ZEROS OF PARTIAL SUMS OF LAURENT SERIES

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To the memory of David L. Williams.

Introduction. The present investigation was prompted by a theorem and a conjecture of Abian. Before stating Abian's problem, I introduce some general assumptions and notations.

Consider a Laurent series

$$\sum_{j=-\infty}^{+\infty} a_j z^j = f(z) \quad (z = re^{i\theta}), \tag{1}$$

whose exact annulus of convergence is

$$\rho < |z| < R \quad (0 \le \rho < R \le +\infty). \tag{2}$$

The *partial sums* of (1):

$$T_n(z) = \sum_{j=-p}^{n} a_j z^j \quad (p = p(n),\ n = 1, 2, 3, \ldots), \tag{3}$$

are defined in terms of a given sequence $\{p(n)\}_{n=1}^{\infty}$ of positive integers. There are no restrictions on $p(n)$ other than

$$\lim_{n \to +\infty} p(n) = +\infty. \tag{4}$$

The sums $T_n(z)$ are also called *sections* or *truncations* of the Laurent series.

I find it convenient to impose, throughout the paper, one additional assumption:

A. *For some* $\xi > 0$, *the compact annulus*

$$\mathfrak{A} = \{z : e^{-\xi} \le |z| \le e^{\xi}\} \quad (\rho < e^{-\xi},\ e^{\xi} < R) \tag{5}$$

contains no zeros of $f(z)$.

It is clear that the condition A has the character of a normalization; it will not affect the generality of my results. Since the zeros of $f(z)$ play a dominant role in this note, it is natural to consider, beside (1), the factored form

$$f(z) = Cz^l \psi(1/z) \varphi(z), \tag{6}$$

where $C \neq 0$ is a constant, l is an integer (not necessarily positive), $\varphi(z)$ and $\psi(z)$ are analytic functions represented by

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(7) $$\varphi(z) = 1 + \sum_{j=1}^{\infty} g_j z^j \quad (|z| < R),$$

(8) $$\psi\left(\frac{1}{z}\right) = 1 + \sum_{j=1}^{\infty} \frac{h_j}{z^j} \quad (|z| > \rho);$$

R and ρ are the exact radii of convergence of the series in (7) and (8). The representation (6), which may be deduced from the elements of complex analysis, is automatically established in the proof of a preliminary result [stated below as Lemma L]; it is therefore unnecessary to take it for granted.

LEMMA L. *Let $f(z)$, $T_n(z)$ and $\mathfrak{A}$ have the same meaning as above and let*

(9) $$f(z) \neq 0 \quad (z \in \mathfrak{A}).$$

In addition, let there be infinitely many positive, as well as infinitely many negative values of j such that

(10) $$a_j \neq 0.$$

Consider all the zeros of $T_n(z)$; denote by

(11) $$z_{n1}, z_{n2}, \ldots, z_{n\sigma} \quad (\sigma = \sigma(n))$$

those of modulus >1 and by

(12) $$\zeta_{n1}, \zeta_{n2}, \ldots, \zeta_{n\tau} \quad (\tau = \tau(n))$$

those of modulus <1.

Introduce the polynomials in z

(13) $$P_n(z) = \prod_{j=1}^{\sigma} \left(1 - \frac{z}{z_{nj}}\right) \quad (\sigma = \sigma(n)),$$

and the polynomials in ζ

(14) $$Q_n(\zeta) = \prod_{j=1}^{\tau} (1 - \zeta\zeta_{nj}) \quad (\tau = \tau(n)).$$

I. *Then, as $n \longrightarrow +\infty$,*

(15) $$P_n(z) \longrightarrow \varphi(z)$$

uniformly throughout the disk $|z| \leq R'$ $(R' < R)$.

Similarly, as $n \longrightarrow +\infty$,

(16) $$Q_n(\zeta) \longrightarrow \psi(\zeta),$$

uniformly throughout the disk $|\zeta| \leq (1/\rho')$ $(\rho' > \rho)$.

II. *The representation* (6) *is valid with φ and ψ defined by* (15) *and* (16). *The integer l is given by*

(17) $$\frac{1}{2\pi} \int_0^{2\pi} \frac{f'(e^{i\theta})}{f(e^{i\theta})} e^{i\theta} d\theta = l,$$

and C by formulae (21) *below.*
III. *Let* $s(n)$ *be the largest integer such that*

(18) $$s = s(n) \leq n, \quad a_s \neq 0,$$

and $q(n)$ *be the largest integer such that*

(19) $$q = q(n) \leq p(n), \quad a_{-q} \neq 0.$$

Then, if n is sufficiently large, say $n > n_0$, *we have*

(20) $$\sigma(n) = s(n) - l > 0, \quad \tau(n) = q(n) + l > 0,$$

and

(21) $$C = \lim_{n \to +\infty} (-1)^{\sigma} a_s z_{n1} z_{n2} \dots z_{n\sigma} = \lim_{n \to +\infty} (-1)^{\tau} a_{-q} (\zeta_{n1} \zeta_{n2} \dots \zeta_{n\tau})^{-1} \neq 0.$$

As might be expected, the proof of the above lemma is completely elementary. The relations (6), (13) and (15) constitute the key to my proofs because they reduce questions concerning $T_n(z)$ to analogous questions concerning the sections of Taylor series and therefore suggest the possibility of extending to Laurent series the fundamental results of Jentzsch, Szegö, Carlson and Rosenbloom describing the distribution of zeros of the partial sums of Taylor series.

