REPRESENTATION OF FUNCTIONS OF SEVERAL VARIABLES DEFINED ON THE TORUS

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Dedicated to Professor Casper Goffman on the occasion of his retirement at Purdue University.

1. Introduction. In answering a question, posed by Lusin, in connection with the representation of measurable functions, Men'šov [2, 8] proved that for any finite almost everywhere (abbreviated a.e.) measurable function f on $T=[0,2\pi]$, there exists a trigonometric series convergent to f a.e.. Bary [2] pointed out such a trigonometric series can be obtained from the term-by-term differentiation of the Fourier series of a primitive for the function f.

The representation of measurable functions of two variables by double trigonometric series was first studied by Dzhavarsheishvili [6]. Subsequently various representation problems for functions of several variables by multiple series were discussed by Dzagnidze ([4] and [5]) and Topuriya [9]. But their results are far from pointwise convergent representation. In connection with the Men'šov and Bary result for functions of several variables, it is natural to ask whether every finite a.e. measurable function f on T^n can be represented by an n-fold trigonometric series convergent to f a.e., summed either by squares or by rectangles [10].

In our previous work [3], we proved that any finite a.e. measurable function f on T^n can be represented by an n-fold trigonometric series convergent to $f(x)$ a.e. summed by squares. In the present article we show that any such a function can be represented by an n-fold trigonometric series convergent to the given function a.e. summed by rectangles.

Even for the case of a single variable, this result is quite deep since even for an integrable function the Fourier series may fail at every point to converge to the function. The result is much deeper in the case of functions of several variables because of the existence of functions continuous on T^n such that the rectangular partial sums of their n-tuple Fourier series diverge everywhere [7]. Neither of the proofs given by Men'šov and Bary can be extended to functions of several variables. We need to develop some fundamental tools. Several of the basic ideas for our main results are taken from our previous work [3]. For convenience and for notational simplicity, we give the proof of our theorem explicitly for functions on T^2.

2. Preliminaries and notations. By the 2-dimensional torus we mean the set of points $x=(x_1,x_2)$ from $T^2=[0,2\pi]\times[0,2\pi]$. Let $m=(m_1,m_2)$ be an integer lattice point of $\mathbf{R}^2$. Then for an integrable function F on T^2 the Fourier series for F is $S[x;F]=\Sigma_m \hat{F}_m e^{im\cdot x}$, where $\hat{F}_m=1/(2\pi)^2\int_{T^2}F(x)e^{-im\cdot x}dx$ with $m\cdot x=m_1x_1+m_2x_2$ and $dx=dx_1\,dx_2$.

To each Fourier series $S[x;F]$, there corresponds a trigonometric series $S'[x;F]=-\Sigma_m(m_1m_2\hat{F}_m e^{im\cdot x})$ obtained by term-by-term mixed differentiation of $S[x;F]$.

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By a rectangular sum for $S[x;F]$ or $S'[x;F]$ we mean a partial sum $S_{l_1 l_2}[x;F]$ or $S'_{l_1 l_2}[x;F]$ with

$$S_{l_1 l_2}[x;F] = \sum_{|m_r| \le l_r} \hat{F}_m e^{im\cdot x}$$

$$S'_{l_1 l_2}[x;F] = -\sum_{|m_r| \le l_r} (m_1 m_2 \hat{F}_m e^{im\cdot x}).$$

We shall make use of the standard equalities

$$S_{l_1 l_2}[x;F] = \frac{1}{\pi^2} \int_{T^2} F(y) D_{l_1}(y_1 - x_1) D_{l_2}(y_2 - x_2)\, dy,$$

$$S'_{l_1 l_2}[x;F] = \frac{1}{\pi^2} \int_{T^2} F(y) \frac{\partial D_{l_1}(y_1 - x_1)}{\partial y_1} \frac{\partial D_{l_2}(y_2 - x_2)}{\partial y_2}\, dy,$$

where

$$D_{l_r}(x_r) = \frac{\sin(l_r + \frac{1}{2})(x_r)}{2\sin(x_r/2)}, \quad (r = 1,2).$$

The following notations are adopted in this article.
$C(T^n)$: the class of continuous functions on T^n,
$CPL(T)$: the class of functions continuous and piecewiselinear on $T = [0, 2\pi]$,
μ_n: the n-dimensional Lebesgue measure in $\mathbf{R}^n$.
Also, for any bounded function F on T^n, we write $\|F\| = \sup\{|F(x)| : x \in T^n\}$.
In the following sections, we shall use the letter A for an absolute constant which may be different from case to case.

