TORSION INVARIANTS AND ACTIONS OF FINITE GROUPS

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Dedicated to the memory of my friend and colleague David L. Williams.

Let G be a finite group and X be a G-CW complex in the sense of T. Matumoto [5] or S. Illman [4]. If X is a finite G-CW complex (i.e., X has only finitely many G-cells), then Illman [4; Section 2] gives a geometric definition for an equivariant Whitehead group $\mathrm{Wh}_G(X)$. Furthermore, he shows [4; Theorem 1.4] that if G is abelian and each component of $X^H=\{x\in X \mid hx=x \text{ for all } h\in H\}$ is simply connected for all subgroups H of G, then $\mathrm{Wh}_G(X)$ is isomorphic to a direct sum of ordinary (i.e., algebraically defined) Whitehead groups. A similar result has been obtained by M. Rothenberg [6; Theorem 1.8].

In a somewhat parallel vein, J. Baglivo [1] considered the following problem. Let X be a G-CW complex which is G-dominated by a finite G-CW complex Y. Does X have the G-homotopy type of a finite G-CW complex? In the approach taken in [1], Baglivo adopts the running hypotheses that $X^G\neq\emptyset$ and that X^H is connected for all subgroups $H\subset G$. Under these conditions, she shows that there exist groups, denoted by $\widetilde{N(H)/H}$ in [1], and elements $w_H(X)\in\tilde{K}_0Z(\widetilde{N(H)/H})$ such that X has the homotopy type of a finite G-CW complex if and only if all the $w_H(X)=0$.

Let X^H_α be a subcomplex of X^H. (In this paper X^H_α will actually be a connectedness component of X^H or a union of such.) Let $G_\alpha=\{g\in G \mid g(X^H_\alpha)=X^H_\alpha\}$ and $N(H)=\{g\in G \mid gHg^{-1}=H\}$ be the normalizer of H. In this paper, we introduce a group $\Gamma(X^H_\alpha,G)$ which fits into a short exact sequence

$$1\longrightarrow \pi_1(X^H_\alpha)\longrightarrow \Gamma(X^H_\alpha,G)\longrightarrow G_\alpha\cap N(H)/H\longrightarrow 1$$

and use these groups to generalize the results of [1], [4], and [6]. In particular, we establish the following theorems:

THEOREM A. *Let G be a finite group and X be a finite G-CW complex. Let $\{H_s \mid s\in S\}$ be a set of representatives for the subgroups of G that are contained in an isotropy subgroup of the action of G on X. Let $\{X^{H_s}_\alpha \mid \alpha\in A_s\}$ be a set of representatives for the connectedness components of X^{H_s}. Then there exists an isomorphism*

$$\Phi:\mathrm{Wh}_G(X)\longrightarrow \sum_{s\in S}\sum_{\alpha\in A_s}\mathrm{Wh}\,\Gamma(X^{H_s}_\alpha,G)$$

THEOREM B. *Let G be a finite group and X be a G-CW complex. Let $\{H_s \mid s\in S\}$ be a set of representatives for the set of isotropy subgroups of the action of G on X. Let $\{X^{H_s}_\alpha \mid \alpha\in A_s\}$ be a set of representatives for the connectedness components of X^{H_s}.*

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i) *If X is G-dominated by a finite G-CW complex, then there exists a family of obstructions* $\{w_\alpha^{H_s}(X) \mid s \in S, \alpha \in A_s\}$ *with* $w_\alpha^{H_s}(X) \in \tilde{K}_0 Z\Gamma(X_\alpha^{H_s}, G)$ *such that X has the G-homotopy type of a finite G-CW complex if and only if* $w_\alpha^{H_s}(X) = 0$ *for all s and α.*

ii) *If X is a finite G-CW complex and* $\{w_\alpha^{H_s} \mid s \in S, \alpha \in A\}$ *is any family of elements with* $w_\alpha^{H_s} \in \tilde{K}_0 Z\Gamma(X_\alpha^{H_s})$, *then there exists a G-CW complex Y, G-dominated by a finite G-CW complex, such that* $w_\alpha^{H_s}(Y) = w_\alpha^{H_s}$.

We recall that if $\mathcal{S}$ is a set of subgroups of G, then $\mathcal{R}$ is said to *represent* $\mathcal{S}$ if each subgroup in $\mathcal{S}$ is conjugate to a unique subgroup in $\mathcal{R}$.

Similarly, if X is a G-CW complex and H is a subgroup of G, then $\{X_\alpha^H \mid \alpha \in A\}$ is a *set of representatives* for the connectedness components of X^H if, for each component X_β^H of X^H, there exists a unique $\alpha \in A$ such that $gX_\beta^H = X_\alpha^H$ for some $g \in G$.

If the action of G on X is free, then the only isotropy subgroup of the action is the trivial group $\{1\}$; hence, the families $\{H_s\}$ of Theorems A and B consist of only the trivial group. If $X = X^{\{1\}}$ is also connected, the family $\{X_\alpha^{H_s}\}$ reduces to just $\{X\}$. Finally, in this case we will show that $\Gamma(X^{\{1\}}, G) = \pi_1(X/G)$ where X/G is the orbit space of the G-action on X. Hence, we obtain the following corollary:

COROLLARY C. *Let G be a finite group and X be a G-CW complex which is connected and on which G acts freely.*

i) *If X is finite, then there is an isomorphism* $\Phi : \mathrm{Wh}_G(X) \longrightarrow \mathrm{Wh}\,\pi_1(X/G)$.

ii) *If X is G-dominated by a finite G-CW complex, then there is a single obstruction* $w^G(X) \in \tilde{K}_0 Z\pi_1(X/G)$ *such that X has the G-homotopy type of a finite G-CW complex if and only if* $w^G(X) = 0$. *Furthermore, any element* $w_0 \in \tilde{K}_0 Z\pi_1(X/G)$ *can occur as this obstruction.*

Corollary C is a reflection of the familiar, and expected, feeling that the orbit space of a free action should capture all the information about the action. For a general action, however, the orbit space does not capture all the information about equivariant Whitehead torsions and equivariant finiteness obstructions even if one stratifies X/G by orbit types. The groups $\Gamma(X_\alpha^{H_s}, G)$ that we introduce are designed to circumvent this difficulty. In particular, they play the same algebraic role for general equivariant torsion invariants that the fundamental group (or $\pi_1(X/G)$, respectively) plays for ordinary torsion invariants (or torsion invariants of free actions, respectively) and, in general, $\Gamma(X, G)$ is not isomorphic to $\pi_1(X/G)$.

We have made an effort in this paper to present the proofs of Theorems A and B as quickly as possible. In particular, we have given in section 1 only the statements of the main results used in proving Theorems A and B and have suppressed the tedious details of the proofs of those results to sections 5 and 6. Section 2 contains some technical results that may be of some independent interest; while sections 3 and 4 contain the proofs of Theorems A and B.

After this paper was written, Soren Illman pointed out to the author that Henning Hauschild had obtained a result similar to Theorem A in his paper [3]. Indeed, the main result of [3, Satz IV.1], gives an algebraic description of $\mathrm{Wh}_G(X)$ valid for any compact Lie group G, not just finite groups, in terms of certain Whitehead groups $\mathrm{Wh}\,\pi_1(EK_\alpha \times_{K_\alpha} X_\alpha^H)$, where we have changed the notation of [3] slightly so that it conforms with the notation of this paper. When we introduce the group $\Gamma(X_\alpha^H, G)$ in section 2, we shall show that $\Gamma(X_\alpha^H, G)$ is isomorphic to $\pi_1(EK_\alpha \times_{K_\alpha} X_\alpha^H)$. If is then easy to see that Theorem A and [3, Satz IV.1] agree completely. Since the methods of proof used in this paper differ from those of [3] and may be of some independent interest, we include a full proof of Theorem A. The author would like to thank Illman for bringing [3] to his attention.

1. The case of a relatively free action. The reader will find that the proofs of the main results of this paper proceed essentially by induction. In this section we outline the results that form the basis for the inductive steps of the main proofs. The proofs of the results in this section are deferred until sections 5 and 6.

Let X be a connected G-CW and $p:\tilde{X}\longrightarrow X$ be its universal cover. We define a prototype, $\Gamma(X, G)$, of the groups $\Gamma(X_\alpha^H, G)$ by $\Gamma(X,G)=\{h \mid h:\tilde{X}\longrightarrow \tilde{X}$ is a homeomorphism such that there exists a $g\in G$ such that $ph=gp\}$. If G acts effectively on X, it is easy to see that there is a short exact sequence

$$1 \longrightarrow \pi_1(X) \longrightarrow \Gamma(X,G) \xrightarrow{\rho} G \longrightarrow 1$$

where $\rho(h)\in G$ satisfies $ph=\rho(h)p$. We note that $\Gamma(X,G)$ acts cellularly on $\tilde{X}$.