Theorem 1 below is an extension to Laurent series of Szegö's sharp form of Jentzsch's classical theorem [8].

THEOREM 1. *Let* (*1*) *and* (*2*) *hold with* $0 \leq \rho < R < +\infty$, *and let condition* A *be satisfied. Let* $\Omega^* = \{n_k\}_{k=1}^{\infty}$ *denote a sequence of positive, strictly increasing integers such that, as* $n \longrightarrow +\infty$, $n \in \Omega^*$, *we have* $|a_n|^{1/n} \longrightarrow R^{-1}$.

Then

I. *If* $N(\varphi_1, \varphi_2; n)$ *denotes the number of zeros of* $T_n(z)$ *in the sector*

$$\Delta = \{z : \varphi_1 \leq \arg z < \varphi_2, |z| > 1\} \quad (\varphi_1 < \varphi_2 \leq \varphi_1 + 2\pi),$$

we have

(22) $$\frac{N(\varphi_1, \varphi_2; n)}{n} \longrightarrow \frac{\varphi_2 - \varphi_1}{2\pi},$$

as

(23) $$n \longrightarrow +\infty, \quad n \in \Omega^*.$$

II. *For any given* $\epsilon > 0$, *there are outside the annulus* $Re^{-\epsilon} \leq |z| \leq Re^{\epsilon}$, *at most* $o(n)$ $(n \in \Omega^*)$ *zeros of* $T_n(z)$ *of modulus* > 1.

The reader will notice that the distribution of arguments described by (22) is the one usually referred to as *equidistribution in the sense of Weyl.*

Theorem 2 below is the extension to Laurent series of the results, announced by Carlson in 1924, first proved by Rosenbloom (1943) and then by Carlson (1948).

THEOREM 2. *Let* (1) *and* (2) *hold with* $0 \leq \rho < R = +\infty$, *let the condition* A *be satisfied, and let* $f(z)$ *have at infinity an essential singularity of order exactly* λ

$0<\lambda\leq+\infty$. *It is then possible to define a sequence* $\Omega=\{n_k\}_{k=1}^{\infty}$, *depending only on the sequence* $\{a_j\}_j$ $(j\geq 0)$, *and having all the following properties.*

I. *If* $\lambda=+\infty$, *and if*

$$n \longrightarrow +\infty, \quad n\in\Omega, \tag{24}$$

then

$$\frac{N(\varphi_1, \varphi_2; n)}{n} \longrightarrow \frac{\varphi_2-\varphi_1}{2\pi}, \tag{25}$$

where $N(\varphi_1, \varphi_2; n)$ *is defined as in Theorem* 1.

II. *If* $0<\lambda<+\infty$ *we may only assert the existence of a constant* $\kappa=\kappa(\varphi_2-\varphi_1, \lambda)>0$, *independent of n, such that*

$$N(\varphi_1, \varphi_2; n) > \kappa n \quad (n\in\Omega, \quad n>n_0). \tag{26}$$

III. *If* $\lambda=+\infty$, *there exists a positive sequence* $\{\rho_n\}$ *such that* $\rho_n\longrightarrow+\infty$ $(n\longrightarrow\infty,$ $n\in\Omega)$, *and such that, for any given* $\epsilon>0$, *there are at most* $o(n)$ *zeros of* $T_n(z)$, *of modulus* >1, *which lie outside the annulus*

$$\rho_n e^{-\epsilon} \leq |z| \leq \rho_n e^{\epsilon}. \tag{27}$$

IV. *If* $0<\lambda<+\infty$, *it is possible that assertion* III *holds without modification. Otherwise there is a positive sequence* $\{\rho_n\}_n$ *such that*

$$\frac{1}{\lambda} \leq \liminf_{\substack{n\to\infty\\ n\in\Omega}} \frac{\log\rho_n}{\log n}, \tag{28}$$

and a constant Λ $(0<\Lambda<+\infty)$ *having the following properties: given* $B>0$, *the partial sum* $T_n(z)$ *has*

(i) *no more than* $\Lambda n/B$ *zeros in the annulus* $1\leq|z|\leq\rho_n e^{-B}$ $(n>n_0,\ n\in\Omega)$, *and*

(ii) *as* $n\longrightarrow\infty$ $(n\in\Omega)$, *no more than* $o(n)$ *zeros in the annulus* $|z|\geq\rho_n e^{B}$.

The assertions of Theorems 1 and 2 concern the zeros of $T_n(z)$ of modulus >1; those of modulus <1 may be treated by the change of variable $\zeta=1/z$ followed by a return to the original variable z. This offers no difficulties provided we observe that our notation (3) for the partial sums $T_n(z)$ does not explicitly indicate the value $p=p(n)$. We are thus led to the introduction of the more detailed notation

$$T_{-p,n} = \sum_{j=-p}^{n} a_j z^j \quad (p>0, n>0).$$

Since the sequences Ω^* and Ω which appear in Theorems 1 and 2 only depend on the behavior of the sequence $\{a_j\}_j$ $(j\geq 0)$ it is clear that we may deduce from our results a number of Corollaries describing the behavior of all the zeros (large and small) of suitable truncations $T_{-p,n}$. It will suffice to state the simplest assertion of this kind.