3. Fundamental lemmas. Several fundamental tools are needed for the proof of our main theorem. The following lemma is due to Men'šov [1]:

LEMMA 1. *Let $[a, b]$ be any subinterval of $T = [0, 2\pi]$, γ be any real number, ϵ be any positive number and $\nu > 8$ be any natural number.*
Then there exist a function $\psi(x)$ and a closed set D such that

(3.1) $\psi(x) \in CPL(T)$ *and* $\psi(x) = 0$ *outside* $[a, b]$,

(3.2) $\|\psi\| \le 2\nu|\gamma|$,

(3.3) $|\int_0^\xi \psi(x)\,dx| < \epsilon$ in $0 \le \xi \le 2\pi$,

(3.4) $\psi(x) = \gamma$ in D,
where $D \subset [a, b]$ *with* $\mu_1(D) > (b-a)(1-(5/\nu))$,

(3.5) *the Fourier series of* ψ *converges uniformly, and* $|S_k[x;\psi]| \le A \min\{k^2\epsilon, \nu|\gamma|\}$ *for* $k \ge 1$ *and* $x \in T$.

Proof. The proof of this lemma is essentially the same as in [1, pp. 488–504].

The next lemma is a generalization of Lemma 1 to T^2, somewhat different from our previous work [3].

LEMMA 2. *Let* $[a_1, b_1] \times [a_2, b_2]$ *be any rectangle in* T^2, *and let* γ, ϵ *and* ν *be given as in Lemma* 1.

Then there exist a function $\psi(x_1, x_2)$ *and a closed set* Λ *such that*

(3.6) $\psi(x_1,x_2)=\psi_1(x_1)\psi_2(x_2)$,
where $\psi_i(x_i)$ *is the function constructed as in Lemma* 1 *except that* $[a, b]$ *and* γ *are replaced by* $[a_i, b_i]$ *and* $\pm\sqrt{|\gamma|}$, *respectively, with the minus sign for* $\psi_2(x_2)$ *if* $\gamma<0$;

(3.7) $\|\psi\| \le 4\nu^2|\gamma|$,

(3.8) $|\int_{[0,\xi]\times[0,\eta]}\psi(x)dx| < \epsilon^2$ *in* $0 \le \xi$, $\eta \le 2\pi$,

(3.9) $\psi(x_1,x_2)=\gamma$ *in* Λ,
where $\Lambda \subset [a_1, b_1] \times [a_2, b_2]$ *and* $\mu_2(\Lambda) > (b_1-a_1)(b_2-a_2)(1-(10/\nu))$,

(3.10) *the rectangular partial sums of* $S[x;\psi]$ *converge uniformly, and for each* $x \in T^2$,

$$|S_{l_1 l_2}[x;\psi]| \le \begin{cases} A\nu^2|\gamma| & \text{for } l_1,\ l_2 \ge 0 \\ A\nu\sqrt{|\gamma|}\ l_1^2\epsilon & \text{for } l_1 \ge 1,\ \ l_2 \ge 0 \\ A\nu\sqrt{|\gamma|}\ l_2^2\epsilon & \text{for } l_2 \ge 1,\ \ l_1 \ge 0. \end{cases}$$

Proof. Let $\psi_1(x_1)$ and D_1 be the function and set, constructed as in Lemma 1, corresponding to $[a_1, b_1]$, $\sqrt{|\gamma|}$, ϵ, ν, while $\psi_2(x_2)$ and D_2 correspond to $[a_2, b_2]$, $\pm\sqrt{|\gamma|}$ (with the minus sign if $\gamma<0$), ϵ, ν.

Let $\psi(x_1,x_2)=\psi(x_1)\psi_2(x_2)$ and $\Lambda=D_1\times D_2$. Then it is easy to see that $\psi(x_1,x_2)$ and Λ have the desired properties (3.6)–(3.10).

For $\psi \in C(T^2)$ we can assign $F_\psi \in C(T^2)$ as follows:

$$F_\psi(x_1,x_2) = \int_{[0,x_1]\times[0,x_2]}\psi(y)dy - g(x_1)\int_{[0,2\pi]\times[0,x_2]}\psi(y)dy$$

$$-g(x_2)\int_{[0,x_1]\times[0,2\pi]}\psi(y)dy + g(x_1)g(x_2)\int_{T^2}\psi(y)dy, \tag{3.11}$$

where $(x_1,x_2)\in T^2$ and g is a monotonic function continuous on T such that $g(0)=0$, $g(2\pi)=1$ and g is constant in all intervals contiguous to some perfect set of measure zero in T and $\lim_{n\to\infty}\int_0^{2\pi}e^{-int}dg(t)=0$, see [2; pp. 406–410]. It follows from the proof of a theorem in [2; pp. 366–367] that the Fourier-Stieltjes series of g converges to zero almost everywhere. We shall fix such a function in this article.

