DE GIORGI PERIMETER, LEBESGUE AREA, HAUSDORFF MEASURE

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To Lamberto Cesari, on the occasion of his 70th birthday.

INTRODUCTION

Let B be an open region in $\mathbf{R}^n$ whose boundary C is a connected, orientable $(n-1)$ dimensional manifold and whose closure is A. For a subset M of $\mathbf{R}^N$ we use $\mu_n(M)$, $H_n^{n-1}(M)$ and $P(M)$ respectively for the Lebesgue measure, the Hausdorff $(n-1)$ dimensional measure and the de Giorgi perimeter of M. We are interested in comparing three "measures" of the size of C. These are

a) the perimeter of A (or of B),

b) the Hausdorff $(n-1)$ dimensional measure of C (or some substitute for C suitable for our purpose),

and

c) the Lebesgue surface area of a mapping whose image is C.

The conjecture is that under rather general conditions the three measures are either all finite or all infinite. The present article is a step toward resolving this problem.

We first observe that the perimeters of A and B need not be equal. Either one can be infinite while the other is finite. We show here that this can occur, for $n = 3$, only when the three dimensional Lebesgue measure of C is positive, and that if $\mu_3(C) > 0$ then at least one of the perimeters $P(A)$, $P(B)$ is infinite. It follows that if $\mu_3(C) = 0$ then $P(A) = P(B)$, both finite or both infinite.

Regarding the Hausdorff $(n-1)$ dimensional measure of C, it is well known this value is generally large compared with other "measures." Suitable substitutes for C do exist in the literature. In [11], Federer considered the reduced boundary, and in [17] Vol'pert considered the essential boundary. It will be shown that the essential boundary of Vol'pert has a topological formulation in the density topology [14], [15].

For Lebesgue surface area we show if the inclusion mapping $i: C \to \mathbf{R}^n$ is collared [2], and C is finitely triangulable then if either A or B has finite perimeter the mapping i has finite integral geometric stable area [8], [9], [10]. For $n = 3$, with the collared hypothesis and the assumption $\mu_3(C) = 0$, we then have the equivalence $P(A)$ is finite if and only if the Lebesgue area of i is finite.

We dedicate this paper to Lamberto Cesari in deep appreciation of the profound

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influence that his vast mathematical output and many generous personal suggestions have had upon the mathematical careers of both of us.

1. THE EQUALITY OF $P(A)$ AND $P(B)$

By modifying the example of Besicovitch [1] one can readily construct examples of open regions B having boundary C with $\mu_3(C) > 0$. Thus, we have, by Theorem 1.1 below, that $P(A) \neq P(B)$. By a simple inversion of $\mathbf{R}^3$ we can change $P(A) > P(B)$ to $P(A) < P(B)$.

THEOREM 1.1. *Let A, B and C be as in the introduction and let* $n = 3$.

a) *If* $\mu_3(C) > 0$ *then either* (i) $P(A) = P(B) = \infty$, (ii) $P(A) = \infty$ *and* $P(B) < \infty$ *or* (iii) $P(A) < \infty$ *and* $P(B) = \infty$.

b) *If* $\mu_3(C) = 0$ *then* $P(A) = P(B)$ *both being either finite or infinite.*

Proof. Suppose $P(A) < \infty$ and $P(B) < \infty$. We show $\mu_3(C) = 0$ and hence $P(A) = P(B)$. First we establish some notation. For each $i = 1, 2, 3$, write $x = (x_1, x_2, x_3) = (x_i, \bar{x}_i)$, where $\bar{x}_i$ is the pair of coordinates orthogonal to x_i. For any subset S of $\mathbf{R}^3$ and any $\bar{x}_i$, let $S(\bar{x}_i) = \{x : x \in S, x = (x_i, \bar{x}_i)\}$. We shall also denote by $S(\bar{x}_i)$ the real-valued function of the real variable x_i given by the characteristic function of the set $S(\bar{x}_i)$. The essential total variation of $S(\bar{x}_i)$ will be denoted by $v(S(\bar{x}_i))$. (The essential total variation of $S(\bar{x}_i)$ is calculated by using partition points which are points of approximate continuity of $S(\bar{x}_i)$. See [13].)