COROLLARY 1.1. *Let the radii of convergence* ρ *and* R *of the series* (1) *be finite and positive.*

Define two sequences $\Omega_1 = \{m_j\}_{j=1}^{\infty}$, $\Omega_2 = \{n_j\}_{j=1}^{\infty}$ *of positive, strictly increasing integers by the conditions*

$$|a_{-m}|^{1/m} \longrightarrow \rho \qquad (m \longrightarrow +\infty,\ m \in \Omega_1),$$

$$|a_n|^{1/n} \longrightarrow R^{-1} \qquad (n \longrightarrow +\infty,\ n \in \Omega_2).$$

Consider all the zeros of $T_{-m,n}$ $(m = m_j,\ n = n_j)$ *which lie in the angle* $\varphi_1 \leq \arg z < \varphi_2$ $(\varphi_1 < \varphi_2 \leq \varphi_1 + 2\pi)$. *Then, as* $j \longrightarrow \infty$, *there are*

$$(1 + o(1)) \frac{\varphi_2 - \varphi_1}{2\pi} m_j$$

of these zeros which have modulus < 1 *and*

$$(1 + o(1)) \frac{\varphi_2 - \varphi_1}{2\pi} n_j$$

of them which have modulus > 1.

It is clear that there are analogous Corollaries covering the remaining cases: $\rho = 0$, $R = +\infty$; $\rho > 0$, $R = +\infty$; $\rho = 0$, $R < +\infty$.

Conjecture of Abian [1]. *Let* (1) *and* (2) *hold with* $0 \leq \rho < R = +\infty$ *and let* $f(z)$ *have at* ∞ *an essential singularity which is not a limit point of zeros of* $f(z)$. *Denote by* $T_n(z)$ *the truncations defined by* (3).

It is then possible, given $\epsilon > 0$, $K > 0$, *to find arbitrarily large values of* n *and complex quantities* c_n *such that simultaneously*

$$|c_n| > K, \quad T_n(c_n) = 0, \quad |f(c_n)| < \epsilon. \tag{29}$$

Abian formulated the above conjecture in connection with his recent proof [1] *that quantities* c_n *satisfying the relations* (29) *do in fact exist for all* $n \geq n_0(\epsilon, K)$, *provided the essential singularity at* ∞ *is a limit point of zeros of* $f(z)$.

I am unable to settle Abian's conjecture if, as $z \longrightarrow \infty$, the growth of $f(z)$ is unrestricted. If $f(z)$ has, at infinity, an essential singularity of finite order, the proof of the conjecture is contained in the following simple consequence of Theorem 2 and of some refinements which are possible because of the perfect regularity of the growth of the functions under consideration.

THEOREM 3. *Let* (1) *and* (2) *hold with* $0 \leq \rho < R = +\infty$ *and let* $f(z)$ *have, at infinity, an essential singularity. Let* $\lambda: 0 \leq \lambda < +\infty$, *be the exact order of this singularity and let there be some* $R_1 < +\infty$ *such that* $f(z) \neq 0$ $(|z| > R_1)$.

I. *Then* λ *is necessarily a positive integer. Writing*

$$M_1(t) = \max_{|z|=t} |f(z)|, \tag{30}$$

we have

$$\lim_{t \to \infty} \frac{\log M_1(t)}{t^{\lambda}} = \kappa_1 \quad (0 < \kappa_1 < +\infty). \tag{31}$$

II. *There exist three positive constants* κ_j $(j=2,3,4)$ *having the following property: for arbitarily large values of n, there will exist, among the zeros*

$$z_{n1}, z_{n2}, \ldots, z_{nj}, \ldots$$

of $T_n(z)$ *no fewer than* $\kappa_2 n$, *which satisfy simultaneously*

$$|f(z_{nj})| < \exp(-\kappa_3 |z_{nj}|^{\lambda}), \quad |z_{nj}| > \kappa_4 n^{1/\lambda}. \tag{32}$$

A closer study of the implications of (31) would enable us to say a good deal more about the values of n whose existence is asserted in Theorem 3. For sake of brevity this aspect of the question is omitted.

Notational conventions. The symbols K, ϵ always denote positive quantities. By $\{\eta_n\}_n$ we denote a sequence (not necessarily positive) such that $\eta_n \longrightarrow 0$ $(n \longrightarrow +\infty)$. Inequalities such as $n > n_0$, $k > k_0, \ldots$ following some relation mean that the relation in question holds for sufficiently large values of $n, k, \ldots$. The symbols K, ϵ, η_n, n_0, $k_0, \ldots$ may have different values in different places.

1. The logarithmic derivatives of f and T_n. Our assumption (9) proves the existence of some Laurent series, convergent for $z \in \mathfrak{A}$ and such that

$$\sum_{j=-\infty}^{+\infty} d_j z^j = \frac{f'(z)}{f(z)}. \tag{1.1}$$

[It is obvious that the compact annulus $\mathfrak{A}$ is a proper subset of the exact annulus of convergence of the series in (1.1).]