The following lemma repeated here for convenience from [3], is one of the most important tools for the proof of our main theorem.

LEMMA 3. *Let* $\psi \in C(T^2)$, *and let* $F_\psi \in C(T^2)$ *be defined as in* (3.11). *Then we have*

(3.12) $F_\psi(x)=0$ *on the boundary of* T^2, *and*

$$\|F_\psi\| \le 4\sup\left\{\left|\int_{[0,\xi]\times[0,\eta]}\psi(x)dx\right| : 0 \le \xi,\ \eta \le 2\pi\right\}, \qquad \textit{and also}$$

(3.13) *if the rectangular partial sums of* $S[x;\psi]$ *converge uniformly, then the rectangular partial sums of* $S'[x;F_\psi]$ *converge to* $\psi(x)$ *for* a.e. $x\in T^2$.

Proof. The conclusions in (3.12) follow immediately from the definition of F_ψ. It remains to prove (3.13).

Observe that

$$S'_{l_1l_2}[x;F_\psi]=\frac{1}{\pi^2}\int_{T^2}F_\psi(y)\,\frac{\partial D_{l_1}(y_1-x_1)}{\partial y_1}\,\frac{\partial D_{l_2}(y_2-x_2)}{\partial y_2}\,dy$$

$$=\frac{1}{\pi^2}\Bigg[\int_{T^2}\psi(y)D_{l_1}(y_1-x_1)D_{l_2}(y_2-x_2)\,dy$$

$$-\int_T D_{l_1}(y_1-x_1)\,dg(y_1)\int_{T^2}\psi(y)D_{l_2}(y_2-x_2)\,dy$$

$$-\int_T D_{l_2}(y_2-x_2)\,dg(y_2)\int_{T^2}\psi(y)D_{l_1}(y_1-x_1)\,dy$$

$$+\int_T D_{l_1}(y_1-x_1)\,dg(y_1)\int_T D_{l_2}(y_2-x_2)\,dg(y_2)\int_{T^2}\psi(y)\,dy\Bigg].$$

From the uniform convergence of $S[x;\psi]$ it is easy to deduce that

$$\int_{T^2}\psi(y)D_{l_r}(y_r-x_r)\,dy$$

also converges uniformly.

So the conclusion follows from the fact that the Fourier-Stieltjes series of g converges to 0 a.e..

REMARK. If $\psi(x)=\psi_1(x_1)\psi_2(x_2)$, where the ψ_i are $CPL(T)$ functions, then the conclusions of Lemma 3 follow readily. We shall apply the above lemma for this class of functions in the proof of our main theorem.

We also need to make use of the Egoroff theorem for a multiple sequence of functions convergent in the sense of Pringsheim [10, p. 68] (i.e. by unrestricted rectangular convergence):

If $\{f_l\}$ is an n-tuple sequence of finite a.e. measurable functions, and if $\{f_l\}$ converges to f a.e. on T^n, then for each $\delta>0$, there exists a set $E\subset T^n$ with $\mu_n(T^n\sim E)<\delta$ such that $\{f_l\}$ converges to f uniformly on E.

4. Representation theorem on T^2. In this section we apply the tools developed in Section 3 to prove our representation theorem on T^2.

THEOREM 1. *For any finite* a.e. *measurable function* f *on* T^2, *there exists* $F\in C(T^2)$ *such that the rectangular partial sums of the double trigonometric series* $-\sum_m(m_1m_2\hat F_m e^{im\cdot x})$ *converge unrestrictedly to* $f(x)$ *for* a.e. $x\in T^2$, *that is* $\lim_{l_1,l_2\to\infty}S'_{l_1l_2}[x;F]=f(x)$ *for* a.e. $x\in T^2$.

Proof. Let us assume that

(4.1) $$\sigma_k = \frac{1}{[10\pi^2 2^{k+2}]}, \quad \nu_k = [10\pi^2 2^{k+2}] + 1.$$

By induction we can obtain:

(a) two sequences of positive numbers $\{q_k\}$, $\{\epsilon_s\}$,

(b) two strictly increasing sequences of natural numbers $\{n_s\}$, $\{e_k\}$,

(c) three sequences of continuous functions $\{\alpha_k\}$, $\{\psi_s\}$, $\{F_s\}$, and a sequence of step functions $\{\beta_k\}$,

(d) a sequence of real numbers $\{\gamma_s\}$,

(e) five sequences of closed sets $\{Y_k\}$, $\{\Lambda_s\}$, $\{P_k\}$, $\{E_s\}$ and $\{X_k\}$,

such that for $e_{k-1} < s \le e_k$ the following hold:

(4.2) $$\alpha_k = f - \sum_{j=1}^{e_{k-1}} \psi_j \quad \text{in} \quad Y_k,$$

where $\mu_2(Y_k) > 4\pi^2 - (1/2^k)$ and $e_0 = 0$,

(4.3) $\|\alpha_k\| \le q_k$ and $q_k \ge 1$ for each k,

(4.4) $$\beta_k = \sum_{e_{k-1}+1}^{e_k} \gamma_s \chi_{\Delta_s} \quad \text{with} \quad \|\beta_k\| \le \|\alpha_k\| \quad \text{and} \quad \|\beta_k - \alpha_k\| \le \sigma_{k+1}^3,$$

(4.5) $$\epsilon_s = \frac{1}{2^s n_s^2 q_k \nu_k} \quad \text{with} \quad n_1 = 1,$$

(4.6) $\psi_s(x_1, x_2) = \psi_{s1}(x_1)\psi_{s2}(x_2)$, where $\psi_s(x_1, x_2)$ is a function as in (3.6) Lemma 2 corresponding to $\bar{\Delta}_s = [a_{s1}, b_{s1}] \times [a_{s2}, b_{s2}]$, γ_s, ϵ_s and ν_k,

(4.7) $\|\psi_s\| \le 4\nu_k^2 |\gamma_s|$,

(4.8) $\left|\int_{[0,\xi]\times[0,\eta]} \psi_s(x)\,dx\right| < \epsilon_s^2$ in $0 \le \xi, \eta \le 2\pi$,

(4.9) $\psi_s(x_1, x_2) = \gamma_s$ in Λ_s,
where $\Lambda_s \subset [a_{s1}, b_{s1}] \times [a_{s2}, b_{s2}]$ and $\mu_2(\Lambda_s) > \mu_2(\Delta_s)(1 - (10/\nu_k))$,

(4.10) the rectangular partial sums of $S[x; \psi_s]$ converge uniformly, and for each $x \in T^2$,

$$|S_{l_1 l_2}[x; \psi_s]| \le \begin{cases} A\nu_k^2 |\gamma_s| & \text{for } l_1, l_2 \ge 0 \\ A\nu_k \sqrt{|\gamma_s|}\, l_1^2 \epsilon_s & \text{for } l_1 \ge 1, \quad l_2 \ge 0; \\ A\nu_k \sqrt{|\gamma_s|}\, l_2^2 \epsilon_s & \text{for } l_2 \ge 1, \quad l_1 \ge 1 \end{cases}$$

(4.11) $F_s(x) = F_{\psi_s}(x) = 0$ on the boundary of T^2, and $\|F_s\| \le 4\epsilon_s^2$,

(4.12) $\lim_{l_1, l_2 \to \infty} S'_{l_1 l_2}[x; F_s] = \psi_s(x)$ for a.e. $x \in T^2$,

(4.13) $\left|\sum_{j=1}^{s} \{S'_{l_1 l_2}[x; F_j] - \psi_j(x)\}\right| < (1/s)$ as $l_1, l_2 \ge n_{s+1}$ and $x \in E_s$, where $E_s \subset T^2$ and $\mu_2(E_s) > 4\pi^2 - (1/s^2)$,

(4.14) $$P_k = Y_k \cap \left(\bigcup_{e_{k-1}+1}^{e_k} \Lambda_s \right) \quad \text{with} \quad \mu_2(P_k) > 4\pi^2 - \frac{1}{2^{k-1}},$$

(4.15) $\left|f(x) - \sum_{j=1}^{e_k} \psi_j(x)\right| \le \sigma_{k+1}^3$ in P_k,

(4.16) $X_k \subset P_k$, $\mu_2(X_k) > 4\pi^2 - (1/2^{k-1})$, and for $x = (x_1, x_2) \in X_k$

$$\left|\int_T D_{l_r}(y_r - x_r)\, dg\,(y_r)\right| < \frac{\sigma_{k+1}^2}{q_{k+1}} \quad \text{if} \quad l_r \ge n_{e_k+1}, \quad \text{and} \quad |S'_{l_1 l_2}[x; F_j]| \le A\sigma_{k+1}$$

whenever $l_1, l_2 \ge n_{e_k+1}$ and $e_k < j \le e_{k+1}$.

Let us sketch how to obtain the above sequences. First, by the Lusin theorem there exists $\alpha_1 \in C(T^2)$ such that

(4.2′) $\alpha_1(x) = f(x)$ in Y_1,
where Y_1 is a closed subset of T^2 and $\mu_2(Y_1) > 4\pi^2 - \frac{1}{2}$.