If (Y,X) is a pair of connected G-CW complexes such that $\pi_1(X)\longrightarrow\pi_1(Y)$ is an isomorphism, then $p^{-1}(X)=\tilde{X}$ is the universal cover of X where $p:\tilde{Y}\longrightarrow Y$ is the universal cover of Y. Hence, there is a homomorphism $r_*:\Gamma(Y,G)\longrightarrow\Gamma(X,G)$ induced by restricting the action of $\Gamma(Y,G)$ on $\tilde{Y}$ to $\tilde{X}$. If the action of G on X is effective, then clearly r_* is an isomorphism.

Suppose now that (Y,X) is a 1-connected pair of G-CW complexes which is *relatively free* (i.e., G acts freely on $Y-X$). Then the action of $\Gamma(Y,G)$ on $\tilde{Y}-\tilde{X}$ is free and the cellular chain complex $C_*=C_*(\tilde{Y},\tilde{X})$ is a chain complex of free $Z\Gamma(Y,G)$ modules. In fact, $C_n(\tilde{Y},\tilde{X})$ has a basis $\{\tilde{e}_i^n\}$ consisting of one n-cell $\tilde{e}_i^n$ out of each $\Gamma(Y,G)$ orbit of n-cells in $\tilde{Y}-\tilde{X}$. This basis is unique up to the usual notion of equivalence of bases and we regard C_* as a based $Z\Gamma(Y,G)$ chain complex. We note that if Y is a finite G-CW complex, then C_* is finitely generated.

If the relatively free, finite G-CW pair (Y,X) has the property that the inclusion $i:X\longrightarrow Y$ is a homotopy (or a G-homotopy) equivalence, then $C_*(\tilde{Y},\tilde{X})$ is a finitely generated, based, acyclic, chain complex of $Z\Gamma(Y,G)$ modules. Hence, its Whitehead torsion is defined and we set $\tau(Y,X)=r_*\tau(C_*(\tilde{Y},\tilde{X}))\in \mathrm{Wh}\,\Gamma(X,G)$. The geometric meaning of the invariant $\tau(Y,X)$ is given in the following theorem.

THEOREM 1.1 *Let (Y,X) be a relatively free, finite, G-CW pair such that the inclusion $i:X\longrightarrow Y$ is a homotopy equivalence. Then $\tau(Y,X)=0$ if and only if there exists a free equivariant formal deformation from Y to X relative to X.*

A equivariant formal deformation is *free* if it is a composite of free elementary equivariant expansions and collapses. The equivariant elementary expansion that sends X to $X\cup b^n\cup b^{n+1}$ is *free* if b^s is a free G-orbit of an s-cell e^s for $s=n, n+1$. The inverse of a free equivariant elementary expansion is a *free* equivariant elementary collapse. The reader is referred to [4] for more complete definitions of equivariant formal deformations and their main properties.

We also prove the following realization theorem.

THEOREM 1.2. *Let X be a finite, connected, G-CW complex on which G acts effectively, and let $\tau_0 \in \mathrm{Wh}\,\Gamma(X, G)$ be arbitrary. Then there exists a relatively free, finite G-CW pair (Y, X) for which the inclusion $i: X \longrightarrow Y$ is a G-homotopy equivalence such that $\tau(Y, X) = \tau_0$.*

Suppose now that (X, A) and (Y, B) are relatively free G-CW pairs. We say that (Y, B) *G-dominates* (X, A) if there exist G-maps $i: (X, A) \longrightarrow (Y, B)$ and $r: (Y, B) \longrightarrow (X, A)$ such that ri is G-homotopic to the identity map as maps of pairs. In this case r is called a *domination* with *section i*.

We note that if X and Y are connected, then $r_*: \pi_1(Y) \longrightarrow \pi_1(X)$ is onto.

Let $p: \tilde{X} \longrightarrow X$ be the universal cover of X and $q: \hat{Y} \longrightarrow Y$ be the pullback of p via r. In fact, $\hat{Y}$ is just the covering space of Y corresponding to $\ker r_*$. Thus $\hat{Y} = \{(x, y) \in \tilde{X} \times Y \mid p(x) = r(y)\}$ and it is easy to see that $\Gamma(X, G)$ acts on $\hat{Y}$ by setting $h(x, y) = (h(x), \rho(h)(y))$ for $h \in \Gamma(X, G)$. Furthermore, this action is free and cellular on $\hat{Y} - q^{-1}(B)$ from which it follows that the cellular chain complex $C_*(\hat{Y}, q^{-1}(B))$ is a chain complex of free $Z\Gamma(X, G)$ modules.

If (Y, B) is a relatively finite, relatively free G-CW pair that G-dominates (X, A), then $C_*(\hat{Y}, q^{-1}(B))$ is a finitely generated free $Z\Gamma(X, G)$ chain complex which dominates $C_*(\tilde{X}, p^{-1}(A))$ in the usual sense (cf. [7] or [8]). Hence the finiteness obstruction $\mathcal{O}(C_*(\tilde{X}, p^{-1}(A))) \in \tilde{K}_0 Z\Gamma(X, G)$ is defined. We set $w_G(X, A) = \mathcal{O}(C_*(\tilde{X}, p^{-1}(A)))$ and call $w_G(X, A)$ the *relative equivariant finiteness obstruction* of (X, A).

THEOREM 1.3. *Let the relatively free G-CW pair (X, A) be G-dominated by the relatively finite, relatively free G-CW (Y, B). Let X and Y be connected, suppose G acts effectively on X and Y, and suppose that Y (or, equivalently, B) is of finite type. Then there exists a relatively finite G-CW pair (Z, A) and a G-homotopy equivalence of pairs $g: (Z, A) \longrightarrow (X, A)$ such that $f \mid A$ is the identity if and only if $w_G(X, A) = 0$.*

The assumption that G acts effectively on X and Y is included only to rule out degenerate cases. If $X - A$ and $Y - B$ are non-empty, then G obviously acts effectively on X and Y.

We also establish a realization theorem in this setting.

THEOREM 1.4. *Let (Z, A) be a relatively finite, relatively free, G-CW pair. Let $w_0 \in \tilde{K}_0 Z\Gamma(Z, G)$ be arbitrary. Suppose that G-acts effectively on Z. Then there exist relatively free G-CW pairs (X, A) and (Y, A) such that (Y, A) is relatively finite, G-dominates (X, A), and $w_G(X, A) = w_0$.*

The complex X constructed to prove 1.4 has the properties that $Z \subset X$ and $\pi_1(Z) \longrightarrow \pi_1(X)$ is an isomorphism. Then the map $\rho: \Gamma(X, G) \longrightarrow \Gamma(Z, G)$ obtained by restriction is an isomorphism and the above equation can be written more precisely as $\rho w_G(X, A) = w_0$.

2. The component structure of a *G*-space. Let X be a G-space. In this section we define the "component structure" of X and collect its main properties. We also define the group $\Gamma(X_\alpha^H, G)$ in this section.

Let X be a G-space. The *component structure* of X is the triple $(\{X_\alpha^H\}, \{G_\alpha\}, \{H_\alpha\})$ where $\{X_\alpha^H \mid H \subset G, \alpha \in A_H\}$ is the set of components of X^H, (A_H is an indexing set depending on H), $G_\alpha = \{g \in G \mid g(X_\alpha^H) = X_\alpha^H\}$, and $H_\alpha = \{g \in G_\alpha \mid g$ is the identity on $X_\alpha^H\}$.

REMARK 2.1. Clearly $H_\alpha = \cap I_x$ where $I_x = \{g \in G \mid gx = x\}$ is the isotropy subgroup of x and the intersection runs over $x \in X_\alpha^H$. Furthermore, $G_\alpha \subset N(H_\alpha)$ where $N(H_\alpha)$ is the normalizer of H_α. Thus H_α is normal in G_α.

LEMMA 2.2. *Let $i: X \longrightarrow Y$ be a G-map. Let X_α^H and Y_β^H be components of X^H and Y^H, respectively, such that $i(X_\alpha^H) \subset Y_\beta^H$. Then*

i) *$G_\alpha \cap N(H) \subset G_\beta$, and*

ii) *If there exists a G-map $r: Y \longrightarrow X$ such that ri is G-homotopic to the identity, then $G_\alpha \cap N(H) = G_\beta \cap N(H)$ and Y_β^H $G_\alpha \cap N(H)$-dominates X_α^H.*

Proof: Let $g \in G_\alpha \cap N(H)$, $y \in Y_\beta^H$ and $z = i(x)$ where $x \in X_\alpha^H$. Let $\omega: I \longrightarrow Y_\beta^H$ be a path such that $\omega(0) = z$ and $\omega(1) = y$. Since $g \in N(H)$, $g\omega$ is a path in Y^H. Since $g \in G_\alpha$, $g\omega(0) = gz = gi(x) = i(g(x)) \in i(X_\alpha^H) \subset Y_\beta^H$. Thus $g\omega$ is a path in Y_β^H since Y_β^H is a component of Y^H. Hence $gy \in Y_\beta^H$ and $g(Y_\beta^H) \subset Y_\beta^H$. But then also $g^{-1} \in G_\alpha \cap N(H)$ and $g^{-1}(Y_\beta^H) \subset Y_\beta^H$. Hence $g(Y_\beta^H) = Y_\beta^H$ and $g \in G_\beta$.

