ZEROS OF SUCCESSIVE DERIVATIVES OF FUNCTIONS ANALYTIC IN A NEIGHBORHOOD OF A SINGLE POLE

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1. Introduction. Pólya [10] defines the *final set* of a meromorphic function f to be the set $S=S(f)$ consisting of all points z such that every disc centered at z contains zeros of infinitely many derivatives of f. In other words, z is in S if and only if $z=\lim_{k\to\infty}(z_k)$, where $\{z_k\}_1^\infty$ is a sequence such that $f^{(n_k)}(z_k)=0$, $n_1<n_2<n_3<\cdots$. The final set determines, roughly speaking, the final position of the zeros of the derivatives of f. Pólya gave a complete description of the final set of an arbitrary meromorphic function in the following remarkable theorem ([9, Theorem 3], [10, p. 180]).

THEOREM. *Let f be meromorphic. If f has two or more poles, then $z\in S$ if and only if z is equidistant from the two poles which are nearest to it. If f has one pole, the final set is empty.*

If z_0 is a pole of f, call the set of points in the plane which are closer to z_0 than to any other pole the "domain" of z_0. The common boundary of the domains of two poles consists of the perpendicular bisector of the line segment joining them. The poles thus can be thought of as repellers of equal strength of the zeros of the derivatives of f.

For proofs of Pólya's Theorem and related material, see [2, p. 98], [7, Theorem 3.6], [11], [12], [14, pp. 32–38].

The main result of the present paper deals with zero-free neighborhoods of the origin of derivatives of functions of the form

$$F(z)=\frac{a_{-1}}{z}+A(z)=\frac{a_{-1}}{z}+a_0+a_1z+a_2z^2+\cdots, \tag{1.1}$$

where $a_{-1}\neq 0$, and where $A(z)$ is analytic in a neighborhood of $z=0$ and possibly entire. It is clear that there corresponds to each derivative $F^{(n)}$ a certain disc about $z=0$ where $F^{(n)}$ is zero-free. We prove several results about these zero-free regions which complement Pólya's Theorem and which are, in a certain sense, stronger.

THEOREM 1. *Let $F(z)$ be given by* (1.1).

(a) *If $A(z)$ is analytic in $|z|<R$ and $\alpha<\frac{1}{2}$, then for all n sufficiently large $F^{(n)}(z)$ has no zero in $|z|\leq\alpha R$. The constant $\frac{1}{2}$ is best possible.*

(b) *If $A(z)$ is entire, and of exponential type T or less, if c_0 denotes the unique real (positive) root of the equation $xe^{1+x}=1$, and if $c<c_0$, then for all n sufficiently large $F^{(n)}(z)$ has no zero in $|z|\leq cT^{-1}(n+1)$. The constant c_0 is best possible ($c_0=.27846$, approximately).*

(c) *If $A(z)$ is entire and of order ρ and type τ $(0<\rho,\ \tau<\infty)$ and $\gamma<[2(e\rho\tau)^{1/\rho}]^{-1}$, then for all n sufficiently large $F^{(n)}(z)$ has no zero in $|z|\leq\gamma(n+1)^{1/\rho}$.*

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To see that $\frac{1}{2}$ cannot be replaced by a larger constant in (a), it is enough to consider the function $F_N(z) = z^{-1} - (1-z^N)^{-1}$, where N is a positive integer. Here $R=1$. By Pólya's Theorem the final set consists of a regular N-sided polygon circumscribing the circle $|z| = \frac{1}{2}$, together with rays extending from the vertices. A disc about $z=0$ of radius $r < \frac{1}{2}$ will contain at most a finite number of zeros of derivatives of $F_N(z)$, and discs of radius $r > \frac{1}{2}$ contain infinitely many such zeros. For ever increasing N, the domain of the point $z=0$ of F_N approaches the disc $|z| < \frac{1}{2}$. To some extent, then, part (a) is an asymptotic version of Pólya's Theorem.

Note that no assumption is made concerning the other singularities of $F(z)$ in (a). In particular, F may have essential singularities on $|z| = R$, or $|z| = R$ could even be a natural boundary. This situation, of course, is beyond the range of Pólya's Theorem.

It is worth emphasizing that the results of Theorem 1 hold for all derivatives past a certain point, while Pólya's Theorem deals only with the behavior of an infinite sequence of derivatives.