By integrating (1.1) round the unit circumference Γ, we see that, since $f(z)$ is single valued, l defined by

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)}\, dz = l = d_{-1}, \tag{1.2}$$

must be an integer. [Note the equivalence of (1.2) and (17).]

Given ρ' and R' such that

$$\rho < \rho' < e^{-\xi}, \quad e^{\xi} < R' < R, \tag{1.3}$$

we have, as $n \longrightarrow +\infty$,

$$T_n(z) \longrightarrow f(z) \quad (\rho' \leq |z| \leq R'). \tag{1.4}$$

Similarly, in view of (9),

$$\frac{T_n'(z)}{T_n(z)} \longrightarrow \frac{f'(z)}{f(z)} \quad (n \longrightarrow +\infty,\ z \in \mathfrak{A}). \tag{1.5}$$

In both (1.4) and (1.5) the convergence is uniform in the regions indicated.

Considering (10), (18) and (19) we see that as $n \longrightarrow +\infty$,

$$s(n) \longrightarrow +\infty, \quad q(n) \longrightarrow +\infty, \tag{1.6}$$

so that, for n large enough,

$$z^q T_n(z) = a_{-q} + a_{-q+1}z + \cdots + a_s z^{s+q} \quad (s = s(n),\ q = q(n),\ a_s a_{-q} \neq 0). \tag{1.7}$$

By the elements of complex analysis

$$\frac{1}{2\pi i}\int_\Gamma \frac{T_n'(z)}{T_n(z)}dz = \tau(n) - q \tag{1.8}$$

and hence, by (1.2), (1.5) and (1.8)

$$l = \tau(n) - q \quad (n > n_0); \tag{1.9}$$

the relations (20) are thus established.

Put

$$V_n(z) = \prod_{j=1}^{\tau(n)}\left(1 - \frac{z}{\zeta_{nj}}\right) \tag{1.10}$$

and note that

$$z^q T_n(z) = a_{-q}P_n(z)V_n(z). \tag{1.11}$$

The Laurent series for T_n'/T_n, convergent in the annulus $\mathfrak{A}$, immediately follows from (1.11); we find

$$\frac{T_n'(z)}{T_n(z)} = \sum_{k=-\infty}^{+\infty} t_{nk}z^k \quad (z \in \mathfrak{A}), \tag{1.12}$$

where

$$t_{nk} = -\sum_{j=1}^{\sigma(n)} z_{nj}^{-k-1} \quad (k = 0, 1, 2, 3, \ldots) \tag{1.13}$$

$$t_{n,-1} = l = d_{-1},\ t_{nk} = \sum_{j=1}^{\tau(n)} \zeta_{nj}^{-k-1} \quad (k = -2, -3, -4, \ldots). \tag{1.14}$$

From (1.1), (1.5), (1.12) and Parseval's relation (also known as Gutzmer's formula in this case) we deduce

$$\sum_{k=-\infty}^{+\infty} |t_{nk} - d_k|^2 r^{2k} = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{T_n'(re^{i\theta})}{T_n(re^{i\theta})} - \frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^2 d\theta \quad (e^{-\xi} \leq r \leq e^{\xi}). \tag{1.15}$$

Similarly, setting

$$G(z) = \sum_{k=0}^{\infty} d_k z^k \quad (|z| \leq e^{\xi}), \tag{1.16}$$

and

$$H(z) = \sum_{k=-2}^{-\infty} d_k z^k \quad (|z| \geq e^{-\xi}), \tag{1.17}$$

we find

$$\sum_{k=0}^{+\infty} |t_{nk} - d_k|^2 r^{2k} = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{P_n'(re^{i\theta})}{P_n(re^{i\theta})} - G(re^{i\theta})\right|^2 d\theta \quad (r \leq e^{\xi}), \tag{1.18}$$

and

(1.19)

$$\sum_{k=-2}^{-\infty} |t_{nk} - d_k|^2 r^{2k} = \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{V_n'(re^{i\theta})}{V_n(re^{i\theta})} - \frac{\tau e^{-i\theta}}{r} - H(re^{i\theta}) \right|^2 d\theta \quad (r \geq e^{-\xi}).$$

Combining (1.5), (1.15), (1.18) and (1.19) we obtain, as $n \longrightarrow +\infty$,

$$\frac{P_n'(z)}{P_n(z)} \longrightarrow G(z) \quad (|z| \leq e^{\xi}), \tag{1.20}$$

$$\left(\frac{V_n'(z)}{V_n(z)} - \frac{\tau}{z}\right) \longrightarrow H(z) \quad (|z| \geq e^{-\xi}), \tag{1.21}$$

uniformly in the regions indicated.

2. Proof of Lemma L. By integrating (1.20) we see that

$$P_n(z) \longrightarrow \exp\left(\int_0^z G(u)\,du\right) \quad (|z| \leq e^{\xi}), \tag{2.1}$$

uniformly in the indicated disk.