Since $\alpha_1 \in C(T^2)$ there exist a positive number $q_1 \ge 1$ and a step function $\beta_1 = \sum_{s=1}^{e_1} \gamma_s \chi_{\Delta_s}$ such that

(4.3′) $\|\alpha_1\| \le q_1$,

(4.4′) $\|\beta_1\| \le \|\alpha_1\|$ and $\|\beta_1 - \alpha_1\| \le \sigma_2^3$
where $\Delta_1, \Delta_2, \ldots, \Delta_{e_1}$ are nonoverlapping rectangles with $\bigcup_{s=1}^{e_1} \Delta_s = T^2$.

Set $\bar{\Delta}_s = [a_{s1}, b_{s1}] \times [a_{s2}, b_{s2}]$.

Let $1 = n_1 < n_2 < \cdots < n_{e_1} < n_{e_1+1}$ be a finite sequence of natural numbers which will be defined inductively, see (4.13′) and (4.16′). Assume that

$$\epsilon_s = \frac{1}{2^s n_s^2 q_1 \nu_1} \quad \text{for} \quad 1 \le s \le e_1. \tag{4.5′}$$

On the basis of Lemma 2, in which we suppose that $[a_1, b_1] \times [a_2, b_2] = [a_{s1}, b_{s1}] \times [a_{s2}, b_{s2}]$, $\gamma = \gamma_s$, $\nu = \nu_1$ and $\epsilon = \epsilon_s$, we can find $\psi_s(x) \in C(T^2)$ and closed set Λ_s such that (4.5)-(4.10) hold for $s = 1, 2, \ldots, e_1$ and $k = 1$.

Let $F_s = F_{\psi_s}$ as in (3.11). Then (4.11)-(4.12) hold for $1 \le s \le e_1$.

By the Egoroff theorem, for $1 \le s \le e_1 - 1$, there are closed sets $E_s \subset T^2$ and natural numbers $n_{s+1} > n_s$ such that $\mu_2(E_s) > 4\pi^2 - (1/s^2)$ and

$$\left|\sum_{j=1}^{s} \left\{S'_{l_1 l_2}[x; F_j] - \psi_j(x)\right\}\right| < \frac{1}{s} \quad \text{in} \quad E_s \quad \text{if} \quad l_1, l_2 \ge n_{s+1}. \tag{4.13′}$$

Set

$$P_1 = Y_1 \cap \left(\bigcup_{s=1}^{e_1} \Lambda_s\right). \tag{4.14′}$$

Then P_1 is closed, $\mu_2(P_1) > 4\pi^2 - 1$ since $\mu_2(Y_1) > 4\pi^2 - \frac{1}{2}$ and

$$\sum_1^{e_1} \mu_2(\Lambda_s) > 4\pi^2(1 - (10/\nu_1)) > 4\pi^2 - (1/2).$$

From (4.2′)-(4.4′) and (4.9) we have

(4.15′) $$\left|f(x) - \sum_{s=1}^{e_1} \psi_s(x)\right| \le \sigma_2^3 \quad \text{in} \quad P_1.$$

Since P_1 is closed, there exists $\delta>0$ such that the set $G=\{x\in T^2 : d(x, P_1)>\delta\}$ is open and $\mu_2(G) > (1-(1/4\pi^2 2^3))\mu_2(T^2 \sim P_1)$.

We can find nonoverlapping closed rectangles $\Delta'_{e_1+1}, \ldots, \Delta'_{k'_1}$ in G such that

$$\sum_{e_1+1}^{k'_1} \mu_2(\Delta'_s) > \left(1 - \frac{1}{4\pi^2 2^3}\right)\mu_2(T^2 \sim P_1).$$

Therefore $T^2 \sim \bigcup_{e_1+1}^{k'_1} \Delta'_s$ can be covered by nonoverlapping rectangles $\Delta'_{k'_1+1}, \ldots, \Delta'_{k_1}$.

Let $f_1 = f - \sum_{s=1}^{e_1} \psi_s$. There exists $q_2 \ge 1$ such that

$$\mu_2(f_1^{-1}[-q_2, q_2] \cap \Delta'_s) > \mu_2(\Delta'_s)(1-\sigma_2) \quad \text{for} \quad e_1 < s \le k'_1.$$

Now we can select a natural number $n_{e_1+1} > n_{e_1}$ and closed sets X_1, E_{e_1} such that

(4.16′) $$\left|\int_T D_{l_r}(y_r - x_r)\, dg\,(y_r)\right| < \frac{\sigma_2^2}{q_2} \quad \text{for each} \quad x=(x_1,x_2)\in X_1,$$

and

(4.13″) $$\left|\sum_{s=1}^{e_1} \{S'_{l_1 l_2}[x; F_s] - \psi_s(x)\}\right| < \frac{1}{e_1} \quad \text{in} \quad E_{e_1},$$

where $X_1 \subset P_1$, $\mu_2(X_1) > 4\pi^2 - 1$, $\mu_2(E_{e_1}) > 4\pi^2 - (1/e_1^2)$ and $l_1, l_2 \ge n_{e_1+1}$.