Concerning (b), each odd-order derivative of the function $z^{-1} + e^z$ has a unique real root z_n, where z_n satisfies the asymptotic formula $z_n \sim c_0(n+1)$, $n \longrightarrow \infty$. Thus the constant c_0 is best possible. The constant $[2(e\rho\tau)^{1/\rho}]^{-1}$ of (c) is apparently not best possible; determination of the correct constant is left as an open problem.

There are several parallels between the results of Theorem 1 and the Whittaker-Gončarov theory of zeros of successive derivatives of entire functions and analytic functions in $|z| < R$. We refer the reader to [4], [5], [6], [13].

If, instead of (1.1), one considers functions of the form

$$F(z) = \frac{a_{-N}}{z^N} + \cdots + \frac{a_{-1}}{z} + A(z), \quad a_{-N} \neq 0, \tag{1.2}$$

then a result similar to Theorem 1 holds. We indicate its precise version below. It is crucial for the present setting, however, that the singularity at $z=0$ be only a pole. The case where F is of the form

$$F(z) = a_N z^N + \cdots + a_1 z + \sum_{k=0}^{\infty} a_{-k} z^{-k}$$

is the subject of [3] and [13]. The authors hope to study the analogous problem for doubly infinite Laurent series in a future paper.

The foregoing results are proved with the aid of the differential operator given by $\theta = z^{1+p}\frac{d}{dz}$. The iterates of operators and those of derivatives are related by a formula $(\theta^n f)(z) = (D^n g)(\omega)$, for $g(\omega) = f(z)$ and an appropriate change of variable $\omega = \omega(z)$.

We now state the main result for the operator $\theta = z^{1+p}\frac{d}{dz}$. Let $\{R_k\}_{k=1}^{\infty}$ be a positive, nondecreasing sequence, let $E(s) = \sum_{k=1}^{\infty} (s/R_{k+1})^{k+1}$ and let $c(E)$ denote the radius of convergence of E $(R_1 \leq c(E) \leq \infty)$. Let the function

$$f(z) = a_{-1}z + a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots \tag{1.3}$$

satisfy the condition

$$\limsup_{n\to\infty} R_{n+1}|a_n|^{1/(n+1)} = 1. \tag{1.4}$$

Note that $f(z)=F(1/z)$, where $F(z)$ is given by (1.1), and that the specific choices $R_n=R$, $R_{n+1}=T^{-1}(n!)^{1/(n+1)}$, and $R_n=(n/e\rho\tau)^{1/\rho}$ lead respectively to cases (a), (b) and (c) of Theorem 1.

THEOREM 1′. *Let $p\geq 1$. Then there is a positive sequence $\{r_n\}_1^\infty$ satisfying $r_n \longrightarrow c(E)$ $(n\longrightarrow\infty)$ such that if $0<\epsilon<1$, then*

$$(\theta^n f)(z) \neq 0 \quad \textit{for} \quad |z|^{-1} < (1-\epsilon)\min\{r_n, R_{n+1}/2\}, \tag{1.5}$$

and for all n sufficiently large.

We obtain Theorem 1 as a direct result of Theorem 1′. Our proof of Theorem 1′ is based on a method suggested by [9] (also see [14, pp. 33–34]) except that we use power series techniques for the most part. One can prove Theorem 1 directly by using the method, although this does not seem to have been done. The advantages of using the more general approach provided by θ and $\{R_k\}$ are that (a), (b) and (c) of Theorem 1 are reduced to essentially one proof, and that we can also obtain results for classes of functions other than (1.1). Specifically, we find zero-free regions for generalized power series given by

$$h(\omega) = a_{-1}\omega^{-1/p} + a_0 + a_1\omega^{1/p} + a_2\omega^{2/p} + \cdots \tag{1.6}$$

and for Dirichlet series of type

$$g(\omega) = a_{-1}e^{\omega} + a_0 + a_1e^{-\omega} + a_2e^{-2\omega} + \cdots. \tag{1.7}$$

This is done in §3.