To integrate (1.21) note that

$$\int_z^A \left(\frac{V_n'(u)}{V_n(u)} - \frac{\tau}{u}\right)du = \sum_{j=1}^{\tau} \int_z^A \left(\frac{1}{u - \zeta_{nj}} - \frac{1}{u}\right)du$$
$$= \sum_{j=1}^{\tau} \left\{\log\left(\frac{A - \zeta_{nj}}{A}\right) - \log\left(\frac{z - \zeta_{nj}}{z}\right)\right\},$$

and let $A \longrightarrow \infty$ along the ray passing through 0 and z. Hence, as in (2.1)

(2.2)

$$\prod_{j=1}^{\tau} \left(1 - \frac{\zeta_{nj}}{z}\right) = Q_n(\zeta) \longrightarrow \exp\left(-\int_z^{\infty} H(u)\,du\right) \quad \left(\zeta = \frac{1}{z},\ |z| \geq e^{-\xi}\right),$$

as $n \longrightarrow +\infty$, and the convergence is uniform in the indicated annulus.

Combining (1.9), (1.11) and (14) we first notice that

$$T_n(z) = \left\{\frac{(-1)^{\tau} a_{-q}}{\zeta_{n1}\zeta_{n2}\cdots\zeta_{n\tau}}\right\} z^l P_n(z) Q_n\left(\frac{1}{z}\right). \tag{2.3}$$

Hence, taking $z \in \mathfrak{A}$, letting $n \longrightarrow +\infty$ and using (1.4), (2.1) and (2.2) we see that

(2.4)

$$f(z) z^{-l} \exp\left(-\int_0^z G(u)\,du\right) \exp\left(\int_z^{\infty} H(u)\,du\right) = \lim_{n\to\infty} \frac{(-1)^{\tau} a_{-q}}{\zeta_{n1}\zeta_{n2}\cdots\zeta_{n\tau}} = C \neq 0.$$

We have thus proved one of the relations (21); an inspection of (1.7) shows that the expression of C as limit of $C_n = (-1)^{\sigma(n)} a_s z_{n1} z_{n2} \ldots z_{n\sigma(n)}$ must also hold.

We now rewrite (2.3) as

$$P_n(z) = C_n^{-1} z^{-l} \left\{ Q_n\left(\frac{1}{z}\right) \right\}^{-1} T_n(z) \tag{2.5}$$

and notice that, if

$$e^{-\xi} \leq |z| \leq R', \tag{2.6}$$

the limiting process $n \longrightarrow +\infty$ yields, in view of (1.4) and (2.2), the uniform convergence of $P_n(z)$ toward a regular limit function which we may call $\varphi(z)$. The uniform convergence of $P_n(z)$ in the disk $|z| \leq e^{\xi}$ was already guaranteed by (2.1). Hence we have proved (15) and also shown that $\varphi(z)$ is the analytic continuation throughout the disk $|z| < R$ of the expression

$$\exp\left(d_0 z + d_1 \frac{z^2}{2} + d_2 \frac{z^3}{3} + \cdots \right).$$

The exact radius of convergence of the series in (7) cannot exceed R since otherwise the representation (6) would imply (against assumption) that $f(z)$ is regular at all points of the circumference $|z| = R$.

The symmetrical treatment of (16) is obvious and will be left to the reader.

The proof of Lemma L is now complete. We have also established the factorization (6).

3. Proof of Theorem 1. Put

$$F(z) = \sum_{j=0}^{\infty} a_j z^j, \quad M(r) = \max_{0 \leq \theta < 2\pi} |F(re^{i\theta})|. \tag{3.1}$$

LEMMA 3.1. *Let the series in* (3.1) *have radius of convergence* R $(0 < R < +\infty)$. *Then, given* $\epsilon > 0$, *it is possible to find* $n_0(\epsilon)$ *such that*

$$\sum_{j=0}^{n} |a_j| R^j \leq e^{\epsilon n} \quad (n > n_0(\epsilon)). \tag{3.2}$$

Proof. By Cauchy's estimate

$$|a_j| R^j e^{-\epsilon j/2} \leq M(Re^{-\epsilon/2}), \quad \sum_{j=0}^{n} |a_j| R^j \leq (n+1) e^{\epsilon n/2} M(Re^{-\epsilon/2}),$$

and (3.2) follows.

Our definition of the sequence Ω^* implies

$$|a_n|^{1/n} \longrightarrow R^{-1} \quad (0 < R < +\infty, \ n \in \Omega^*, \ n \longrightarrow +\infty).$$

Hence in view of Lemma 3.1, we have, for $n \in \Omega^*$,

$$|a_n| R^n = \exp(\eta_n n), \quad \sum_{j=1}^{n} |a_j| R^j = \exp(\eta_n n). \tag{3.3}$$

[The validity of (3.3) is restricted by our notational convention regarding the symbol η_n.]