Now observe that $\mu_2(\bigcup_{e_1+1}^{k'_1} f_1^{-1}[-q_2, q_2] \cap \Delta'_s) > \mu_2(T^2 \sim P_1) - (1/2^2)$ and $\mu_2(f_1^{-1}[-\sigma_2^3, \sigma_2^3] \cap (T^2 \sim \bigcup_{e_1+1}^{k'_1} \Delta'_s)) \ge \mu_2(P_1)$. Hence, for each $s = e_1+1, \ldots, k_1$, there exist $\bar\alpha_s \in C(T^2)$ and closed set $Y'_s \subset \Delta'_s$ such that $\bar\alpha_s(x) = 0$ for $x \notin \Delta'_s$ and

$$\|\bar\alpha_s\| \le \begin{cases} q_2, & e_1 < s \le k'_1 \\ \sigma_2^3 & k'_1 < s \le k_1, \end{cases}$$

$$\bar\alpha_s(x) = f_1(x) \quad \text{in} \quad Y'_s,$$

where $\mu_2(\bigcup_{e_1+1}^{k_1} Y'_s) > 4\pi^2 - (1/2^2)$.

Set $Y_2 = \bigcup_{e_1+1}^{k_1} Y'_s$ and $\alpha_2 = \sum_{e_1+1}^{k_1} \bar\alpha_s$. Then

(4.2″) $\alpha_2 \in C(T^2)$ and $\alpha_2(x) = f_1(x) = f(x) - \sum_{s=1}^{e_1} \psi_s(x)$ in Y_2.

(4.3″) $\|\alpha_2\| \le q_2$

(4.14″) $\mu_2(Y_2) > 4\pi^2 - (1/2^2)$.

Now for each $e_1 < s \le k_1$, there exists a step function $\bar\beta_s = \sum_{j=1}^{j_s} \gamma_{sj} \chi_{\Delta_{sj}}$ such that $\|\bar\beta_s\| \le \|\bar\alpha_s\|$ and $\|\bar\beta_s - \bar\alpha_s\| \le \sigma_3^3$, where $\Delta_{s1}, \ldots, \Delta_{sj_s}$ are nonoverlapping rectangles such that $\bigcup_{j=1}^{j_s} \Delta_{sj} = \Delta'_s$, and if $\bar\Delta_{sj} = [a_{sj1}, b_{sj1}] \times [a_{sj2}, b_{sj2}]$, then

$$\max\{(b_{sjr} - a_{sjr}) : r = 1, 2\} < \sigma_2^3 \frac{\sin(\delta/2\sqrt{2})}{q_2}$$

for each $e_1 < s \le k'_1$ and $1 \le j \le j_s$.

The rectangles Δ_{sj}, $e_1 < s \le k_1$ and $1 \le j \le j_s$ are enumerated in the order $\Delta_{e_1+1}, \ldots, \Delta_{e_1'}$ for $e_1 < s \le k_1'$, and in the order $\Delta_{e_1'+1}, \ldots, \Delta_{e_2}$ for $k_1' < s \le k_1$.

We may rewrite the sum of the step function $\bar{\beta}_s$ as

$$\beta_2 = \sum_{e_1+1}^{k_1} \bar{\beta}_s = \sum_{e_1+1}^{e_2} \gamma_s \chi_{\Delta_s}, \tag{4.4''}$$

and therefore $\|\beta_2\| \le \|\alpha_2\|$ and $\|\beta_2 - \alpha_2\| \le \sigma_3^3$.

Next, by the same procedure as above for the step function β_2, we can define: a finite sequence of natural numbers $n_{e_1+1} < \cdots < n_{e_2}$, a finite sequence of positive numbers

$$\left\{\epsilon_s = \frac{1}{2^s n_s^2 q_2 \nu_2}\right\}_{e_1+1}^{e_2},$$

two sequences of continuous functions $\{\psi_s\}_{e_1+1}^{e_2}$, $\{F_s\}_{e_1+1}^{e_2}$, and two sequences of closed sets $\{\Lambda_s\}_{e_1+1}^{e_2}$, $\{E_s\}_{e_1+1}^{e_2-1}$ such that for each $s = e_1+1, \ldots, e_2$ and $k=2$, the properties (4.6)-(4.13) hold.