2. A differential operator. Let p be a nonnegative number, and let θ be the operator defined by $\theta = z^{1+p}\dfrac{d}{dz}$. The iterates of θ onto the various powers of z are given by

$$\begin{aligned}\theta^n(z^k) &= (k)(k+p)(k+2p)\cdots(k+(n-1)p)z^{k+np}\\ &= C_{kn}z^{k+np};\quad n=1,2,3,\ldots;\quad k=0,\pm1,\pm2,\ldots.\end{aligned} \tag{2.1}$$

For $k\geq 1$ and $n\geq 1$, clearly we have

$$0 \leq C_{kn} \leq k^n(1+np)^n, \qquad \text{and} \qquad |C_{-k,n}| \leq C_{kn}. \tag{2.2}$$

Also, $C_{0n}=0$.

Let $R>0$ and let the function f be given by

$$f(z) = \sum_{k=-1}^{\infty} a_k z^{-k}, \quad |z|>\frac{1}{R}. \tag{2.3}$$

Then

$$\begin{aligned}(\theta^n f)(z) &= \sum_{k=-1}^{\infty} a_k\theta^n(z^{-k}) = \sum_{k=-1}^{\infty} a_k C_{-k,n}z^{-k+np}\\ &= z^{1+np}\sum_{k=-1}^{\infty} a_k C_{-k,n}z^{-k-1}, \quad |z|>\frac{1}{R}.\end{aligned} \tag{2.4}$$

We will express $(\theta^n f)(z)$ in another way. Let $G_n(w)$ be the function defined by

$$G_n(w) = \sum_{k=-1}^{\infty} C_{-k,n} w^{-k-1}, \quad n = 1,2,3,\ldots.$$

By (2.2), we see that the series for $G_n(w)$ converges in the range $|w|>1$, and in fact $G_n(w)$ is analytic there. Fix a number r, $0<r<R$, let $|t|=r$ and $|z|>r^{-1}>R^{-1}$. Then

$$G_n(zt) = C_{1n} + \frac{C_{-1,n}}{z^2t^2} + \frac{C_{-2,n}}{z^3t^3} + \cdots$$

and it follows from the Cauchy Integral Formula that

$$C_{-k,n} z^{-k-1} = (2\pi i)^{-1} \int_{|t|=r} G_n(zt) t^k dt, \quad |z| > \frac{1}{r}, \quad k = -1,0,1,2,\ldots. \tag{2.5}$$

Substituting this into (2.4) yields

$$(\theta f)(z) = z^{1+np} \sum_{k=-1}^{\infty} a_k (2\pi i)^{-1} \int_{|t|=r} G_n(zt) t^k dt, \tag{2.6}$$

valid over the ranges indicated by (2.5). In what follows, a prime will indicate that the term corresponding to $k=0$ is omitted from a sum. Interchanging integration and summation in (2.6), justified by uniform convergence, we obtain

$$\begin{aligned}(\theta^n f)(z) &= z^{1+np}(2\pi i)^{-1} \int_{|t|=r} \sum_{k=-1}^{\infty}{}' a_k t^k G_n(zt)\, dt \\ &= z^{1+np}(2\pi i)^{-1} \int_{|t|=r} (f(1/t) - a_0) G_n(zt)\, dt,\end{aligned} \tag{2.7}$$

and this is valid for $|z|>1/r$ and $n=1,2,3,\ldots.$ If we replace z by a new variable,

$$x = C_{2n}/C_{1n} z, \tag{2.8}$$

and factor out the quantity C_{1n}, (2.7) becomes

$$(\theta^n f)(z) = C_{1n} z^{1+np} (2\pi i)^{-1} \int_{|t|=r} (f(1/t) - a_0) \left\{ \sum_{k=-1}^{\infty}{}' \frac{C_{-k,n}}{C_{1n}} \left[\frac{C_{1n}x}{C_{2n}t}\right]^{k+1} \right\} dt, \tag{2.9}$$

whenever z and x are related by (2.8) and $|x|<C_{2n}r/C_{1n}$, $n=1,2,3,\ldots.$ The integral in (2.9) times the factor $(2\pi i)^{-1}$ will be denoted by $I_n(x)$,

$$I_n(x) = (2\pi i)^{-1} \int_{|t|=r} (f(1/t) - a_0) \left\{ \sum_{k=-1}^{\infty}{}' \frac{C_{-k,n}}{C_{1n}} \left[\frac{C_{1n}x}{C_{2n}t}\right]^{k+1} \right\} dt,$$

and so we can write

$$(\theta^n f)(z) = C_{1n} z^{1+np} I_n(x), \quad |C_{2n} z^{-1}| = |C_{1n} x| < C_{2n} r. \tag{2.10}$$

The following lemma pertains to the asymptotic behavior of the sequence $\{I_n(x)\}$.