To prove ii), let X_γ^H be the component of X^H such that $r(Y_\beta^H) \subset X_\gamma^H$. Let $F: X \times I \longrightarrow X$ be a G-homotopy from the identity to ri. Then $F(X_\alpha^H \times I)$ is contained in a component of X^H and, in fact, this component must be X_α^H since $F \mid X \times 0$ is the identity. Hence $ri(X_\alpha^H) = (F \mid X \times 1)(X_\alpha^H) \subset X_\alpha^H$. But also $ri(X_\alpha^H) \subset r(Y_\beta^H) \subset X_\gamma^H$; from which it follows that $X_\alpha^H \cap X_\gamma^H \neq \emptyset$. Since X_α^H and X_γ^H are components of X^H, this implies that $X_\gamma^H = X_\alpha^H$ and that $r: Y_\beta^H \longrightarrow X_\alpha^H$. The rest of ii) now follows from i) and by restricting F to $X_\alpha^H \times I$. □

COROLLARY 2.3. *Let $f: X \longrightarrow Y$ be a G-homotopy equivalence. Let X_α^H and Y_β^H be components of X^H and Y^H, respectively, such that $f(X_\alpha^H) \subset Y_\beta^H$. Then*

i) *$f \mid X_\alpha^H: X_\alpha^H \longrightarrow Y_\beta^H$ is a $G_\alpha \cap N(H)$-homotopy equivalence*

ii) *If $X \subset Y$ and f is the inclusion map, then the inclusion*

$$X_\alpha^H \cup [Y_\beta^H \cap \bigcup_{K \supsetneq H} Y^K] \longrightarrow Y_\beta^H$$

is a $G_q \cap N(H)$-homotopy equivalence.

Proof. The first part of the corollary is an immediate consequence of 2.2. To prove the second part, we note first that $Y_\beta^H \cap \bigcup_{K \supsetneq H} Y^K = \bigcup_{K \supsetneq H} Y_\delta^K$ is the union of the components of $Y^K (K \supsetneq H)$ that are contained in Y_β^H. Similarly,

$$X_\alpha^H \cap [Y_\beta^H \cap \bigcup_{K \supsetneq H} Y^K] = \bigcup_{K \supsetneq H} X_\gamma^K$$

is the union of the components of $X^K (K \supsetneq H)$ contained in X_α^H.

Define $i_*: \{X_\gamma^K\} \longrightarrow \{Y_\delta^K\}$ by $i_*(X_\gamma^K) = Y_\delta^K$ if $X_\gamma^K \subset Y_\delta^K$. Since the inclusion $i: X \longrightarrow Y$ is a G-homotopy equivalence, i_* is a well-defined bijection such that $i \mid X_\gamma^K: X_\gamma^K \longrightarrow Y_\delta^K = i_*(X_\gamma^K)$ is a homotopy equivalence for all K and γ. A straightforward, but tedious, Van Kampen and Mayer-Vietoris argument will now complete the proof of ii). □

We now turn to the definition of the group $\Gamma(X_\alpha^H, G)$. We observe first that if X_α^H is a component of X^H, then G_α/H_α acts effectively on X_α^H. Thus the group $\Gamma(X_\alpha^H, G_\alpha/H_\alpha)$ is defined. On the other hand, $G_\alpha \cap N(H) \subset G_\alpha$ and $H \subset H_\alpha$. Thus there is an obvious homomorphism $\sigma: G_\alpha \cap N(H)/H \longrightarrow G_\alpha/H_\alpha$. We now define $\Gamma(X_\alpha^H, G)$ to be $\sigma^*\Gamma(X_\alpha^H, G_\alpha/H_\alpha)$. More precisely, $\Gamma(X_\alpha^H, G)$ is defined to be the pull back group making the diagram below commute

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(X_\alpha^H) & \longrightarrow & \Gamma(X_\alpha^H, G) & \longrightarrow & G_\alpha \cap N(H)/H & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow \sigma & & \\ 1 & \longrightarrow & \pi_1(X_\alpha^H) & \longrightarrow & \Gamma(X_\alpha^H, G_\alpha/H_\alpha) & \longrightarrow & G_\alpha/H_\alpha & \longrightarrow & 1 \end{array}$$

We remark that if there exists a point $x \in X_\alpha^H$ with $I_x = H$, then $H_\alpha = H$ and $G_\alpha \subset N(H)$. Thus σ is an isomorphism and $\Gamma(X_\alpha^H, G) = \Gamma(X_\alpha^H, G_\alpha/H_\alpha)$.

We shall now show that $\Gamma(X_\alpha^H, G)$ is isomorphic to the group $\pi_1(EK_\alpha \times_{K_\alpha} X_\alpha^H)$ of [3; Section 4]. Since the group K_α of [3] is just the subgroup of $W(H) = N(H)/H$ that maps the component X_α^H to itself, it is easy to see that $K_\alpha = G_\alpha \cap N(H)/H$. To simplify our notation, we write K_α instead of $G_\alpha \cap N(H)/H$. Thus there is a pull back diagram

$$\begin{array}{ccc} \Gamma(X_\alpha^H, G) & \xrightarrow{\bar{\rho}} & K_\alpha \\ \bar{\sigma}\downarrow & & \sigma\downarrow \\ \Gamma(X_\alpha^H, G_\alpha/H_\alpha) & \xrightarrow{\rho} & G_\alpha/H_\alpha \end{array}$$

Using this diagram, define an action of $\Gamma(X_\alpha^H, G)$ on $EK_\alpha \times \tilde{X}_\alpha^H$ by setting $\gamma(x, y) = (\bar{\rho}(\gamma)(x), \bar{\sigma}(\gamma)(y))$ for $\gamma \in \Gamma(X_\alpha^H, G)$. It is easy to verify that this action is free and that the orbit space is $EK_\alpha \times_{K_\alpha} X_\alpha^H$. The existence of the claimed isomorphism is now clear.

3. The proof of Theorem A. In this section we prove Theorem A. The first step is to define the map Φ.

To do this, let $x \in \mathrm{Wh}_G(X)$ be represented by the pair (Y, X) of connected, G-finite, G-CW complexes where the inclusion $i: X \longrightarrow Y$ is a G-homotopy equivalence. Let $H = H_s$ be one of the distinguished (i.e., in the set $\{H_s \mid s \in S\}$ of Theorem A) subgroups of G and let X_α^H be one of the distinguished components of X^H. Let Y_β^H be the component of Y^H containing X_α^H.

Let $p: \tilde{Y}_\beta^H \longrightarrow Y_\beta^H$ be the universal cover of Y_β^H and notice that $p^{-1}(X_\alpha^H) = \tilde{X}_\alpha^H$ is the universal cover of X_α^H. We claim that $C_* = C_*(\tilde{Y}_\beta^H, \tilde{X}_\alpha^H \cup p^{-1}(Y_\beta^H \cap \bigcup_{K \supsetneq H} Y^K))$ is a finitely generated, acyclic, based chain complex of free $Z\Gamma(X_\alpha^H, G)$ modules. Indeed, it follows from 2.3 that the inclusion $i: X_\alpha^H \cup (Y_\beta^H \cap \bigcup_{K \supsetneq H} Y^K) \longrightarrow Y_\beta^H$ is a $G_\alpha \cap N(H)$-homotopy equivalence; hence, C_* is acyclic. If we now set,

$$Z = Y_\beta^H - [X_\alpha^H \cup (Y_\beta^H \cap \bigcup_{K \supsetneq H} Y^K)],$$

then $I_z = H$ for all $z \in Z$. Hence, $G_\alpha \cap N(H)/H$ acts freely on Z and relatively freely on $(Y_\beta^H, X_\alpha^H \cup (Y_\beta^H \cap \bigcup_{K \supsetneq H} Y^K))$. By the analysis of section 1, it now follows that C_* is a finitely generated, free, based chain complex of $Z\Gamma(Y_\beta^H, G_\alpha \cap N(H)/H)$ modules. On the other hand, the map $r_*: \Gamma(Y_\beta^H, G_\alpha \cap N(H)/H) \longrightarrow \Gamma(X_\alpha^H, G_\alpha/H_\alpha)$

obtained by restricting the homeomorphism h of $\tilde{Y}^H_\beta$ to $\tilde{X}^H_\alpha$ is easily seen to induce an isomorphism $r_*:\Gamma(Y^H_\beta, G_\alpha\cap N(H)/H)\longrightarrow\Gamma(X^H_\alpha, G)$ and the claim follows.

