LEVEL-SETS OF SPECIAL BLASCHKE PRODUCTS

Charles Belna, William Cohn, George Piranian, and Kenneth Stephenson

Dedicated to the memory of David L. Williams

There exist bounded holomorphic functions f in the unit disk Δ such that for uncountably many positive numbers λ the λ-level-set $\{z \in \Delta : |f(z)| = \lambda\}$ has infinite length ([4] and [7]). Also, there exist Blaschke products and singular inner functions possessing one level-set of infinite length ([3] and [6]). The present paper describes Blaschke products whose λ-level-set has infinite length for each λ in a preassigned set of capacity 0 and of type F_σ on the interval (0, 1). This solves Problem 5 in [6].

The properties of Riemann surfaces that we use in our proofs are described in Chapters 9 and 10 of [2].

THEOREM 1. *Let K be a set of capacity* 0 *on the interval* $(0,1)$, *and let it be closed with respect to Δ. Then there exists a Blaschke product whose λ-level-set has infinite length if and only if $\lambda \in K$.*

In the proof, we omit the trivial case where K is empty. Let $\mathcal{S}$ and π denote the universal covering surface of the domain $\Omega = \Delta - K$ and the natural projection of $\mathcal{S}$ onto $\Delta - K$. By Koebe's uniformization theorem, there exists a conformal mapping H of Δ onto $\mathcal{S}$ such that the composite function $\phi = \pi \circ H$ satisfies the condition $\phi(0) = 0$. Because the set of asymptotic values of ϕ is the set $K \cup \partial\Delta$ and K has capacity 0, the function ϕ is a Blaschke product (see for example [5, Theorem 2 on p. 33 and Footnote 1 on p. 72]).

Let Γ denote the group of automorphisms T of Δ that satisfy the functional equation $\phi \circ T = \phi$ throughout Δ. Then Γ is isomorphic to the fundamental group $\pi_1(\Omega)$, and it consists of a set of Möbius transformations of the form

$$T_n(z) = e^{i\theta_n} \frac{z - a_n}{1 - \bar{a}_n z}.$$

The points a_n are precisely the points where $\phi(z) = 0$; therefore

$$\phi(z) = e^{i\theta} \prod_1^\infty \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z}.$$

Let λ_0 be the least number in the set K, and let ω denote the subdomain of Δ that contains the origin and whose image under the mapping ϕ is the slit disk $\Delta - [\lambda_0, 1)$. Let λ denote a number in (0, 1), let C_λ denote the circle $|w| = \lambda$ (minus the point λ, in case $\lambda \in K$), and let α denote the portion of the inverse image $\phi^{-1}(C_\lambda)$ that lies in $\bar{\omega}$. Then the λ-level-set of ϕ is the union of the arcs $T_n(\alpha)$ $(n = 1, 2, \ldots)$, and it follows that the length of the λ-level-set of ϕ is

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$$l(\phi^{-1}(C_\lambda)) = \sum_{n=1}^{\infty} \int_\alpha |T_n'(z)|\,|dz| = \int_\alpha \sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|1-\bar{a}_n z|^2}\,|dz|.$$

If $\lambda \notin K$, the arc α is bounded away from $\partial\Delta$; therefore it is rectifiable and the integrand in the last expression is bounded on α. It follows that $l(\phi^{-1}(C_\lambda)) < \infty$.

If $\lambda \in K$, every sequence $z_1, z_2, \ldots$ on α with the property $\phi(z_k) \longrightarrow \lambda$ has the property $|z_k| \longrightarrow 1$, since f is continuous and omits λ; therefore the arc α reaches the circle $\partial\Delta$. To estimate our integrand on the arc α, we denote by ϕ_j the jth factor of the Blaschke product ϕ, and we make the substitution $A_j = |\phi_j|^2$ in the identity

$$1 - \prod_{j=1}^{n} A_j = (1-A_1) + A_1(1-A_2) + \cdots + A_1 A_2 \cdots A_{n-1}(1-A_n).$$