Since $P(A) < \infty$ and $P(B) < \infty$ we have for μ_2-almost every $\bar{x}_i$ that $v(A(\bar{x}_i)) < \infty$ and $v(B(\bar{x}_i)) < \infty$. Since $A \setminus B = C$, we have for μ_2-almost every $\bar{x}_i$ that $v(C(\bar{x}_i)) < \infty$. The compactness of $C(\bar{x}_i)$ together with $\mu_1(C(\bar{x}_i)) > 0$ and $v(C(\bar{x}_i)) < \infty$ imply that

$(*_i)$ $C(\bar{x}_i)$ can be decomposed in a union of a nonempty, finite, disjointed collection of nondegenerate intervals and a set of μ_1 measure zero.

We derive a contradiction from $\mu_3(C) \neq 0$. Suppose $\mu_3(C) > 0$. Then $(*_i)$ holds for μ_3-almost every $x \in C$ $(i = 1,2,3)$. Let $i = 1$. From $(*_i)$, there are numbers $a < b$ and a set $\bar{X}_1$ of positive μ_2 measure such that $F = \{(x_1, \bar{x}_1) : a < x_1 < b, \bar{x}_1 \in \bar{X}_1\}$ is a subset of C. We have μ_3-almost every $x = (x_1, x_2, x_3) \in F$ is a point of linear density of C in the x_2 direction. From $(*_2)$ we have μ_3-almost every $x^0 = (x_1^0, x_2^0, x_3^0)$ in F is contained in an arc $I(x^0)$ contained in $C(\bar{x}_2^0)$. For $x^0 \in F$, let $J(x^0)$ be the arc $\{(x_1, x_2^0, x_3^0) : a \leq x_1 \leq b\}$. Then $T(x^0) = I(x^0) \cup J(x^0)$ contains a simple triod when $I(x^0)$ exists. Clearly, $\{x_3 : \exists x^0 \in F \ni T(x^0) \text{ exists and } x_3^0 = x_3\}$ has positive μ_1 measure. Consequently, one can find an uncountable disjointed collection of simple triods contained in the compact two dimensional manifold C. This contradicts Moore's Triod Theorem [16]. Hence $\mu_3(C) = 0$ and the theorem is proved.

2. PERIMETER AND MEASURE OF BOUNDARIES

Let M be a measurable set. It is known that the Hausdorff $(n-1)$ dimensional measure of the usual boundary of M does not determine the finiteness of the perimeter of M. We show that a more suitable topology on $\mathbf{R}^n$ is the density topology,

[14] and [15]. The density topology on $\mathbf{R}^n$ is generated by the approximately continuous real-valued functions on $\mathbf{R}^n$. A set M is d-open if it is measurable and each point of M is a point of density one of M. As in general topology, a regular open set is one which is equal to the interior of its closure.

The perimeter of a measurable set is invariant under Lebesgue measure equivalence. Also, every measurable set is equivalent to its d-closure and to its d-interior. Consequently, for the purposes of perimeter, it is sufficient to consider only regular open sets in the density topology.

PROPOSITION 2.1. *If M is a regular open set in the density topology, then the d-boundary $\partial_d M$ of M is the essential boundary of Vol'pert.*

Proof. According to Vol'pert [17], the essential boundary of a measurable set M is the set of points $x \in \mathbf{R}^n$ such that x is neither a point of density of M nor a point of rarefaction of M. Let M be a regular open set in the density topology. It is clear that the set of points of density of M is M itself and the set of points of rarefaction of M is the complement of the d-closure of M.

PROPOSITION 2.2. *If M is a regular open set in the density topology and $P(M) < \infty$ then $H_n^{n-1}(\partial_d M) < \infty$.*

Proof. This is immediate from [17], Theorem, page 228.

THEOREM 2.1. *If M is a regular open set in the density topology then $P(M) < \infty$ if and only if $H_n^{n-1}(\partial_d M) < \infty$.*

Proof. Due to Proposition 2.2 above we need only show that $H_n^{n-1}(\partial_d M) < \infty$ implies $P(M) < \infty$. This implication follows from Federer [12], Theorem 4.5.11, page 506, since the Hausdorff $(n-1)$ dimensional measure is no smaller than the integral geometric $(n-1)$ dimensional measure.