We set $\tau^H_\alpha(Y,X)=r_*\tau(C_*(\tilde{Y}^H_\beta,\tilde{X}^H_\alpha\cup p^{-1}(Y^H_\beta\cap\bigcup_{K\supsetneq H}Y^K))$ in $\mathrm{Wh}\,\Gamma(X^H_\alpha, G)$ and define the homomorphism Φ by $\Phi(x)=(\tau^{H_t}_\alpha(Y,X))$ where (Y,X) represents $x\in\mathrm{Wh}_G(X)$ and $H_t\in\{H_s\mid s\in S\}$ and $\alpha\in A_t$ in the notation of Theorem A. It is clear from the definition of addition in $\mathrm{Wh}_G(X)$ that Φ is a homomorphism if it is well defined.

LEMMA 3.1. *The map Φ is well defined.*

Proof. It suffices to show that if Z is obtained from Y by an equivariant elementary expansion relative to X, then $\tau^{H_t}_\alpha(Z,X)=\tau^{H_t}_\alpha(Y,X)$ for all $H_t\in\{H_s\mid s\in S\}$ and all $\alpha\in A_t$. Without loss of generality, we may assume that $Z=Y\cup b^n\cup b^{n+1}$ $(n\geq 0)$, where b^r is an equivariant r-cell $(r=n,\ n+1)$, and that there is a characteristic G-map $\phi: G/H_t\times I^{n+1}\longrightarrow Z$ for b^{n+1} such that $\phi\mid: G/H_t\times I^n\longrightarrow Z$ is a characteristic G-map for b^n and $\phi(G/H_t\times J^n)\subset Y$ where H_t is one of the distinguished subgroups of G. For simplicity of notation, let $H=H_t$.

Clearly $\phi(H\times J^n)\subset Y^H$ and since J^n is connected, $\phi(H\times J^n)$ lies in a single component Y^H_β of Y^H. Let X^H_α be the component of X^H contained in Y^H_β. We may further assume that X^H_α is one of the distinguished components of X^H.

Let Z^H_γ be the component of Z^H containing Y^H_β and let $e^r=\phi(H\times I^r)$. By 2.2 if $g\in G_\beta\cap N(H)$, then $ge^r\in Z^H_\gamma$. On the other hand, if $ge^r\in Z^H_\gamma$, then clearly $g\in N(H)$, $g(Y^H_\beta)$ is a component of Y^H, and an argument similar to 2.2 shows that $g(Y^H_\beta)=Y^H_\beta$. Thus $g\in G_\beta\cap N(H)$. It follows that Z^H_γ is obtained from Y^H_β by a $G_\beta\cap N(H)$-equivariant elementary expansion, and that, if we pass to $G_\beta\cap N(H)/H$ this expansion is free. Since $G_\beta\cap N(H)/H=G_\alpha\cap N(H)/H$ by 2.2, it follows from 5.1 below that $\tau^H_\alpha(Y,X)=\tau(Y^H_\beta,X^H_\alpha)=\tau(Z^H_\gamma,X^H_\alpha)=\tau^H_\alpha(Z,X)$.

If either $\delta\neq\alpha$ or $s\neq t$, then for $H=H_s$, the component Y^H_β of Y^H containing X^H_α is unaffected by the equivariant elementary expansion to Z. That is, $Y^H_\beta=Z^H_\gamma$ where Z^H_γ is the component of Z^H containing Y^H_β. Hence $\tau^H_\delta(Y,X)=\tau^H_\delta(Z,X)$ for all $H=H_s$ and all δ. Thus Φ is well defined. □

LEMMA 3.2. *The map Φ is injective.*

Proof. Let the connected G-CW pair (Y,X) represent an element $x\in\mathrm{Wh}_G(X)$ for which $\Phi(x)=0$. Let $H=H_t$ be a distinguished subgroup of G of minimal order that is also an isotropy subgroup for some point in $Y-X$. Let X^H_α be a distinguished component of X^H which does not equal the component Y^H_β of Y^H that contains it. Let $B^H_\alpha=X^H_\alpha\cup(Y^H_\beta\cap\bigcup_{K\supsetneq H}Y^K)$.

It follows from 2.1 and 2.2 that the pair (Y^H_β, B^H_α) is $G_\alpha\cap N(H)/H$ relatively free. Furthermore, the inclusion $B^H_\alpha\longrightarrow Y^H_\beta$ is a $G_\alpha\cap N(H)/H$ homotopy equivalence by 2.3 and $\tau(Y^H_\beta, B^H_\beta)=\tau^H_\alpha(Y,X)=0$. Hence, there is a free $G_\alpha\cap N(H)/H$-equivariant formal deformation of Y^H_β to B^H_α relative to B^H_α. This deformation can also be regarded as a $G_\alpha\cap N(H)$-equivariant formal deformation and as such it extends to a G-equivariant formal deformation from GY^H_β to $GB^H_\alpha=GX^H_\alpha\cup(GY^H_\beta\cap\bigcup_{K\supsetneq H}GY^K)$ relative to GB^H_α by a standard equivariant cell attaching argument.

Now let $Z=X\cup\bigcup Y^K\cup\bigcup GY^H_\delta$ where the first union runs over subgroups K of G satisfying either $|K|>|H|$ or $|K|=|H|$ and K is not conjugate to H and the second union runs over the components of Y^H other than Y^H_β. Then Z is a G-invariant subcomplex of Y and $Y-Z=GY^H_\beta$. But also $GY^H_\beta \cap Z=GB^H_\alpha$. Thus there is a G-equivariant formal deformation of Y to Z relative to Z.

Now the pair (Z, X) of G-CW complexes also represents the element $x \in \mathrm{Wh}_G(X)$, but is less "complex" than (Y, X) in the sense that Z differs from X either in fewer components of Z^H than Y^H or in fewer fixed point sets of distinguished subgroups of G than Y. Since there are only finitely many places such differences can occur, we can now proceed by induction to obtain an equivariant formal deformation of Z to X relative to X. Hence $x=0$ and Φ is injective. □

LEMMA 3.3. *The map Φ is surjective.*

Proof. Let $\tau_0 \in \mathrm{Wh}\,\Gamma(X^{H_s}_\alpha, G)$ be an arbitrary element. It suffices to show there exists an element $x \in \mathrm{Wh}_G(X)$ such that the component of $\Phi(x)$ in $\mathrm{Wh}\,\Gamma(X^{H_s}_\alpha, G)$ is τ_0 and all other components of $\Phi(x)$ are zero. Let $H=H_s$. There are now two cases.

Case I. There exists a point $z \in X^H_\alpha$ such that $I_z=H$. In this case $H_\alpha=H$ and $G_\alpha \subset N(H)$ by 2.1. Hence $G_\alpha \cap N(H)/H$ acts effectively on X^H_α, and by 1.2 there exists a finite $G_\alpha \cap N(H)/H$-CW complex Y^H_β containing X^H_α for which the inclusion $X^H_\alpha \longrightarrow Y^H_\beta$ is a $G_\alpha \cap N(H)/H$-homotopy equivalence and such that $\tau(Y^H_\beta, X^H_\alpha)=\tau_0$ in $\mathrm{Wh}\,\Gamma(X^H_\alpha, G_\alpha \cap N(H)/H)$.

Let $Z=G\times_{G'} Y^H_\beta$ be the balanced product of G and Y^H_β over $G'=G_\alpha \cap N(H)$ where $g \in G'$ acts on G by right multiplication and on Y^H_β via the homomorphism $G_\alpha \cap N(H) \longrightarrow G_\alpha \cap N(H)/H$. Then Z is a left G-space and there is an obvious G-map $\phi: G\times_{G'} X^H_\alpha \longrightarrow X$. Let $Y=X \cup_\phi Z$. It is a routine, but tedious, matter to verify that (Y, X) is G-CW pair representing a class $x \in \mathrm{Wh}_G(X)$ with the desired properties.

Case II. There is no point $z \in X^H_\alpha$ such that $I_z=H$. In this case, let $Z=X \cup b^n \cup b^{n+1}$ $(n \geq 2)$ be obtained from X via a equivariant elementary expansion with the property that there exists a characteristic G-map $\phi: G/H\times I^{n+1} \longrightarrow Z$ for b^{n+1} for which $\phi|: G/H\times I^n \longrightarrow Z$ is a characteristic G-map for b^n and for which

$$\phi(G/H\times J^n) = Z_0 \subset X^H_\alpha.$$

Clearly such a G-CW complex Z exists. Furthermore, if Z^H_γ is the component of Z^H containing X^H_α, $G_\gamma=G_\alpha \cap N(H)$ and $H_\gamma=H$ by 2.2, the observation that restriction defines an injection $G_\gamma \longrightarrow G_\alpha$, and 2.1. Hence, case I may be applied to Z to complete the proof of 3.3. □

Theorem A is a now trivial consequence of 3.1–3.3.