Now set

$$P_2 = Y_2 \cap \left(\bigcup_{e_1+1}^{e_2} \Lambda_s \right). \tag{4.14''}$$

Then P_2 is closed with $\mu_2(P_2) > 4\pi^2 - (1/2)$, and

$$\left| f(x) - \sum_{s=1}^{e_2} \psi_s(x) \right| = \left| f_1(x) - \sum_{e_1+1}^{e_2} \psi_s(x) \right| \le \sigma_3^3 \quad \text{in} \quad P_2, \tag{4.15''}$$

Finally, we show that for each $x = (x_1, x_2) \in X_1$

$$|S_{l_1 l_2}'[x; F_s]| \le A\sigma_2 \quad \text{whenever} \quad e_1 < s \le e_2 \quad \text{and} \quad l_1, l_2 \ge n_{e_1} + 1. \tag{4.16''}$$

Write $S_{l_1 l_2}'[x; F_s] = (1/\pi^2)\left[\sum_{j=1}^4 W_{sj}(x)\right]$, where

$$W_{s1}(x) = \int_{T^2} \psi_s(y) D_{l_1}(y_1 - x_1) D_{l_2}(y_2 - x_2)\, dy,$$

$$W_{s2}(x) = -\int_T D_{l_1}(y_1 - x_1)\, dg(y_1) \int_{T^2} \psi_s(y) D_{l_2}(y_2 - x_2)\, dy,$$

$$W_{s3}(x) = -\int_T D_{l_2}(y_2 - x_2)\, dg(y_2) \int_{T^2} \psi_s(y) D_{l_1}(y_1 - x_1)\, dy,$$

and

$$W_{s4}(x) = \int_{T^2} \psi_s(y)\, dy \prod_{r=1}^{2} \int_T D_{l_r}(y_r - x_r)\, dg(y_r).$$

Set $\bar{\Delta}_s = [a_{s1}, b_{s1}] \times [a_{s2}, b_{s2}]$. Since $d(X_1, \Delta_s) > \delta$ as $s = e_1+1, \ldots, e_1'$, hence either

$$x_1 \notin [a_{s1} - (\delta/\sqrt{2}),\ b_{s1} + (\delta/\sqrt{2})] \quad \text{or} \quad x_2 \notin [a_{s2} - (\delta/\sqrt{2}),\ b_{s2} + (\delta/\sqrt{2})].$$

It suffices to consider the case $x_1 \notin [a_{s1} - (\delta/\sqrt{2}),\ b_{s1} + (\delta/\sqrt{2})]$. Now

$$\begin{aligned}|W_{s1}(x)| &\leq A\nu_2\sqrt{|\gamma_s|}\left|\int_{a_{s_1}}^{b_{s_1}}\psi_{s_1}(y_1)D_{l_1}(y_1-x_1)\,dy_1\right| \\ &\leq A\nu_2\sqrt{|\gamma_s|}\;\frac{\nu_2\sqrt{|\gamma_s|}}{\sin(\delta/2\sqrt{2})}(b_{s1}-a_{s1}) \\ &\leq A\nu_2^2|\gamma_s|\frac{1}{\sin(\delta/2\sqrt{2})}\frac{\sigma_2^3\sin(\delta/2\sqrt{2})}{q_2} \\ &\leq A\sigma_2,\end{aligned}$$

since $|\gamma_s| \leq q_2$ [see (4.3″), (4.4″)].

By (4.16′) it is easy to see $|W_{sj}(x)| \leq A\sigma_2$, $(j=2,3,4)$. As for $e_1' < s \leq e_2$, we have $|W_{sj}(x)| \leq A\nu_2^2|\gamma_s| \leq A\nu_2^2\sigma_2^3 \leq A\sigma_2$, $(j=1,2,3)$, $|W_{s4}(x)| \leq A\sigma_2^4\epsilon_s^2 \leq A\sigma_2$. So the inequality (4.16″) is proved.

Continuing the above process, we obtain the sequences (a)-(e) satisfying required conditions (4.1)-(4.16).

Let $F(x)=\sum_{s=1}^{\infty}F_s(x)$, $x\in T^2$. It follows from (4.11) that $\sum_{s=1}^{\infty}F_s$ converges uniformly on T^2, and so $F\in C(T^2)$. Therefore, $S'_{l_1l_2}[x;F]=\sum_{s=1}^{\infty}S'_{l_1l_2}[x;F_s]$ for each l_1, $l_2 \geq 0$ and $x\in T^2$.

Suppose $\Omega = \underline{\lim}_{k\to\infty}E_k \cap \underline{\lim}_{k\to\infty}X_k$. Then $\mu_2(\Omega)=4\pi^2$, since $\mu_2(E_k) > 4\pi^2-(1/k^2)$ and $\mu_2(X_k) > 4\pi^2-(1/2^{k-1})$ for any $k\geq 1$.