Clearly, the identity implies that

$$1 - |\phi|^2 = \sum_{n=1}^{\infty} (1-|\phi_n|^2) \prod_{j<n} |\phi_j|^2$$

(see [1, p. 116]). On α, the left member has the constant value $1-\lambda^2$ and each of the finite products in the right member has modulus less than 1. From the relation

$$1 - |\phi_n(z)|^2 = \frac{(1-|a_n|^2)(1-|z|^2)}{|1-\bar{a}_n z|^2}$$

we deduce the inequality

$$\sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|1-\bar{a}_n z|^2} > \frac{1-\lambda^2}{1-|z|^2}.$$

Because the function $1/(1-|z|^2)$ is not integrable on any arc that reaches the unit circle, $l(\phi^{-1}(C_\lambda)) = \infty$ if $\lambda \in K$. This concludes the proof of Theorem 1. □

Small changes in the proof yield a more general theorem.

THEOREM 1′. *Let K be a set of capacity* 0 *in the punctured disk* $\Delta - \{0\}$, *and let it be closed relative to* Δ. *Let B denote a Blaschke product, with* $B(0) = 0$, *whose Riemann surface is the universal covering surface of* $\Delta - K$. *If γ is a rectifiable arc in* $\{|z| < \rho < 1\}$, *the inverse image* $B^{-1}(\gamma)$ *has infinite length if and only if the closure of γ meets the set K.*

THEOREM 2. *If K is a set of capacity* 0 *and of type* F_σ *on the interval* (0, 1), *then there exists a Blaschke product whose λ-level-set has infinite length for each λ in K.*

Let Ω be the universal covering surface of $\Delta - \{-1/2\}$, let $S_1, S_2, \ldots$ be an enumeration of its sheets, and let σ be a conformal mapping of Δ onto Ω that carries the point 0 to a point of Ω over 0. Write $K = \bigcup F_n$, where each set is compact and none is empty. Set $\tilde{K} = \bigcup \sigma^{-1}[S_n \cap \pi^{-1}(F_n)]$, where π is the natural projection map.

Let B denote a Blaschke product, with $B(0) = 0$, whose Riemann surface is the universal covering surface of $\Delta - \tilde{K}$, and define ϕ to be the composition $\pi \circ \sigma \circ B$. It is well-known that the composition of two inner functions is an inner function, and it is easy to see that ϕ does not have the asymptotic value 0. Therefore ϕ is a Blaschke product. Now suppose that $\lambda \in F_n$, and let β denote the arc on S_n that lies above the arc $\{\lambda e^{i\theta} : 0 < \theta < \pi/2\}$. If $\gamma = \sigma^{-1}(\beta)$, then the λ-level-set of ϕ contains the inverse image $B^{-1}(\gamma)$. By Theorem 1′, this inverse image has infinite length, and Theorem 2 is proved. □

The hypothesis of Theorem 2 does not imply the existence of a holomorphic function in Δ whose λ-level-set has infinite length if and only if $\lambda \in K$. This is clear from the two observations that if f is holomorphic in Δ, then the set of values λ for which the λ-level-set of f has infinite length is type G_δ, and that such a set can not be both countable and dense on an interval.

Theorems 1 and 2 remain valid when we replace the words "Blaschke product" with "singular inner function". Because the proofs are similar, we shall deal only with the extension of Theorem 2.

Let T be the Möbius transformation that maps the three points $-1, 0, 1$ onto $-1, 1/2, 1$, respectively. If we modify the proof of Theorem 2 by taking $\tilde{K} = \bigcup \sigma^{-1}[S_n \cap \pi^{-1}(T(F_n) \cup \{1/2\})]$, the proof yields a Blaschke product ϕ omitting the value 1/2 and such that, for each path τ in $\Delta - T(K)$ ending at a point of $T(K)$, the inverse image $\phi^{-1}(\tau)$ has infinite length. If a path in $\Delta - K$ ends at a point of K, then its inverse image under the singular mapping $T^{-1} \circ \phi$ has infinite length. This proves the extension of Theorem 2.

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Department of Mathematics
Syracuse University
Syracuse, New York 13210

and

Department of Mathematics
Wayne State University
Detroit, Michigan 48202

Department of Mathematics
University of Michigan
Ann Arbor, Michigan 48109

and

Department of Mathematics
University of Tennessee
Knoxville, Tennessee 37916