4. The proof of Theorem B. This section contains only the proof of Theorem B.

Let X and Y be G-CW complexes with Y finite and let $r: Y \longrightarrow X$ be a G-domination with section $i: X \longrightarrow Y$. Let $H=H_s$ be one of the distinguished isotropy subgroups (i.e., H is in the set $\{H_s \mid s \in S\}$ of representatives of the isotropy subgroups of the action of G on X given in Theorem B) and let X^H_α be a distinguished component of X^H. Let Y^H_β be the component of Y^H such that $i(X^H_\alpha) \subset Y^H_\beta$. It then follows from 2.2 that $G_\alpha \cap N(H)=G_\beta \cap N(H)$ and that Y^H_β $G_\alpha \cap N(H)$-dominates X^H_α.

If we now consider the pairs $(X^H_\alpha, X^H_\alpha \cap \bigcup_{K \supsetneq H} X^K)$ and $(Y^H_\beta, Y^H_\beta \cap \bigcup_{K \supsetneq H} Y^K)$, we see easily that they are relatively free $\bar{G} = G_\alpha \cap N(H)/H$-$CW$ pairs and that the former is $\bar{G}$-dominated by the latter which, in turn, is relatively finite. We now define $w^{H_s}_\alpha(X)$ to be the invariant $w_{\bar{G}}(X^H_\alpha, X^H_\alpha \cap \bigcup_{K \supsetneq H} X^K)$ of 1.3.

If X has the G-homotopy type of a finite G-CW complex, then, for all $s \in S$ and $\alpha \in A_s$, $(X^{H_s}_\alpha, X^{H_s}_\alpha \cap \bigcup_{K \supsetneq H} X^K)$ has the $\bar{G} = G_\alpha \cap N(H)/H$ homotopy type of a relatively finite, relatively free $\bar{G}$-CW pair (Y, B). Hence, for all $s \in S$ and $\alpha \in A_s$, $w^{H_s}_\alpha(X) = 0$ by 1.3 and the necessity part of Theorem B i) follows.

Suppose now that $w^{H_s}_\alpha(X) = 0$ for all $s \in S$ and all $\alpha \in A_s$. Let $H_1, H_2, \ldots, H_m$ be the distinguished subgroups of G ordered in such a way that if H_i is conjugate to a subgroup of H_j, then $j \leq i$. Let $X^{H_i}_{\alpha_1}, \ldots, X^{H_i}_{\alpha_{n_i}}$ be the distinguished components of X^H_α. Order the pairs (p, q); $1 \leq p \leq m$, $1 \leq q \leq n_p$ lexicographically.

We shall construct by induction G-homotopy equivalences $f_{p,q}: Z_{p,q} \longrightarrow X$ where $Z_{p,q}$ is a G-CW complex such that $(Z_{p,q})^H$ is finite for any subgroup H of G conjugate to some H_i with $1 \leq i \leq p-1$ and such that $G(Z_{p,q})^{H_p}_{\beta_j}$ is finite for any component $(Z_{p,q})^{H_p}_{\beta_j}$ of $(Z_{p,q})^{H_p}$ corresponding to $X^{H_p}_{\alpha_j}$ under $f_{p,q}$ for $1 \leq j \leq q$. Then Z_{m,n_m} will be a finite G-CW complex having the G-homotopy type of X and the sufficiency part of Theorem B i) will follow.

We let $Z_{0,0} = X$ and $f_{0,0}$ be the identity. Suppose now that $f_{p,q}: Z_{p,q} \longrightarrow X$ satisfying the above conditions has been constructed. To simplify notation, we will identify X with $Z_{p,q}$ via $f_{p,q}$ and will assume that X^H is finite for any H conjugate to H_i with $1 \leq i \leq p-1$ and that $GX^{H_p}_{\alpha_j}$ is finite for $1 \leq j \leq q$.

Case I: $q < n_p$. In this case we simplify notation further by letting $H = H_p$, $\alpha = \alpha_{p+1}$, and $\bar{G} = G_\alpha \cap N(H)/H$. Since $(X^H_\alpha, X^H_\alpha \cap \bigcup_{K \supsetneq H} X^K)$ is $\bar{G}$-dominated by a finite $\bar{G}$-CW pair, $0 = w^H_\alpha(X) = w_{\bar{G}}(X^H_\alpha, X^H_\alpha \cap \bigcup_{K \supsetneq H} X^K)$ by hypothesis, and

$$X^H_\alpha \cap \bigcup_{K \supsetneq H} X^K$$

is finite by the inductive hypothesis, by 1.3 there exists a $\bar{G}$-homotopy equivalence of pairs $f: (Y, X^H_\alpha \cap \bigcup_{K \supsetneq H} X^K) \longrightarrow (X^H_\alpha, X^H_\alpha \cap \bigcup_{K \supsetneq H} X^K)$ such that Y is a finite $\bar{G}$-CW complex and such that $f \mid X^H_\alpha \cap \bigcup_{K \supsetneq H} X^K$ is the identity.

Let $h: (X^H_\alpha, X^H_\alpha \cap \bigcup_{K \supsetneq H} X^K) \longrightarrow (Y, X^H_\alpha \cap \bigcup_{K \supsetneq H} X^K)$ be a $\bar{G}$-homotopy inverse for f whose restriction to $X^H_\alpha \cap \bigcup_{K \supsetneq H} X^K$ is the identity. Since there is a natural epimorphism $G' = G_\alpha \cap N(H) \longrightarrow G_\alpha \cap N(H)/H = \bar{G}$, we can, and will regard f and h as $G_\alpha \cap N(H)$-maps which are mutual $G_\alpha \cap N(H)$-homotopy inverses. It is then clear that the maps $1 \times f$ and $1 \times h$ induce G-homotopy equivalences f' and h' that make the following diagram commute

$$\begin{array}{ccc} G \times Y & \underset{1 \times h}{\overset{1 \times f}{\rightleftarrows}} & G \times X^H_\alpha \\ \sigma \downarrow & & \downarrow \tau \\ G \times_{G'} Y & \underset{h'}{\overset{f'}{\rightleftarrows}} & G \times_{G'} X^H_\alpha \end{array}$$

where $G \times_{G'} Y$ is obtained from $G \times Y$ by identifying (gg', y) with $(g, g'y)$ for

$g' \in G' = G_\alpha \cap N(H)$, $G \times_{G'} X_\alpha^H$ is defined similarly, and σ and τ are the obvious quotient maps. Note that f' and h' are the identity on the subspace

$$G \times_{G'} (X_\alpha^H \cap \bigcup_{K \supsetneq H} X^K).$$

We now define an equivalence relation $\sim$ on $G \times_{G'} (X_\alpha^H \cap \bigcup_{K \supsetneq H} X^K)$ by setting $[g, x] = [g', x']$ whenever $gx = g'x'$ in X where $[g, x] = \sigma(g, x) = \tau(g, x)$. Extend $\sim$ to equivalence relations on $G \times_{G'} Y$ and $G \times_{G'} X^H$ by identifying no points outside $G \times_{G'} (X_\alpha^H \cap \bigcup_{K \supsetneq H} X^K)$. Let $Z = G \times_{G'} Y/\sim$ and note that $G \times_{G'} X_\alpha^H/\sim\, = GX_\alpha^H = \bigcup_{g \in G} gX_\alpha^H \subset X$. Then Z is a finite G-CW complex and f' and h' induce G-maps $f'': Z \longrightarrow GX_\alpha^H$, $h'': GX_\alpha^H \longrightarrow Z$ which are easily seen to be G-homotopy equivalences. Furthermore, $G(X_\alpha^H \cap \bigcup_{K \supsetneq H} X^K) \subset Z$ and f'' is the identity on this subspace.

It is now a routine matter to extend f'' to a G-homotopy equivalence $f_{p,q}: Z_{p,q} \longrightarrow X$ by using the techniques of [1; Section 4] and [4; Section 4]. Clearly, this map has all the properties needed to complete the proof of case I.

Case II. $q = n_p$. The proof of this case is identical with the proof of case I except that the notation is changed so that $H = H_{p+1}$ and $\alpha = \alpha_1$.

This completes the proof of Theorem B i).

The proof of Theorem B ii) proceeds by an induction argument similar in spirit to the one just given. The key idea is to use 1.4 to complete the inductive step. The tedious, but by now clear, details are left to the reader.

5. The proofs of Theorems 1.1 and 1.2. In this section we give the proofs of Theorems 1.1 and 1.2. The proof of 1.1 requires the following lemma.