Throughout the remainder of this section we only pass to the limit $n \longrightarrow +\infty$ under the restriction $n \in \Omega^*$. Consequently, by (18) and (20),

$$s = s(n) = n, \quad \sigma(n) = n - l; \tag{3.4}$$

moreover the relations (3.3) are valid and consequently

$$\max_{|z|=1} |T_n(Rz)| \leq \exp(\eta_n n) + K \leq \exp(\eta_\sigma \sigma). \tag{3.5}$$

By (2.2)

$$\max_{|z|=1} \left| \frac{1}{Q_n(1/Rz)} \right| \leq K \quad (n > n_0). \tag{3.6}$$

Combining (3.5), (3.6), (2.6), (2.5), (21), and writing

$$U_\sigma(z) = P_n(Rz) = 1 + u_{\sigma 1} z + u_{\sigma 2} z^2 + \cdots + u_{\sigma\sigma} z^\sigma, \tag{3.7}$$

we find

$$\max_{|z|=1} |U_\sigma(z)| = \exp(\eta_\sigma \sigma). \tag{3.8}$$

By (3.7), (13), (21), (3.3) and (3.4) we obtain

$$|u_{\sigma\sigma}| = \frac{R^\sigma}{|z_{n1} z_{n2} \cdots z_{n\sigma}|} = \frac{|a_n| R^n}{|C_n|} R^{-l} = \exp(\eta_\sigma \sigma). \tag{3.9}$$

An inspection of (3.8) and (3.9) shows that a well known equidistribution condition [5; p. 14] is immediately applicable and leads to assertion I of Theorem 1.

Assertion II, concerning the moduli of the zeros z_{nj}, is an immediate consequence of (3.8), (3.9) and Jensen's identity applied to the polynomials $U_\sigma(z)$ and $z^\sigma U_\sigma(1/z)$. Consider, for instance the latter polynomial. Its zeros are R/z_{nj}; let there be N_n' of them in the disk $|z| \leq \exp(-\epsilon)$ ($\epsilon > 0$, is fixed, otherwise arbitrary). Then, by Jensen's theorem and (3.8) $\log |u_{\sigma\sigma}| + \epsilon N_n' \leq \eta_\sigma \sigma$ and hence, in view (3.9), $N_n' \leq \eta_\sigma \sigma$ ($\sigma \longrightarrow +\infty$).

We thus see that there are "few" zeros of $P_n(z)$ in the annulus $|z| \geq e^\epsilon R$.

The same arguments applied to $U_\sigma(z)$ show that there are few zeros of $P_n(z)$ in the disk $|z| \leq e^{-\epsilon} R$.

Assertion II of Theorem 1 immediately follows. The proof of Theorem 1 is now complete.

4. Maximum term and central index of $F(z)$. Throughout the remainder of the paper we assume that $F(z)$ defined by (3.1), is an entire, transcendental function of order exactly $\lambda \leq +\infty$. We adopt the notations (3.1) with this new meaning of $F(z)$.

We take for granted the notion of maximum term

$$\mu(r) = \max_{0 \leq j} |a_j| r^j, \tag{4.1}$$

and central index $\nu(r)$ of the series in (3.1). What little we need to know about $\mu(r)$ and $\nu(r)$ is entirely covered by Pólya and Szegö [6; pp. 1–9]. The following lemma is contained in [6; p. 9, ex. 60].

LEMMA 4.1. *If* $0<\lambda<+\infty$, *and* ϵ *is given* $(0<\epsilon<\lambda)$, *there exist arbitrarily large values of r such that*

$$\frac{\log \mu(r)}{\nu(r)} < \frac{1}{\lambda-\epsilon}. \tag{4.2}$$

In the limiting case $\lambda=+\infty$,

$$\liminf_{r\to\infty} \frac{\log \mu(r)}{\nu(r)} = 0. \tag{4.3}$$

Take $n_k = \nu(r_k)$ and note that, by definition

$$|a_n| r_k^n = \mu(r_k) \quad (n=n_k). \tag{4.4}$$

Using Lemma 4.1 we select a sequence of values r_k such that

$$r_k \longrightarrow +\infty \quad (k\longrightarrow +\infty), \tag{4.5}$$

and such that

$$\lim_{k\to+\infty} \frac{\log \mu(r_k)}{n_k} = A_1 \leq \frac{1}{\lambda} \quad (0\leq A_1<+\infty). \tag{4.6}$$

In the special case $\lambda=+\infty$, we take $A_1=0$.

We now consider the sequence of positive integers $\Omega: n_1, n_2, n_3, \ldots$. We write n for any member of this sequence and let $n\longrightarrow +\infty$ by values of the sequence Ω. Consequently, in view of (20), $\sigma\longrightarrow\infty$ means that σ takes on, successively, the values $\sigma(n_k)=n_k-l$ $(k=1,2,3,\ldots)$. Using the above notations, we have

$$\sum_{j=0}^{n} |a_j| r_k^j \leq (1+n_k)\mu(r_k),$$

and hence

$$\max_{|z|=1} |T_n(r_k z)| \leq (1+n_k)\mu(r_k)+1 < 2n_k\mu(r_k) \quad (n=n_k,\ k>k_0). \tag{4.7}$$

Putting

$$W_\sigma(z) = P_n(r_k z) = 1 + w_{\sigma 1} z + \cdots + w_{\sigma\sigma} z^\sigma \quad (n=n_k), \tag{4.8}$$

and using (2.6) and (4.7), we see that the arguments which lead to (3.8) now imply

$$\max_{|z|=1} |W_\sigma(z)| \leq K r_k^{|l|} n_k \mu(r_k) \quad (k>k_0,\ \sigma=\sigma(n_k)). \tag{4.9}$$

By Liouville's theorem, we know (since $F(z)$ is transcendental) that

$$\frac{\log \mu(r_k)}{\log r_k} \longrightarrow +\infty \quad (k\longrightarrow +\infty). \tag{4.10}$$

