the Ricci curvature and volume of $M$.*

(ii) *For any minimal $M$, $l \leq n'e^{b} \operatorname{vol} M(C_1 + C_2)$, where $C_1, C_2$ are as in Theorem* 3.3. *If $\partial M = \phi$, then $C_1 + C_2$ may be bounded above given a lower bound for the injectivity radius and an upper bound for the sectional curvature of $M$.*

(iii) *If $\partial M \neq \phi$, then $l \leq n'e^{b} \operatorname{vol} M C_4$, where $C_4$ is as given in Theorem* 3.6.

Corollary 4.6 applies directly to give lower bounds for the positive eigenvalues of the second variation operator $\Delta$. For the sake of brevity, we omit writing this out in detail.

**6. Laplacian for differential forms.** Let $M$ be a compact Riemannian manifold with boundary $\partial M$. For convenience, we assume that $M$ is oriented. All our results generalize to the non-orientable case by passing to the orientable double cover. Let $\Delta = d\delta + \delta d$ denote the Hodge Laplacian acting on smooth differential $p$-forms, $1 \leq p \leq n-1$. The operator $\Delta$ is positive semi-definite on $C_0^{\infty}M$.

The results of Section 4 would apply to $\Delta$ if one imposed Dirichlet or Neumann boundary conditions. However, it is more interesting to consider the Hodge Laplacian with absolute or relative boundary conditions. Let $\nu$ be an inward pointing normal covector along $\partial M$. If $a \in \Lambda^p M$ is a smooth $p$-form, then along $\partial M$ we may decompose $a$ into its tangential and normal components, $a = a_{\tan} + a_{\mathrm{norm}} \wedge \nu$, with $a_{\tan} \in \Lambda^p \partial M$, $a_{\mathrm{norm}} \in \Lambda^{p-1}\partial M$. The form $a$ is said to satisfy relative boundary conditions if $a_{\tan} = (\delta a)_{\tan} = 0$ and $a$ is said to satisfy absolute boundary conditions if $a_{\mathrm{norm}} = (da)_{\mathrm{norm}} = 0$. Here $d$ is the exterior derivative, $d : \Lambda^p M \to \Lambda^{p+1}M$, and $\delta$ is the adjoint of $d$. Clearly, the Hodge star operator $*$ maps forms satisfying absolute boundary conditions to those satisfying relative boundary conditions,

$$* : \Lambda^p M \to \Lambda^{n-p} M.$$

Recall that $a$ is said to be harmonic if $\Delta a = 0$. The significance of absolute and relative boundary conditions stems from the well-known [13]:

THEOREM 6.1. (i) *The singular cohomology group $H^p M$ is isomorphic to the space of harmonic p-forms satisfying absolute boundary conditions.*

(ii) *The singular cohomology group $H^p(M, \partial M)$ is isomorphic to the space of harmonic p-forms satisfying relative boundary conditions.*

Suppose that $\gamma_1, \ldots, \gamma_{n-1}$ are eigenvalues of the second fundamental form of $\partial M$. The $\gamma_i$ are functions on $\partial M$ and we may define a real number $\sigma_p$ by

$$\sigma_p = \max_{x \in \partial M} \max_{I} \gamma_{i_1} + \cdots + \gamma_{i_p},$$

where $I = (i_1, \ldots, i_p)$ is a multi-index. The following technical lemma will be required:

LEMMA 6.2. *Denote $N$ to be an inward pointing normal vector along $\partial M$. Let $a \in \Lambda^p M$, then $\nabla_N \|a\|^2 \geqslant 0$ if either:*

(i) *The p-form $a$ satisfies absolute boundary conditions and $\sigma_p \leqslant 0$.*

(ii) *The p-form $a$ satisfies relative boundary conditions and $\sigma_{n-p} \leqslant 0$.*

*Proof.* Since $*$ induces an isomorphism from relative to absolute boundary conditions, it suffices to establish (ii).

Fix $x \in \partial M$. To compute $\nabla_N \|a\|^2$, we choose a co-frame field $\omega_1, \ldots, \omega_{n-1}, \omega_n$, so that the second fundamental form is diagonalized at $x$. In coordinates with respect to this frame, let $a_{i_1 \ldots i_p}$ be the components of $a_{\tan}$ and $a_{j_1 \ldots j_{p-1} n}$ the components of $a_{\text{norm}}$, where the indices $i, j$ run from 1 to $n-1$.

The relative boundary conditions read $a_{i_1 \ldots i_p} = 0$ for $a_{\tan} = 0$ and

$$\sum_{\alpha} a_{k_1 \ldots k_p \alpha, \alpha} + a_{k_1 \ldots k_{p-1} n, n} = 0$$

for $(\delta a)_{\tan} = 0$. Here, the index $\alpha$ runs from 1 to $n-1$. An index following a comma means covariant differentiation.

Now

$$\tfrac{1}{2} \nabla_N \|a\|^2 = (\nabla_N a, a) = \sum_J a_{j_1 \ldots j_{p-1} n} a_{j_1 \ldots j_{p-1} n, n}$$

since $a_{\tan} = 0$.

Using that $(\delta a)_{\tan} = 0$ gives $\frac{1}{2} \nabla_N \|a\|^2 = -\sum_{J, \alpha} a_{j_1 \ldots j_{p-1} \alpha, \alpha} a_{j_1 \ldots j_{p-1} n}$.