LEMMA 2.1. *The quotient* (C_{1n}/C_{2n}) *converges to* 0 *as* $n \longrightarrow \infty$. *For each fixed* $k=1,2,3,\ldots$ *the sequence* $S_{-k,n}$ *defined by*

$$S_{-k,n} = (C_{-k,n}/C_{1n})[C_{1n}/C_{2n}]^{k+1}, \quad n = 1,2,3,\ldots$$

is convergent to 0.

Proof. Recall the following properties of the gamma function Γ [1, pp. 256–258]

$$(1+z)(2+z)\cdots(n-1+z)\Gamma(1+z) = \Gamma(n+z), \tag{2.11}$$

$$\Gamma(1-z)\Gamma(z) = \pi\csc(\pi z), \tag{2.12}$$

and

$$\lim_{n\to\infty} n^{b-a}\frac{\Gamma(n+a)}{\Gamma(n+b)} = 1. \tag{2.13}$$

By (2.11)

$$\begin{aligned} C_{1n} &= (1)(1+p)(1+2p)\cdots(1+(n-1)p) \\ &= p^{n-1}(1+p^{-1})(2+p^{-1})\cdots(n-1+p^{-1}) \\ &= p^{n-1}\Gamma(n+p^{-1})/\Gamma(1+p^{-1}) \end{aligned}$$

Similarly,

$$\begin{aligned} C_{2n} &= 2p^{n-1}(1+2p^{-1})(2+2p^{-1})\cdots(n-1+2p^{-1}) \\ &= 2p^{n-1}\Gamma(n+2p^{-1})/\Gamma(1+2p^{-1}). \end{aligned}$$

Consequently,

$$C_{1n}/C_{2n} = \{\Gamma(1+2p^{-n})/2\Gamma(1+p^{-n})\}\Gamma(n+p^{-1})/\Gamma(n+2p^{-1}),$$

and this tends to 0 with order $n^{-1/p}$ by (2.13).

Using the same idea on $C_{-k,n}$, now along with (2.12),

$$\begin{aligned} C_{-k,n} &= (-k)p^{n-1}\Gamma(n-kp^{-1})/\Gamma(1-kp^{-1}) \\ &= (-k)p^{n-1}\pi^{-1}\Gamma(n-kp^{-1})\Gamma(kp^{-1})\sin(k\pi/p). \end{aligned}$$

Then we can write

$$\frac{C_{-k,n}}{C_{1n}} = (-k/\pi)\sin(k\pi/p)\Gamma(1+p^{-1})\frac{\Gamma(kp^{-1})\Gamma(n-kp^{-1})}{\Gamma(n+p^{-1})}. \tag{2.14}$$

In view of (2.13) again, this tends to 0 for fixed k at the rate $n^{-(k+1)/p}$. Therefore $S_{-k,n}$ tends to 0 at the rate $n^{-2(k+1)/p}$, which completes the proof. □

The functions $I_n(x)$ are analytic in x for $|x|<C_{1n}^{-1}C_{2n}r$. By Lemma 2.1, the sequence $\{I_n(x)\}_{n=1}^{\infty}$ converges uniformly on compact sets to the constant function a_{-1}. This leads to the following result.

THEOREM 2.1. *Let* f *be given by* (2.3) *and let* $\beta>0$ *be an arbitrary constant. Then for all* n *sufficiently large,* $(\theta^n f)(z)$ *has no zero which satisfies* $|z|\geq C_{2n}/C_{1n}\beta$.

Proof. On the contrary, suppose there exists a subsequence of points z_{n_k} for which $(\theta^{n_k}f)(z_{n_k})=0$ and $|z_{n_k}|\geq C_{2n_k}/C_{1n_k}\beta$, $k=1,2,3\ldots$. Put $x_{n_k}=C_{1n_k}^{-1}C_{2n_k}z_{n_k}^{-1}$, so that $|x_{n_k}|\leq\beta$. By (2.8) and (2.10), $I_{n_k}(x_{n_k})=0$. Thus a subsequence of $\{x_{n_k}\}$ must converge to a point x_0 where the limit function, the constant a_{-1}, vanishes. This contradiction proves the theorem. □

3. Main proofs and applications. We require some technical results concerning the sequence r_n of Theorem 1′.

Define the sequence δ_n for $n=1,2,3,\ldots$ by $\delta_n=\max_{1\leq k\leq[(n-1)p]}|C_{-k,n}/C_{1n}|$, where $[\eta]$ denotes the greatest integer not exceeding η.