We shall prove that

$$\lim_{l_1,l_2\to\infty} S'_{l_1l_2}[x;F] = f(x) \quad \text{for each} \quad x\in\Omega. \tag{4.17}$$

Now for each $x\in\Omega$, there exists j_0 such that $x\in E_k\cap X_k$ if $k\geq j_0$. So $x\in P_k$ if $k\geq j_0$. Observing (4.15) and noting that at most one term of the sum $\sum_{e_{k-1}+1}^{e_k}\psi_s(x)$ is nonzero, we obtain $f(x)=\sum_{s=1}^{\infty}\psi_s(x)$ in Ω. Given $\epsilon>0$, there exists k_0 such that $k_0-1\geq j_0$, $1/(k_0-1)<\epsilon$, and $j>e_{k_0-1}$, then $|\sum_{s=j}^{\infty}\psi_s(x)|<\epsilon$.

We want to show that

$$|S'_{l_1l_2}[x;F]-f(x)| \leq A\epsilon \quad \text{whenever} \quad l_1,\, l_2 > n_{e_{k_0}}. \tag{4.18}$$

If l_1, $l_2 > n_{e_{k_0}}$, then either $l_2 \geq l_1 > n_{e_{k_0}}$ or $l_1 \geq l_2 > n_{e_{k_0}}$. Only the case $l_2\geq l_1 > n_{e_{k_0}}$ need be treated. For this case there exist natural numbers k,j such that $k\geq k_0-1$, $e_k<j\leq e_{k+1}$ and $n_j<l_1\leq n_{j+1}$

$$\begin{aligned}|S'_{l_1l_2}[x;F]-f(x)| &= \left|\sum_{s=1}^{\infty}S'_{l_1l_2}[x;F_s]-\sum_{s=1}^{\infty}\psi_s(x)\right| \\ &\leq \left|\sum_{s=1}^{j-1}\{S'_{l_1l_2}[x;F_s]-\psi_s(x)\}\right| + |S'_{l_1l_2}[x;F_j]| \\ &\quad + \left|\sum_{s=j+1}^{\infty}S'_{l_1l_2}[x;F_s]\right| + \epsilon \\ &\leq \frac{1}{j-1} + A\sigma_{k+1} + \left|\sum_{s=j+1}^{\infty}S'_{l_1l_2}[x;F_s]\right| + \epsilon,\end{aligned}$$

[see (4.13), (4.16)].

Write $S'_{l_1 l_2}[x;F_s] = (1/\pi^2)\left[\sum_{t=1}^4 W_{st}(x)\right]$ $(s \geq j+1)$. If $e_{j'-1} < s \leq e_{j'}$, then

$$|W_{s1}(x)| \leq A\nu_{j'}\sqrt{|\gamma_s|}\; l_1^2 \epsilon_s \qquad \text{[see (4.10)]}$$

$$= A\nu_{j'}\sqrt{|\gamma_s|}\; l_1^2 \frac{1}{2^s n_s^2 q_{j'}\nu_{j'}} \leq \frac{A}{2^s},$$

$$|W_{st}(x)| \leq A\nu_{j'}\sqrt{|\gamma_s|}\; \epsilon_s \leq \frac{A}{2^s}, \qquad (t=2,3)$$

$$|W_{s4}(x)| \leq \epsilon_s^2 \leq \frac{A}{2^s}.$$

Hence $\sum_{s=j+1}^{\infty} |S'_{l_1 l_2}[x;F_s]| \leq A \sum_{s=j+1}^{\infty} (1/2^s) \leq A\epsilon$, so that the inequality (4.18) is proved, and consequently the existence of the limit in (4.17), and thus the proof of Theorem 1 is complete. □

5. Representation theorem on T^n. The basic ideas for proving Theorem 1 in the preceding section are the lemmas stated in Section 3. It is easy to extend these lemmas to higher dimensions. Therefore, parallel to the proof for Theorem 1, we have the following representation theorem on T^n.

THEOREM 2. *For any finite* a.e. *measurable function f on T^n, there exists $F \in C(T^n)$ such that the rectangular partial sums of the n-tuple trigonometric series $\sum_m (i)^n (m_1 m_2 \ldots m_n)\hat{F}_m e^{im\cdot x}$ converge to $f(x)$ for* a.e. *$x \in T^n$, where $m = (m_1, m_2, \ldots, m_n)$ is an integer lattice point of $\mathbf{R}^n$, and $\hat{F}_m$ is the m-th Fourier coefficient of F.*

Finally, we remark that it is still an open question whether our representation theorem holds in the sense of spherical summation.

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