LEMMA 5.1. *Let (Y, X) be a relatively free, finite, G-CW pair such that the inclusion $i: X \longrightarrow Y$ is a homotopy equivalence. Let the G-CW complex Z be obtained from Y by a free equivariant elementary expansion. Then $\tau(Z, X)$ is defined and $\tau(Z, X) = \tau(Y, X)$.*

Proof. Clearly (Z, X) is relatively free and the inclusion $j: X \longrightarrow Z$ is a homotopy equivalence. Hence $\tau(Z, X)$ is defined. On the other hand, if we regard $C_*(\tilde{Y}, \tilde{X})$ as a chain complex of $Z\Gamma(Z, G)$ modules via the restriction map

$$r_*'': \Gamma(Z, G) \longrightarrow \Gamma(Y, G),$$

then $0 \longrightarrow C_*(\tilde{Y}, \tilde{X}) \longrightarrow C_*(\tilde{Z}, \tilde{X}) \longrightarrow C_*(\tilde{Z}, \tilde{Y}) \longrightarrow 0$ is a short exact sequence of finitely generated, based, acyclic chain complexes over $Z\Gamma(Z, G)$. Hence, in $\mathrm{Wh}\,\Gamma(Z, G)$ we have $\tau(C_*(\tilde{Z}, \tilde{X})) = \tau(C_*(\tilde{Y}, \tilde{X})) + \tau(C_*(\tilde{Z}, \tilde{Y}))$.

We claim that $\tau(C_*(\tilde{Z}, \tilde{Y})) = 0$. For there exist equivariant s-cells b^s $(s = n, n+1)$ such that $Z = Y \cup b^n \cup b^{n+1}$ and a characteristic G-map $\phi: G \times I^{n+1} \longrightarrow Z$ for b^{n+1} such that $\phi \mid G \times I^n$ is a characteristic G-map for b^n. Let $\tilde{\phi}: I^{n+1} \longrightarrow \tilde{Z}$ be a lift of $\phi \mid 1 \times I^{n+1}$. Then $\tilde{e}^s = \tilde{\phi}(I^s)$ $(s = n, n+1)$ can be taken to the basis of $C_s(\tilde{Z}, \tilde{Y})$. Since $C_s(\tilde{Z}, \tilde{Y}) = 0$ for $s \neq n, n+1$ and $\partial: C_{n+1}(\tilde{Z}, \tilde{Y}) \longrightarrow C_n(\tilde{Z}, \tilde{Y})$ has matrix the identity relative to the bases described, $\tau(C_*(\tilde{Z}, \tilde{Y})) = 0$.

It now follows that $\tau(C_*(\tilde{Z}, \tilde{X})) = \tau(C_*(\tilde{Y}, \tilde{X}))$ in $\mathrm{Wh}\,\Gamma(Z, G)$. Thus $\tau(Z, X) = r_*'\tau(C_*(\tilde{Y}, \tilde{X}))$ in $\mathrm{Wh}\,\Gamma(X, G)$ where $r': \Gamma(Z, G) \longrightarrow \Gamma(X, G)$ is induced by restriction. Since $r_*'\tau(C_*(\tilde{Y}, \tilde{X}))$ clearly equals $\tau(Y, X)$, the lemma follows. □

Proof of Theorem 1.1. Let the G-CW pair (Y, X) satisfy the hypothesis of 1.1 and suppose $\tau(Y, X)=0$. The arguments given in [4; Section 4] show that there exists a free equivariant formal deformation from Y to Z relative to X such that Z is in "simplified form." In particular, $Z=X\cup\bigcup b_i^n\cup\bigcup b_i^{n+1}$ $(n\geq 2,\ 1\leq i\leq t)$ where b_i^s has a characteristic G-map $\phi_i^s: G\times I^s\longrightarrow Z$ $(s=n,\ n+1)$ satisfying $\phi_i^s(1\times \dot{I}^s)=x_i\in X$ if $s=n$ and $\phi_i^s(1\times J^{s-1})=y_i$ if $s=n+1$. Furthermore, $\tau(Z, X)=0$ by 3.1 or 5.1.

It now follows that $\tau(Z, X)$ is represented by the matrix M of

$$\partial: C_{n+1}(\tilde{Z}, \tilde{X}) \longrightarrow C_n(\tilde{Z}, \tilde{X})$$

where we identify $\Gamma(Z, G)$ with $\Gamma(X, G)$ via the restriction. Furthermore M may be reduced to the identity by a finite sequence of matrix operations of the following types:

1) Multiply a row or column by ± 1;
2) Multiply a row or column by $a\in\Gamma(Z, G)$;
3) Replace the matrix N by the matrix $\left(\begin{smallmatrix} N & 0\\ 0 & 1\end{smallmatrix}\right)$ or the inverse operation;
4) Replace a row ρ_i by $\rho_i+\rho_j$ for $i\neq j$.

Matrix operations of types 1) and 2) may be realized by changing the basis of $C_n(\tilde{Z}, \tilde{X})$ or $C_{n+1}(\tilde{Z}, \tilde{X})$ in an appropriate fashion. Thus they may be trivially covered by a free formal deformation. The proof of 5.1 shows that operations of the first type in 3) are covered by a free equivariant elementary expansion.

In order to cover the inverse of the operation that sends M to $\left(\begin{smallmatrix} M & 0\\ 0 & 1\end{smallmatrix}\right)$ by a free formal deformation, we first pick characteristic maps $\tilde{\psi}^s: I^s\longrightarrow \tilde{Z}$ for the cells $\tilde{e}^s$ $(s=n,\ n+1)$ for which $\partial[\tilde{e}^{n+1}]=[\tilde{e}^n]$ where $[\tilde{e}^s]$ denotes the class of $\tilde{e}^s$ in $C_s(\tilde{Z}, \tilde{X})$. We may, in fact, assume that $p\tilde{\psi}^s=\phi_t^s\,|\,1\times I^s=\psi^s$ where $p:\tilde{Z}\longrightarrow Z$ is the projection of the universal cover and ϕ_t^s is the characteristic G-map for b_t^s described above.

But now the following diagram commutes

$$\begin{array}{ccccccc}
\pi_n(A, X) & \underset{\approx}{\overset{p_*}{\longleftarrow}} & \pi_n(\tilde{A}, \tilde{X}) & \underset{\approx}{\overset{h_*}{\longrightarrow}} & H_n(\tilde{A}, \tilde{X}) & = & C_n(\tilde{Z}, \tilde{X})\\
\partial\uparrow & & \partial\uparrow & & \partial\uparrow & & \partial\uparrow\\
\pi_{n+1}(Z, A) & \underset{\approx}{\overset{p_*}{\longleftarrow}} & \pi_{n+1}(\tilde{Z}, \tilde{A}) & \underset{\approx}{\overset{h_*}{\longrightarrow}} & H_{n+1}(\tilde{Z}, \tilde{A}) & = & C_{n+1}(\tilde{Z}, \tilde{X})
\end{array}$$

where $A=X\cup\bigcup b_i^n$ $(1\leq i\leq t)$ and h_* is the Hurewicz homomorphism. Since the indicated maps are isomorphisms, $\partial[\psi^{n+1}]=[\psi^n]$. Now define $\eta^{n+1}: \dot{I}^{n+1}\longrightarrow A$ by $\eta^{n+1}\,|\,I^n=\psi^n$ and $\eta^{n+1}\,|\,J^n=c$ the constant map to x_t. Since $[\psi^{n+1}\,|\,I^n]=\partial[\psi^{n+1}]$, we know that $\psi^{n+1}\,|\,I^n$ is homotopic to ψ^n as maps into A. But then also $\psi^{n+1}\,|\,\dot{I}^{n+1}$ is homotopic to η^{n+1} as maps into A.

If we now define $\bar{\eta}^{n+1}: G\times\dot{I}^{n+1}\longrightarrow A$ by $\bar{\eta}(g, t)=g\eta(t)$, then clearly $\bar{\eta}^{n+1}$ and ϕ_t^{n+1} are G-homotopic. By [4; Lemma 4.1] there is a free formal deformation of Z to W relative to X where $W=A\cup\bigcup b_i^{n+1}\cup c^{n+1}$ $(1\leq i\leq t-1)$ and c^{n+1} is an equivariant $(n+1)$-cell with attaching G-map $\bar{\eta}^{n+1}$. Clearly W is obtained from $V=X\cup\bigcup b_i^n\cup\bigcup b_i^{n+1}$ $(1\leq i\leq t-1)$ by a free equivariant elementary expansion.

The free equivariant formal deformation from Z to W to V is then relative to X and covers the given type 3) operation.

The proof that operations of type 4) can be covered by free equivariant formal deformations proceeds along lines similar to those of the argument above. Namely, one considers the restriction $\psi_i = \phi_i^{n+1} \mid 1 \times \dot{I}^{n+1}$ of a characteristic G-map for b_i^{n+1}, homotopes ψ_i nonequivariantly to replace ρ_i by $\rho_i + \rho_j$ (cf. [2; pp. 63–64]), and then symmetrizes over G to obtain a G-homotopy to which [4; Lemma 4.1] may be applied. The details are left to the reader.