3. PERIMETER AND AREAS

When the $(n-1)$ dimensional manifold C is finitely triangulable the inclusion map $i: C \to \mathbf{R}^n$ has an $(n-1)$ dimensional Lebesgue area $L_{n-1}(i)$ associated with it. Federer has established the following.

THEOREM 3.1. [11]. *Let A, B and C be as in the introduction, C be finitely triangulable and $i: C \to \mathbf{R}^n$ be the inclusion map. If $\mu_n(C) = 0$ and $L_{n-1}(i) < \infty$ then $P(A) < \infty$.*

We investigate a converse to the above theorem. First we give a definition. The manifold C is said to be collared [2] if there is an embedding $f: C \times [0,1] \to \mathbf{R}^n$ with $f(x,0) = i(x)$ for $x \in C$. For convenience we shall assume the collaring is such that $f[C \times [0,1]] \subset A$. Hence $A \setminus f[C \times [0,1]]$ has positive distance from C when C is compact. We refer the reader to [8], [9], [10] for the definition of the integral geometric stable area of the mapping i.

THEOREM 3.2. *Let A, B and C be as in the introduction, C be finitely triangulable and collared. If $P(A)$ or $P(B)$ is finite then the integral geometric stable area of the mapping i is finite.*

To prove the theorem, we need several lemmas.

Let $X = X_1 \cup X_2 \subset \mathbf{R}^{n-1} \times [-1,1]$, where

$$X_1 = S^{n-2} \times [-1,1] \cup B^{n-1} \times \{-1\},$$
$$S^{n-2} = \{z \in \mathbf{R}^{n-1} : |z| = 1\},$$
$$B^{n-1} = \{z \in \mathbf{R}^{n-1} : |z| \leq 1\},$$

(X_2,X_3) is an oriented relative $(n-1)$ manifold [7] with $X_2 \setminus X_3$ connected,

$$X_3 = X_1 \cap X_2,$$
$$X_2 \subset B^{n-1} \times [1/2,1].$$

Denote by π the natural projection of $\mathbf{R}^n = \mathbf{R}^{n-1} \times \mathbf{R}^1$ onto $\mathbf{R}^{n-1}$ given by $\pi(z,s) = z$. We will use Čech cohomology with integer coefficients.

LEMMA 3.1. *Let X, X_1, X_2, X_3 be as above. Then $\pi | X_2 : (X_2,X_3) \to (B^{n-1},S^{n-2})$. If the origin of $\mathbf{R}^n = \mathbf{R}^{n-1} \times \mathbf{R}^1$ is in the unbounded component of $\mathbf{R}^n \setminus X$ then the homomorphism*

$$(\pi | X_2)^* : H^{n-1}(B^{n-1},S^{n-2}) \to H^{n-1}(X_2,X_3)$$

is trivial.

Proof. Let $h : X \times [0,1] \to S^{n-1}$ be the continuous map given by

$$h((z,s),t) = \begin{cases} \dfrac{(z,s)}{[|z|^2 + |s|^2]^{1/2}} & ,-1 \leq s \leq 0 \\ \dfrac{(z,2ts + (1-t)s)}{[|z|^2 + |2ts + (1-t)s|^2]^{1/2}} & ,0 \leq s \leq 1/2 \\ \dfrac{(z,t + (1-t)s)}{[|z|^2 + |t + (1-t)s|^2]^{1/2}} & ,1/2 \leq s \leq 1 \end{cases}$$

Define the maps α and β by

$$\alpha(z,s) = h((z,s),1), \qquad \beta(z,s) = h((z,s),0).$$

Next, let $\gamma : B^{n-1} \to S^{n-1}$ be a homeomorphism into S^{n-1} such that

$$(\gamma^{-1} \circ \alpha)(z,s) = z, \qquad (z,s) \in X_2.$$

Hence $\pi | X_2 = \gamma^{-1} \circ \alpha$. In order to calculate the homomorphism $(\pi | X_2)^*$ we define three more sets E^+, E^- and S.

$$E^+ = \left\{ \frac{(z,1)}{[|z|^2 + 1]^{1/2}} : z \in B^{n-1} \right\},$$
$$E^- = \text{closure of } S^{n-1} \setminus E^+,$$
$$S = E^+ \cap E^-.$$

The compact pairs (S^{n-1},E^-) and (E^+,S) are relative $(n-1)$ cells and the inclusion map $i_1:(E^+,S)\to(S^{n-1},E^-)$ induces an isomorphism of $H^{n-1}(S^{n-1},E^-)$ onto $H^{n-1}(E^+,S)$. Moreover the inclusion map $i_2:(X_2,X_3)\to(X,X_1)$ induces an isomorphism of $H^{n-1}(X,X_1)$ onto $H^{n-1}(X_2,X_3)$.