LEMMA 3.1. *The sequence $\delta_n n^{1/p}$ is bounded as $n\longrightarrow\infty$; in order-of-magnitude notation, $\delta_n=O(n^{-1/p})$, $n\longrightarrow\infty$.*

Proof. We use the representation (2.14), where $1\leq k\leq[(n-1)p]$. Consider the term $\Gamma(kp^{-1})\Gamma(n-kp^{-1})$. Let $y(x)=\Gamma(x)\Gamma(n-x)$, $0<x<n$. Note that $p^{-1}\leq kp^{-1}\leq[(n-1)p]p^{-1}\leq n-1$, for $1\leq k\leq[(n-1)p]$, so that the values k/p fall within the domain of $y(x)$. Now $y'(x)=\Gamma'(x)\Gamma(n-x)-\Gamma(x)\Gamma'(n-x)$, and this is negative if and only if $\frac{\Gamma'}{\Gamma}(x)<\frac{\Gamma'}{\Gamma}(n-x)$, that is, $\psi(x)<\psi(n-x)$ where ψ is the "Digamma" function $\psi=\Gamma'/\Gamma$ [1, pg. 258]. Since $\psi(x)$ is increasing, we must have $y'(x)<0$ if and only if $0<x<n/2$. Therefore, $y(x)$ is decreasing on $(0,n/2)$ and increasing on $(n/2,n)$. By symmetry, if $A<B$ and B is nearer to 1 than A is to 0, then $y(A)<y(B)$. Let k_0 be the first integer (independent of n) such that $k_0/p=|(k_0/p)-0|\geq n-(n-1)=1$. Then $y(k_0/p)\leq y(n-1)$. Also $y(k/p)\leq y(n-1)$ for $k_0\leq k\leq[(n-1)p]$. Thus if $k_0\leq k\leq[(n-1)p]$, then (2.14) and the above discussion give

$$\begin{aligned}\left|\frac{C_{-k,n}}{C_{1n}}\right| &\leq (n/\pi)\Gamma(1+p^{-1})y(k/p)/\Gamma(n+p^{-1})\\ &\leq (n/\pi)\Gamma(1+p^{-1})y(n-1)/\Gamma(n+p^{-1})\\ &= \{\pi^{-1}\Gamma(1+p^{-1})\Gamma(1)\}n\Gamma(n-1)/\Gamma(n+p^{-1}).\end{aligned}$$

Noting (2.13), we see that

$$|C_{-k,n}/C_{1n}|\leq m_0 n^{-1/p},\quad k_0\leq k\leq[(n-1)p], \tag{3.1}$$

where m_0 is a constant independent of k. Now for fixed k, $1\leq k\leq k_0$, the proof of Lemma 2.1 shows that $|C_{-k,n}/C_{1n}|\leq m_1 n^{-(k+1)p}\leq m_1 n^{-2/p}$, where m_1 is an absolute constant. Considering (3.1) and the definition of δ_n, this completes the proof. □

Let f be given by (1.3), suppose (1.4) holds, and let $0<\epsilon<1$, $L=(1-\epsilon)^{-1}$. Then there is a constant $M=M(f,\epsilon)$ such that

$$|a_k|\leq M(L/R_{k+1})^{k+1},\quad k=0,1,2,\ldots. \tag{3.2}$$

Define the sequence $r_n=r_n(f,M,\{R_k\}_1^\infty,\epsilon)$ by the formula

$$r_n=\sup\{s\,|\,0<s<c(E),\ \ \delta_n ME(s)\leq|a_{-1}|/2\},\quad n=1,2,3\ldots. \tag{3.3}$$

Since $\delta_n\longrightarrow 0$, clearly $r_n\longrightarrow c(E)$ as $n\longrightarrow\infty$.