This completes the proof of 1.1. □

Proof of Theorem 1.2. Let $\tau_0 \in \mathrm{Wh}\Gamma(X, G)$ be represented by the matrix

$$(\lambda_{ij}) \in \mathrm{GL}(m, Z\Gamma(X, G)).$$

Let $n \geq 2$ and set $Z = X \cup \bigcup b_i^n$ $(1 \leq i \leq m)$ where b_i^n is a free equivariant n-cell with characteristic G-map $\phi_i^n : G \times I^n \longrightarrow Z$ satisfying $\phi_i^n(1 \times \dot{I}^n) = x_0 \in X$. We note that there is an obvious equivariant retraction $\rho : Z \longrightarrow X$.

Let $p : \tilde{Z} \longrightarrow Z$ be the universal cover of Z. Since $n \geq 2$, $\pi_1(X) \longrightarrow \pi_1(Z)$ is an isomorphism and $p^{-1}(X) = \tilde{X}$ is the universal cover of X. Thus, there is a string of isomorphisms

$$\pi_n(Z, X) \underset{\approx}{\overset{p_*}{\longleftarrow}} \pi_n(\tilde{Z}, \tilde{X}) \underset{\approx}{\overset{h_*}{\longrightarrow}} H_n(\tilde{Z}, \tilde{X}) = C_n(\tilde{Z}, \tilde{X}).$$

On the other hand, ρ_* splits the exact sequence

$$\longrightarrow \pi_n(X) \overset{\rho_*}{\underset{\longrightarrow}{\dashleftarrow}} \pi_n(Z) \longrightarrow \pi_n(Z, X) \longrightarrow$$

and we can (and will) regard $\pi_n(Z, X)$ as a direct summand of $\pi_n(Z)$.

Let $f_i : (\dot{I}^{n+1}, 0) \longrightarrow (Z, x_0)$ be a map representing the class $a_i \in \pi_n(Z)$ corresponding to the image of $\sum_j \lambda_{ij} e_j^n$ and extend f_i to $F_i : G \times \dot{I}^{n+1} \longrightarrow Z$ by setting $F_i(g, t) = g f_i(t)$. If we now let $Y = Z \cup \bigcup b_i^{n+1}$ where F_i is the G-attaching map of the free G-cell b_i^{n+1}, then the pair (Y, X) has all the properties required by 1.2.

6. The proofs of Theorems 1.3 and 1.4. In this section we prove Theorems 1.3 and 1.4. We begin with an elementary, but very useful, observation about $\pi_*(M, Y)$ and $H_*(M, Y)$ where (M, Y) is a G-CW pair.

LEMMA 6.1. *Let (M, Y) be a G-CW pair and $p : \tilde{M} \longrightarrow M$ be the universal cover of M. Then*

(*)
$$\cdots \to H_{q+1}(\tilde{M}, p^{-1}(Y)) \to H_q(p^{-1}(Y)) \to H_q(\tilde{M}) \to H_q(\tilde{M}, p^{-1}(Y)) \to \cdots$$

is an exact sequence of $Z\Gamma(M, G)$ modules. If $\pi_1(Y) \longrightarrow \pi_1(M)$ is an isomorphism, then

(**)
$$\cdots \to \pi_{q+1}(M, Y) \to \pi_q(Y) \to \pi_q(M) \to \pi_q(M, Y) \to \cdots$$

is an exact sequence of $Z\Gamma(M, G)$ modules.

Proof. We prove only the statement about homotopy since the other part is obvious. To do this we first define $Z\Gamma(M, G)$ module structures on the various terms in (**). We note first that $p^{-1}(Y)=\tilde{Y}$ is the universal cover of Y since $\pi_1(Y) \longrightarrow \pi_1(M)$ is an isomorphism and that any homeomorphism $h \in \Gamma(M, G)$ restricts to a homeomorphism $h| : \tilde{Y} \longrightarrow \tilde{Y}$.

We now define an action of $\Gamma(M, G)$ on $\pi_q(\tilde{M}, \tilde{Y}, \tilde{y}_0)$, where $\tilde{y}_0 \in \tilde{Y}$ is a base point, by letting $h \in \Gamma(M, G)$ act via the composite.

$$\pi_q(\tilde{M}, \tilde{Y}, \tilde{y}_0) \xrightarrow{h_*} \pi_q(\tilde{M}, \tilde{Y}, h(\tilde{y}_0)) \xrightarrow{\alpha_{h_\#}} \pi_q(\tilde{M}, \tilde{Y}, \tilde{y}_0)$$

where $\alpha_{h_\#}$ is the change of basepoint homomorphism induced by a path α_h in $\tilde{Y}$ from $\tilde{y}_0$ to $h(\tilde{y}_0)$. Since $\tilde{Y}$ is simply connected, $\alpha_{h_\#}$ is independent of the choice of α_h, and the action of h is well defined. The action of $\Gamma(M, G)$ on $\pi_q(M, Y, y_0)$, where $y_0 = p(\tilde{y}_0)$, is now obtained by requiring the isomorphism

$$p_* : \pi_q(\tilde{M}, \tilde{Y}, \tilde{y}_0) \longrightarrow \pi_q(M, Y, y_0)$$

to be a $Z\Gamma(M, G)$ isomorphism. The action of $\Gamma(M, G)$ on the other terms in (**) is defined similarly.

The claim that (**) is an exact sequence of $Z\Gamma(M, G)$ modules is now obvious. □

If (M, Y) is a G-CW pair and $p : \tilde{M} \longrightarrow M$ is the universal cover of M we shall call the composite $\eta = Hp_*^{-1}$ the Hurewicz homomorphism, where

$$\pi_q(M, Y, y_0) \xleftarrow{p_*} \pi_q(\tilde{M}, p^{-1}(Y), \tilde{y}_0) \xrightarrow{H} H_q(\tilde{M}, p^{-1}(Y))$$

and H is the usual Hurewicz map. If $\pi_1(Y) \longrightarrow \pi_1(M)$ is an isomorphism, η is a map of $Z\Gamma(M, G)$ modules.

Let X and Y be G-CW complexes and $f: Y \longrightarrow X$ be a G-map. Let $M = M(f)$ be the mapping cylinder of f and $p : \tilde{M} \longrightarrow M$ be its universal cover. We define $\pi_q(f)$ and $H_q(f)$ respectively, by $\pi_q(M, Y)$ and $H_q(\tilde{M}, p^{-1}(Y))$, respectively. As usual, we may represent an element $\alpha \in \pi_q(f)$ by a pair of maps (s, t) where $s : I^q \longrightarrow X$, $t : \dot{I}^q \longrightarrow Y$, and $ft = s | \dot{I}^q$.

If G acts effectively on X, then the homomorphism $\rho : \Gamma(M, G) \longrightarrow \Gamma(X, G)$ defined by restricting $h \in \Gamma(M, G)$ to $\tilde{X}$ is an isomorphism. In this case, if $f_* : \pi_1(Y) \longrightarrow \pi_1(X)$ is an isomorphism and we identify $\Gamma(M, G)$ and $\Gamma(X, G)$ via ρ, and $\pi_q(M)$ and $H_q(\tilde{M})$ with $\pi_q(X)$ and $H_q(\tilde{X})$, respectively, in the obvious ways, we obtain the commutative diagram

$$\begin{array}{ccccccccc}
\longrightarrow & \pi_{q+1}(f) & \longrightarrow & \pi_q(Y) & \xrightarrow{f_*} & \pi_q(X) & \longrightarrow & \pi_q(f) & \longrightarrow \\
 & \downarrow \eta & & \downarrow \eta & & \downarrow \eta & & \downarrow \eta & \\
\longrightarrow & H_{q+1}(f) & \longrightarrow & H_q(\tilde{Y}) & \xrightarrow{\tilde{f}_*} & H_q(\tilde{X}) & \longrightarrow & H_q(f) & \longrightarrow
\end{array}$$

of $Z\Gamma(X, G)$ modules, where $\tilde{f}: \tilde{Y} \longrightarrow \tilde{X}$ is a lift of f.

Suppose now that $\alpha_i \in \pi_n(f)$ $(i = 1, \ldots, k)$ is represented by the pair of maps (s_i, t_i). Let $Y_1 = Y \cup b_1^n \cup \cdots \cup b_k^n$ where b_i^n is a free equivariant n-cell with characteristic G-map $\phi_i : G \times I^n \longrightarrow Y_1$ satisfying $\phi | g \times \dot{I}^n = gt_i$ for $g \in G$, $i = 1, \ldots, k$. Extend f to $f_1 : Y_1 \longrightarrow X$ by setting $f_1 | \phi_i(g \times I^n) = gs_i$. We shall say that f_1 is obtained from f by attaching free equivariant n-cells to Z via α_i $(i = 1, \ldots, k)$.