However, differentiating along $\partial M$,

$$(\nabla_\alpha a)_{\tan} = \sum_J a_{j_1 \ldots j_{p-1} n} \omega_{j_1} \wedge \cdots \wedge \omega_{j_{p-1}} \wedge (\nabla_\alpha \omega_n)_{\tan}.$$

From the definition of the second fundamental form

$$(\nabla_\alpha a)_{\tan} = \sum_J a_{j_1 \ldots j_{p-1} n} \gamma_\alpha \omega_{j_1} \wedge \cdots \wedge \omega_{j_{p-1}} \wedge \omega_\alpha$$

where no sum on $\alpha$ is intended in the last equation.

Thus $\frac{1}{2} \nabla_N \|a\|^2 = -\sum_J \sum_{\alpha \notin J} \gamma_\alpha (a_{j_1 \ldots j_{p-1} n})^2$ where $J = (j_1, \ldots, j_{p-1})$ is summed over all increasing multi-indices.

By definition of $\sigma_{n-p}$: $\frac{1}{2} \nabla_N \|a\|^2 \geqslant -\sigma_{n-p} \sum_J (a_{j_1 \ldots j_{p-1} n})^2$.

So that along $\partial M$, $\frac{1}{2} \nabla_N \|a\|^2 \geqslant -\sigma_{n-p} \|a\|^2 \geqslant 0$.

This proves Lemma 6.2. □

The Weitzenbock formula reads $\Delta = -D + R_p$ where $D$ is Bochner's Laplacian and $R_p : \Lambda^p M \to \Lambda^p M$ is an endomorphism depending upon the curvature tensor of $M$. Suppose that $(R_p v, v) \geqslant -b_p (v, v)$ for some real number $b_p$. By $*$-duality, one has $b_p = b_{n-p}$.

Let $K_a$ denote the heat kernel for the Hodge Laplacian with absolute boundary conditions and $K_r$ the analogous kernel for relative boundary conditions. Let $K_{n,0}$ be the heat kernel on functions with Neumann boundary conditions.

The main result of this section is:

THEOREM 6.4. (i) *If* $\sigma_p \leqslant 0$, *then for the heat kernel on p-forms* $|K_a| \leqslant e^{b_p t} K_{n,0}$, $1 \leqslant p \leqslant n-1$.

(ii) *If* $\sigma_{n-p} \leqslant 0$, *then for the heat kernel on p-forms* $|K_r| \leqslant e^{b_p t} K_{n,0}$, $1 \leqslant p \leqslant n-1$.

(iii) *If* $\partial M = \phi$, *no side condition is required to assure that* $|K| \leqslant e^{b_p t} K_0$, $1 \leqslant p \leqslant n-1$. *Here K denotes the heat kernel on p-forms and* $K_0$ *the heat kernel on functions.*

*Proof.* This is entirely analogous to the proof of Theorem 4.3. In applying Duhamel's principle, a boundary integral arises. The sign of the integrand is controlled with Lemma 6.2.

Using Lemma 2.4 and Theorems 6.1, 6.4, one obtains upper bounds for the betti numbers of $M$. Set $\beta_p M = \dim H^p M$ and $\beta_p(M, \partial M) = \dim H^p(M, \partial M)$. Then one has

COROLLARY 6.5. (i) *If* $\sigma_p \leqslant 0$, *then* $\beta_p M \leqslant \binom{n}{p} e^{b_p}(1 + C_5 \operatorname{vol} M)$.

(ii) *If* $\partial M = \phi$ *then* $\beta_p M \leqslant \binom{n}{p} e^{b_p} \operatorname{vol} M(C_1 + C_2)$.

*Here* $C_1$, $C_2$, $C_5$ *have the geometric dependence described in Section* 3. *The analogous bounds to* (i) *for* $\beta_p(M, \partial M)$ *follow by Lefschetz duality.*

Similarly, for the positive eigenvalues of the Hodge Laplacian, $0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots$, one has

COROLLARY 6.6. (i) *Suppose that* $\sigma_p \leqslant 0$ *and one imposes absolute boundary conditions on the Hodge Laplacian* $\Delta$. *Then for some m and all* $k \geqslant m$, $\lambda_k \geqslant C_6 k^{2/n}$. *Here* $m, C_6$ *depend only on* $n$, $p$, $b_p$, $\operatorname{vol} M$, *and* $C_5$ *of Theorem* 3.5.

(ii) *If* $\partial M = \phi$, *then for some m and all* $k \geqslant m$, $\lambda_k \geqslant C_7 k^{2/n}$. *Here* $m, C_7$ *depend only on* $n$, $p$, $b_p$, $\operatorname{vol} M$, *and* $C_1, C_2$ *of Theorem* 3.3

The results analogous to (i) for relative boundary conditions follow by $*$-duality.

Corollary 6.6 follows from Theorem 6.4 using Lemmas 2.5 and 2.6.

REMARK. If $M$ is a domain in the flat plane $\mathbf{R}^2$ then $b_1 = 0$. Set $p = 1$, in Part (i) of Theorem 6.4, so that the conclusion reads $|K_a| \leqslant K_{n,0}$. Taking the trace and letting $t \to \infty$ yields $\beta_1 M \leqslant 2$. One observes that a domain with $l$-holes will have $\beta_1 M = l$. This shows that some condition on the second fundamental form, such as $\sigma_1 \leqslant 0$, is required for Theorem 6.4 to hold.

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