LEMMA 3.2. *If n is sufficiently large, $r_n \geq R_1(1-2\delta_n M|a_{-1}|^{-1})$. If for some $\rho>0$, $R_n/n^{1/\rho}$ has a finite and positive limit as $n \longrightarrow \infty$, then $r_n \geq \lambda(\ln n)^{1/(\rho+\epsilon)}$ for all n sufficiently large, where $\lambda>0$ is a constant.*

Proof. We have $\delta_n ME(s) \leq \delta_n M \sum_{k=1}^{\infty} (s/R_1)^{k+1} \leq \delta_n M(1-sR_1^{-1})^{-1}|a_{-1}|/2$ if $s \leq R_1(-2\delta_n M|a_{-1}|^{-1})$. This proves the first part. If $0<\lim_{n\to\infty}(R_n/n^{1/\rho})<\infty$, then $E(s)$ is an entire function of order ρ. Thus $E(s) \leq K\exp(s^{\rho+\epsilon})$, for a constant K, and the result follows on taking logarithms and noting the asymptotic form of δ_n. □

Proof of Theorem 1′. We are given $0<\epsilon<1$ and an $f(z)$ which satisfies (1.3), (1.4) and (3.2). Let r_n be defined by (3.3). By (2.8) and (2.10), $(\theta^n f)(z) \neq 0$ if and only if $I_n(x) \neq 0$. Using the power series for $f(z)$ and integrating, we find that

$$I_n(x) = a_{-1} + \sum_{k=1}^{\infty} a_k (C_{-k,n}/C_{1n}) z^{-k-1}. \tag{3.4}$$

We will show that $I_n(x) \neq 0$ for appropriately restricted z.

We break the sum in (3.4) into two parts at the point $k=[(n-1)p]$, square brackets denoting the greatest integer function. If $k \geq [(n-1)p]+1$, then $k>(n-1)p \geq n-1$, since $p \geq 1$, so $k \geq n$ in this range. Also for $k \geq [(n-1)p]+1$ and $p \geq 1$,

$$\left|\frac{C_{-k,n}}{C_{1n}}\right|$$

$$= \frac{k!}{n!\,(k-n)!} \cdot \frac{(k)(k-p)\cdots(k-(n-1)p)}{(k)(k-1)\cdots(k-(n-1))} \cdot \frac{(1)(2)\cdots(n)}{(1)(1+p)\cdots(1+(n-1)p)}$$

$$\leq \frac{k!}{n!\,(k-n)!} = \binom{k}{n},$$

the binomial coefficient. Using (3.2) and the fact that $\{R_n\}$ is decreasing, there follows

$$\sum_{[(n-1)p]+1}^{\infty} |a_k(C_{-k,n}/C_{1n})z^{-k-1}| \leq M \sum_{k=n}^{\infty} \binom{k}{n} |L/zR_{k+1}|^{k+1}$$

$$\leq M \sum_{k=n}^{\infty} \binom{k}{n} (L/R_{n+1}|z|)^{k+1} \tag{3.5}$$

$$= M(L/R_{n+1}|z|)^{n+1}(1-(L/R_{n+1}|z|)^{-(n+1)}.$$

Putting $X=(L/R_{n+1}|z|)$, the above expression is $M(X/1-X)^{n+1}$. This quantity tends to 0 as $n \longrightarrow \infty$ if $0<X<1/2$. Therefore, (3.5) can be made arbitrarily small if n is sufficiently large and $|z|^{-1}<(R_{n+1}/2L)=(1-\epsilon)R_{n+1}/2$. Now consider the terms of (3.4) corresponding to $1 \leq k \leq [(n-1)p]$. We have

$$\sum_{k=1}^{[(n-1)p]} |a_k(C_{-k,n}/C_{1n})z^{-k-1}| \leq M\delta_n \sum_{k=1}^{[(n-1)p]} (L/R_{n+1}|z|)^{k+1}$$

$$\leq \delta_n ME(L/|z|) < |a_{-1}|/2$$

provided that $(L/|z|) < r_n$, that is $|z|^{-1} < (1-\epsilon)r_n$. It follows that $I_n(x) \neq 0$ if $|z|^{-1} < (1-\epsilon)\min\{r_n, R_{n+1}/2\}$, and this completes the proof of Theorem 1′. □