The following, easily verified lemma is the key ingredient in the proofs of 1.3 and 1.4:

LEMMA 6.2. *If* $q<n$, *then* $\pi_q(f)=\pi_q(f_1)$. *If* $n=q\geq 2$ *and* $f_*:\pi_1(Y)\longrightarrow\pi_1(X)$ *is an isomorphism, there is an exact sequence of* $Z\Gamma(X,G)$ *modules.*

$$\ldots\pi_{n+1}(f)\longrightarrow\pi_{n+1}(f_1)\longrightarrow\pi_n(\tilde{Y}_1,\tilde{Y})$$

$$\xrightarrow{\partial}\pi_n(f)\longrightarrow\pi_n(f_1)\longrightarrow 0$$

in which $\pi_n(\tilde{Y}_1,\tilde{Y})$ *is free over* $Z\Gamma(X,G)$ *on generators* β_i $(i=1,\ldots,k)$ *satisfying* $\partial\beta_i=\alpha_i$.

Proof of Theorem 1.3. Let f: $(Z,C)\longrightarrow(X,A)$ be a G-homotopy equivalence of pairs where (Z,C) is a relatively finite, relatively free G-CW pair. Let M be the mapping cylinder of M. Then $Z\subset M\supset X$ and we have isomorphisms

$$\pi_1(Z)\xrightarrow{j_*}\pi_1(M)\xleftarrow{i_*}\pi_1(X)$$

induced by inclusion. Since G acts effectively on X and Y, the homomorphisms $j^*:\Gamma(M,G)\longrightarrow\Gamma(Z,G)$ and $i^*:\Gamma(M,G)\longrightarrow\Gamma(X,G)$ obtained by restricting the action are isomorphisms. We identify $\Gamma(Y,G)$ with $\Gamma(X,G)$ via $i^*(j^*)^{-1}$.

It is now easy to see that if $\tilde{f}$: $\tilde{Z}\longrightarrow\tilde{X}$ covers f, then

$$\tilde{f}_*:C_*(\tilde{Z},p^{-1}(B))\longrightarrow C_*(\tilde{X},q^{-1}(B))$$

is a chain equivalence of chain complexes over $Z\Gamma(X,G)$ where $p:\tilde{Z}\longrightarrow Z$ and $q:\tilde{X}\longrightarrow X$ are the universal covers of Z and X, respectively. Hence,

$$w_G(X,A)=\mathcal{O}(C_*(\tilde{X},q^{-1}(A)))=\mathcal{O}(C_*(\tilde{Z},p^{-1}(B)))=0$$

since the latter chain complex is a finitely generated chain complex of free $Z\Gamma(X,G)$ modules. The necessity part of 1.3 follows.

Suppose now that (X,A) is G-dominated by (Y,B) as in 1.3 and that $w_G(X,A)=0$. Let r: $(Y,B)\longrightarrow(X,A)$ be a domination with section i: $(X,A)\longrightarrow(Y,B)$. By combining 6.2 with the arguments of [1; Section 4] and [4; Section 4], we may assume that $r|B:B\longrightarrow A$ is a G-homotopy equivalence and that r: $Y\longrightarrow X$ is n-connected where $n=\max(\dim(Y-B),3)$. It then follows from [9; Lemma 2.3] that $H_{n+1}(r)=\pi_{n+1}(r)$ is a projective $Z\Gamma(X,G)$ module that represents $w_G(X,A)$. Since $w_G(X,A)=0$, it follows that there exist finitely generated, free $Z\Gamma(X,G)$ modules. F_1 and F_2 such that $\pi_{n+1}(r)\oplus F_1=F_2$.

Suppose F_1 has rank k. Let $\alpha_i\in\pi_n(r)$ $(i=1,\ldots,k)$ be the trivial class and let r_1: $(Y_1,B)\longrightarrow(X,A)$ be obtained from r by attaching free equivariant n-cells to (Y,B) via α_i $(i=1,\ldots,k)$. It follows from 6.2 that $\pi_{n+1}(r_1)=\pi_{n+1}(r)\oplus F_1$. Hence $\pi_{n+1}(r_1)=F_2$ is a free $Z\Gamma(X,G)$ module. We may now attach free equivariant $(n+1)$-cells to Y_1 via a free basis for $\pi_{n+1}(r_1)$ to obtain a G-map r_2: $(Y_2,B)\longrightarrow(X,A)$ such that $r_2:Y_2\longrightarrow X$ is a homotopy equivalence and $r_2|B:B\longrightarrow A$ is a G-homotopy equivalence. Since (Y_2,B) and (X,A) are relatively free, this implies that r_2: $(Y_2,B)\longrightarrow(X,A)$ is a G-homotopy equivalence of pairs by [5]. Note that (Y_2,B) is relatively finite.

Now let $h: A \longrightarrow B$ be a G-homotopy inverse for $r_2 | A$. Using the techniques of [1; Section 4] and [4; Section 4], we may extend h to a G-homotopy equivalence of pairs $h: (Z, A) \longrightarrow (Y_2, B)$ with (Z, A) relatively free and relatively finite. Then $r_2 h$ is G-homotopic to a G-homotopy equivalence $f: (Z, A) \longrightarrow (X, A)$ such that $f|A$ is the identity. This completes the proof of 1.3. □

Proof of 1.4. Let P_0 and P_1 be finitely generated, projective $Z\Gamma(Z, G)$ modules such that $(-1)^n [P_0] = w_0 \in \tilde{K}_0 Z\Gamma(Z, G)$, where $n = \max(\dim(Z-A), 3)$, and such that $P_0 \oplus P_1 = F$ is finitely generated and free. Let $\rho_i: F \longrightarrow P_i$ $(i=0,1)$ be the projection. Let $C_* = \{C_q, \partial_q\}$ be the chain complex with $C_q = 0$ for $q<n$, $C_q = F$ for $q \geq n$; $\partial_{n+2q-1} = \rho_1$, and $\partial_{n+2q} = \rho_0$ for $q \geq 1$. We shall construct relatively free G-CW pairs (X, Z) and (Y, Z) such the $C_* = C_*(\tilde{X}, \tilde{Z})$ and $C_q(\tilde{Y}, \tilde{Z}) = 0$ for $q \neq n$.

We let $Y = Z \vee b_1^n \vee \cdots \vee b_k^n$ where $k = \operatorname{rank} F$ and b_i^n is a free equivariant n-cell with characteristic map $\phi_i: G \times I^n \longrightarrow Y$ satisfying $\phi_i(g \times \dot{I}^n) = g y_0$ where $y_0 \in Y$ is a basepoint. Then $C_q(\tilde{Y}, \tilde{Z}) = 0$ for $q \neq n$ and $C_n(\tilde{Y}, \tilde{Z}) = F$ as a $Z\Gamma(Z, G)$ module.

We now construct by induction a space X_q containing Z, and a map $i_q: X_q \longrightarrow Y$ such that $C_*(\tilde{X}_q, \tilde{Z})$ realizes C_* up to dimension $n+q-1$ and Q is a direct summand of $\pi_{n+q}(i_q)$ where $Q = P_1$ if q is even and $Q = P_0$ if q is odd. To construct X_1, consider the inclusion map $i_0: Z \longrightarrow Y$. There is an exact sequence

$$\longrightarrow \pi_n(Y) \longrightarrow \pi_n(i_0) \xrightarrow{\partial} \pi_{n-1}(Z) \longrightarrow \pi_{n-1}(Y) \longrightarrow$$

with $\pi_n(i_0) = F$ as $Z\Gamma(Z, G)$ modules and $\partial = 0$. Let ρ_1 be the composite

$$F \xrightarrow{\rho_1} P_1 \subset F$$

and let $\alpha_j = \rho_1(\zeta_j) \in F = \pi_n(i_0)$ where ζ_j $(j=1, \ldots, k)$ is a free basis for F. Let $i_1: X_1 \longrightarrow Y$ be obtained from i_0 by attaching free equivariant n-cells to Z via α_j $(j=1, \ldots, k)$. It follows immediately from 6.2, that P_0 is a direct summand of $\pi_{n+1}(i_1)$.

We remark that since the boundary operator in the above exact sequence is trivial, the attaching maps for the n-cells of X_1 are equivariantly trivial. Hence, they are equivariantly homotopic to the attaching maps for Y and there exists a G-homotopy equivalence $r_1: Y \longrightarrow X_1$.

Once $i_q: X_q \longrightarrow Y$ has been constructed with the desired properties, i_{q+1} is obtained by attaching free equivariant $(n+q)$-cells to X_q via the classes $\rho_s(\zeta_j)$ $(j=1, \ldots, k)$ where $\rho^s: F \longrightarrow Q = P_s$ is the projection onto $Q = P_s$ $(s=0$ or $1)$ and again invoking 6.2.

Let $X = \bigcup_q X_q$ and Y be as above. Then the reader may verify that the composite r given by

$$Y \xrightarrow{r_1} X_1 \subset X$$

is a domination with section i satisfying $i | X_q = i_q$ and that (X, A) and (Y, A) have the properties claimed in 1.4. This completes the proof of 1.4. □

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