Consequently (4.6), (4.9) and (4.10) yield

$$\max_{|z|=1} |W_\sigma(z)| \leq \exp((A_1+\eta_\sigma)\sigma) \quad (\sigma\longrightarrow +\infty). \tag{4.11}$$

With regard to the leading term of $W_\sigma(z)$ we have, by (20), (2.5), (21) and (4.4),

$$|w_{\sigma\sigma}| = \frac{|a_n| r_k^n}{|C_n|} r_k^{-l} \geq K\mu(r_k) r_k^{-|l|}, \tag{4.12}$$

and by (4.6) and (4.10)

$$|w_{\sigma\sigma}| \geq \exp((A_1 + \eta_\sigma)\sigma) \quad (\sigma \longrightarrow +\infty). \tag{4.13}$$

5. Proof of assertions I and III of Theorem 2. Since $F(z)$ is of infinite order we may take $A_1 = 0$ in (4.11) and (4.13). The latter relations may be given the same form as (3.8) and (3.9). Our proof of Theorem 1 holds with the fixed quantity R replaced by the quantities r_k which appear in (4.4). The members of Ω are the integers $n_k = \nu(r_k)$, and we may complete the change of notation by setting $\rho_n = r_k$ $(n = n_k)$. It is now obvious that the proof of assertions I and III is almost identical to the proof of Theorem 1. [The reader will have no difficulty in performing the necessary minor adaptations of our arguments.]

6. Proof of assertion II of Theorem 2. It is advantageous to introduce the auxiliary polynomial

$$X_\sigma(z) = \frac{r_k^l z^\sigma}{\mu(r_k)} P_n\left(\frac{|C_n|^{1/\sigma} r_k}{z}\right). \tag{6.1}$$

By (20), (2.5) and (4.4)

$$|X_\sigma(0)| = 1. \tag{6.2}$$

By (4.8), (4.9) and Cauchy's estimate $|C_n|^{j/\sigma}|w_{\sigma j}| \leq K r_k^{|l|} n_k \mu(r_k)$ $(k > k_0)$, and hence (6.1) implies $\max_{|z|=1} |X_\sigma(z)| \leq K r_k^{2|l|} n_k^2$. Using again (4.6) and (4.10) we find

$$\max_{|z|=1} |X_\sigma(z)| \leq \exp(\eta_\sigma \sigma) \quad (\sigma \longrightarrow +\infty). \tag{6.3}$$

[Note that, in view of (6.2), it is necessary that $\eta_\sigma > 0$ in (6.3).]

Let $x_{\sigma\sigma}$ be the coefficient of the leading term of the polynomial $X_\sigma(z)$; by (6.1)

$$|x_{\sigma\sigma}| = \frac{r_k^l}{\mu(r_k)},$$

and by (4.6) and (4.10) we finally obtain

$$|x_{\sigma\sigma}| = \exp(-(A_1 + \eta_\sigma)\sigma) \quad (\sigma \longrightarrow +\infty). \tag{6.4}$$

If $A_1 = 0$, (6.2), (6.3) and (6.4) enable us to reduce the treatment of $X_\sigma(z)$ to that of $W_\sigma(z)$, given in §5. In this case the very precise relations (25) hold and, a fortiori, the relations (26) must also hold. If $A_1 > 0$ we apply to $X_\sigma(z)$ the following

THEOREM OF CARLSON-ROSENBLOOM. *Let* $X(z) = 1 + x_1 z + x_2 z^2 + \cdots + x_\sigma z^\sigma$ *be a polynomial of degree* σ *such that*

$$|x_\sigma| = e^{-\alpha\sigma} \quad (\alpha \geq 0, \ \sigma \geq 1) \tag{6.5}$$

and such that

(6.6) $$\max_{|z|=1} |X_\sigma(z)| \le e^{\eta\sigma} \quad (0 < \eta).$$

Denote by $\mathfrak{N}$ *the number of zeros of* $X(z)$ *in the angle*

$$\varphi_1 \le \arg z \le \varphi_2 \quad (0 < \varphi_2 - \varphi_1 = \pi/\gamma,\ \gamma > 1/2).$$

Define

(6.7) $$\kappa = \kappa(\alpha, \gamma) = \frac{1}{4}\left\{\frac{\sin(\pi/12\gamma)}{11 + 3\alpha}\right\}^{\omega} \quad (\omega = \exp(\gamma(2\alpha + 9))).$$

Then, if

(6.8) $$\eta \le \kappa,$$

we have

(6.9) $$\mathfrak{N} \ge \sigma\kappa.$$

It is clear that if we select in (6.5) and (6.7)

(6.10) $$\alpha = A_1 + \eta_\sigma$$

and in (6.6),

(6.11) $$\eta = \eta_\sigma,$$

we have, for σ large enough

(6.12) $$\eta_\sigma < \kappa$$

and hence (6.9) follows.

For sake of clarity we observe that η_σ has the same meaning in (6.3), (6.11) and in the left-hand side of (6.12). In view of our notational convention, the other symbols η_σ may have different values.

The arguments of the zeros of $X_\sigma(z)$ only differ in sign from the arguments of the zeros of the corresponding polynomial $P_n(z)$. Assertion 2 of Theorem 2 is now obvious.

With regard to the Theorem of Carlson-Rosenbloom we note that a proof of the statement given above will be found in [4; pp. 78–82, Th. 3].