Applications. (1) *Proof of Theorem* 1. Let $F(z)$ be given by (1.1), $f(z) = F(1/z)$ by (1.3), and let $p=1$. Then $\theta = z^2 \frac{d}{dz}$, and if $\omega = 1/z$ then $(\theta f)(z) = (DF)(\omega)$, $(\theta^n f)(z) = (-1)^n (D^n F(\omega))$. Since $p=1$, $C_{-k,n} = 0$ for $k<n$, and so $\delta_n = 0$ for all n. In this case $r_n = c(E) \geq R_{n+1}$, and so r_n does not figure into condition (1.5). For part (a) take $R_n = R$. The resulting form of (1.5) is equivalent to (a), with z now replaced by $\omega = 1/z$, z satisfying (1.5). For (c) take $R_n = (n/e\rho\tau)^{1/\rho}$. Recall the remarks made just after (1.4). To prove (b) we let $R_{n+1} = T^{-1}(n!)^{1/(n+1)}$. Condition (1.5) gives zero-free regions for $F^{(n)}(\omega)$ which have size approximately $|\omega| < (n+1)/2e$. To get the best constant, let $c < c_0$ and pick $\epsilon > 0$ so that $cL = (1-\epsilon)^{-1}c = c' < c_0$. Next, put $R_{k+1} = T^{-1}(k!)^{1/(k+1)}$ into the second member of (3.5). Then (3.5) does not exceed

$$\frac{M}{n!}(TL/|z|)^{n+1}\exp(TL/|z|).$$

If $T|\omega| = (T/|z|) < c(n+1)$, then $(TL/|z|) < c'(n+1)$. In this case (3.5) does not exceed

$$\frac{M}{n!}(c'(n+1))^{n+1}e^{c'(n+1)} < M(e/n)^n(n+1)^{n+1}(c'e^{c'})^{n+1}$$
$$= Me^{-1}(1+n^{-1})^n(n+1)(c'e^{1+c'})^{n+1}.$$

This quantity tends to 0 as $n \longrightarrow \infty$ by definition of c_0 and c'. Therefore $(\theta^n f)(z) \neq 0$, all large n, when $|z|^{-1} < T^{-1}c(n+1)$. Thus $(D^n F)(\omega) \neq 0$ for all large n provided $|\omega| < T^{-1}c(n+1)$. This completes the proof of Theorem 1. □

(2) *Dirichlet Series.* Let $p=0$ (we do not need Theorem 1′ for this example). The operator θ becomes $\theta = z\frac{d}{dz}$, and this corresponds to the change of variable $f(z) = g(\operatorname{Ln} z)$, that is, $(\theta^n f)(z) = g^{(n)}(\operatorname{Ln} z)$ when f and g are so related. If we start with a function given by the Dirichlet series (1.7), defined for

$$\operatorname{Re}(\omega) > \operatorname{Ln} R^{-1} (\text{e.g., } e^{\omega}\coth(\omega/2))$$

then $f(z) = g(\operatorname{Ln} z)$ satisfies (2.3). Theorem 2.1 implies that the derivatives of g have zero-free regions. Since $C_{1n} = 1$ and $C_{2n} = 2^n$, the zero-free regions are of the form $\operatorname{Re}(\omega) \geq n\operatorname{Ln}(2) - \operatorname{Ln}(\beta)$, for all n sufficiently large, where β is any fixed constant.

(3) *Noninteger Power Series.* Let $p \geq 1$ and $\theta = z^{1+p}\frac{d}{dz}$. Let $h(\omega)$ be given by (1.6). Zero-free regions in this example are understood to be intersections of discs with the cut plane, and $\omega^{1/p}$ will denote the principle branch of the power function. Putting $\omega = z^{-p}$ and $f(z) = h(\omega) = h(z^{-p})$, we can apply Theorem 1′ to f. Note that $(\theta f)(z) = (-p)(Dh)(\omega)$, $(\theta^n f)(z) = (-p)^n(D^n h)(\omega)$. Therefore, $(D^n h)(\omega) \neq 0$ for all n sufficiently large when $|\omega| < (1-\epsilon)\min\{r_n^p, (R_{n+1}/2)^p\}$. It is of interest that (by Lemma 3.2) the growth rates of r_n and R_{n+1} as $n \longrightarrow \infty$ are compatible, at least in the two principal cases $R_n \sim R$ and $R_n \sim n^{1/\rho}$.

Finally, if $F(z)$ is given by (1.2) instead of (1.1), then the results analogous to Theorem 1 and Theorem 2.1 are obtained by replacing $(n+1)$ by $(n+N)/N$ and C_{2n}/C_{1n} by $C_{N+1,n}/C_{Nn}$, respectively.

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