As to the substance of the theorem, Carlson announced it (without proof) in 1924 [2]. The first proof to appear in the literature was presented by Rosenbloom in his remarkable doctoral dissertation [7]. Carlson's original proof was published in 1948 [3].

7. Proof of assertion IV of Theorem 2. If $A_1 = 0$ (in this case the arguments of the zeros are equidistributed) there is nothing to change in the proof of assertion III of Theorem 2.

If $A_1 > 0$ we deduce from (4.6)

(7.1) $$(A_1 + \eta_k)n_k \le \log M(r_k) \le r_k^{\lambda + \eta_k}$$

and defining, as in §5,

$$r_k = \rho_n, \tag{7.2}$$

the relation (28) follows.

We now apply Jensen's formula to the polynomials $W_\sigma(z)$ defined in (4.8) and use (4.11). Given $B>0$, we denote by N_n'' the number of zeros of $P_n(z)$ in the disk

$$|z| \leq \rho_n \exp(-B) \quad (n \in \Omega), \tag{7.3}$$

and thus find

$$BN_n'' \leq \sum_{|z_{nj}| \leq \rho_n \exp(-B)} \log\left|\frac{\rho_n}{z_{nj}}\right| \leq (A_1 + \eta_n)n. \tag{7.4}$$

Taking $\Lambda = A_1 + \epsilon$, we obtain assertion IV(i) of Theorem 2.

To obtain assertion IV(ii) we apply the proof given in this section to $X_\sigma(z)$ instead of $W_\sigma(z)$. Assertion IV(ii) is somewhat more precise than assertion IV(i) because the right-hand side of (6.3) is smaller than the right-hand side of (4.11).

8. Proof of Theorem 3. The functions $M_1(t)$ in (30), $M(t)$ in (3.1) and

$$M_2(t) = \max_{|z|=t} |\varphi(z)| \tag{8.1}$$

do not coincide. On the other hand, since $F(z)$ is transcendental (by assumption) we have, in view of Liouville's theorem

$$\frac{\log M(t)}{\log t} \longrightarrow +\infty \quad (t \longrightarrow +\infty), \tag{8.2}$$

and taking into account (6) and a fundamental property of the maximum term [6; p. 8, ex. 54]

$$1 = \lim_{t\to\infty} \frac{\log M_1(t)}{\log M(t)} = \lim_{t\to\infty} \frac{\log M_2(t)}{\log M(t)} = \lim_{t\to\infty} \frac{\log \mu(t)}{\log M(t)}. \tag{8.3}$$

By (8.3) the order of $\varphi(z)$ is exactly $\lambda < +\infty$. By assumption, $f(z)$ has no zeros in the annulus $|z| > R_1$; hence, by (6), $\varphi(z)$ has at most finitely many zeros and is consequently of the form

$$\varphi(z) = Y(z)\exp(D(z)). \tag{8.4}$$

where $Y(z)$ $(\not\equiv 0)$ and $D(z)$ are polynomials. Let d be the degree of $D(z)$. We cannot have $d=0$ since, in this case, $f(z)$ would not have an essential singularity at ∞. Hence

$$1 \leq d = \lambda < +\infty, \tag{8.5}$$

and using (8.3) we obtain (31) as well as

$$\lim_{t\to\infty} \frac{\log \mu(t)}{t^\lambda} = \kappa_1. \tag{8.6}$$

The above relation and the elementary identity [6; p. 5, ex. 33]

$$\log\mu(t_2)-\log\mu(t_1)=\int_{t_1}^{t_2}\frac{\nu(x)}{x}\,dx\quad(0<t_1<t_2), \tag{8.7}$$

yield, by a standard tauberian argument [9; p. 47], $\nu(t)\sim\lambda\kappa_1 t^{\lambda}$ $(t\longrightarrow\infty)$, and hence

$$\lim_{t\to\infty}\frac{\log\mu(t)}{\nu(t)}=\frac{1}{\lambda}. \tag{8.8}$$

We now return to the proof in §7, and note that (8.8) enables us to replace (7.1) by the more precise relation

$$\rho_n\sim n^{1/\lambda}(\kappa_1\lambda)^{-1/\lambda}\quad(n\longrightarrow+\infty). \tag{8.9}$$

It immediately follows, from elementary considerations, that (8.4) and (8.5) imply the existence of a sector

$$D=\{re^{i\theta}:\varphi_1\le\theta\le\varphi_2,\ r>0\} \tag{8.10}$$

with $\varphi_2-\varphi_1>\pi/(\lambda+\epsilon)$ $(\epsilon>0,\ \epsilon$ arbitrary) such that

$$|f(z)|\le\exp(-Kr^{\lambda})\quad(z\in D,\ r>r_0). \tag{8.11}$$

By (8.5), assertions (ii) and (iv) of Theorem 2 are applicable to the polynomials $P_n(z)$.

It is now immediate (in view of (8.9)) that for large, suitably chosen values of n, there will be at least $\kappa_2 n$ zeros of $T_n(z)$, of modulus $\ge\kappa_4 n^{1/\lambda}$, which fall in the sector (8.10). Returning to (8.11) we see that the relations (32) must be satisfied. This completes the proof of Theorem 3. □

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