Since X_1 and E^- are contractible and $\alpha:(X,X_1)\to(S^{n-1},E^-)$, we have the commuting diagram where the rows are exact.

$$\begin{array}{ccccccc}
0=H^{n-2}(X_1) & \to & H^{n-1}(X,X_1) & \overset{j_2^*}{\to} & H^{n-1}(X) & \to & H^{n-1}(X_1)=0 \\
 & & \uparrow\alpha^* & & \uparrow\alpha^* & & \\
0=H^{n-2}(E^-) & \to & H^{n-1}(S^{n-1},E^-) & \overset{j_1^*}{\to} & H^{n-1}(S^{n-1}) & \to & H^{n-1}(E^-)=0
\end{array}$$

From the commuting diagram of continuous maps

$$\begin{array}{ccccc}
 & \overset{\pi|X_2}{\swarrow} & (X_2,X_3) & \overset{i_2}{\to} & (X,X_1) \\
 & & \downarrow\alpha_2=\alpha|X_2 & & \downarrow\alpha \\
(B^{n-1},S^{n-2}) & \overset{\gamma}{\to} & (E^+,S) & \overset{i_1}{\to} & (S^{n-1},E^-)
\end{array}$$

we have the following diagram of homomorphisms commutes, where the horizontal homomorphisms are isomorphisms.

$$\begin{array}{ccccccc}
\mathbf{Z} & \approx & H^{n-1}(X_2,X_3) & \overset{i_2^*}{\leftarrow} & H^{n-1}(X,X_1) & \overset{j_2^*}{\to} & H^{n-1}(X)\approx\mathbf{Z} \\
 & \overset{(\pi|X_2)^*}{\nearrow} & \uparrow\alpha_2^* & & \uparrow\alpha^* & & \uparrow\alpha^* \\
\mathbf{Z}\approx H^{n-1}(B^{n-1},S^{n-2}) & \overset{\gamma^*}{\leftarrow} & H^{n-1}(E^+,S) & \overset{i_1^*}{\leftarrow} & H^{n-1}(S^{n-1},E^-) & \overset{j_1^*}{\to} & H^{n-1}(S^{n-1})\approx\mathbf{Z}
\end{array}$$

Since h defines the homotopy $h:(X,X_1)\times[0,1]\to(S^{n-1},E^-)$ between α and β, we have $\beta^*=\alpha^*$ and hence

$$\begin{array}{ccc}
H^{n-1}(X_2,X_3) & \overset{\approx}{\leftarrow} & H^{n-1}(X) \\
\uparrow(\pi|X_2)^* & & \uparrow\beta^* \\
H^{n-1}(B^{n-1},S^{n-2}) & \overset{\approx}{\leftarrow} & H^{n-1}(S^{n-1})
\end{array}$$

commutes. Consequently, if the origin of $\mathbf{R}^n=\mathbf{R}^{n-1}\times\mathbf{R}^1$ is in the unbounded component of $\mathbf{R}^n\setminus X$ then $(\pi|X_2)^*$ is trivial since β^* would be trivial by Borsuk's Theorem, [7] page 302. Lemma 3.1 is now proved.

Let g be a component of $(\pi\circ i)^{-1}(z)$ and $\epsilon>0$. We define the sets

$$\begin{aligned}
B(z,\epsilon) &= \{\zeta\in\mathbf{R}^{n-1}:|\zeta-z|\le\epsilon\}, \\
U(z,\epsilon) &= \{\zeta\in\mathbf{R}^{n-1}:|\zeta-z|<\epsilon\}, \\
S(z,\epsilon) &= \{\zeta\in\mathbf{R}^{n-1}:|\zeta-z|=\epsilon\}, \\
V(g,\epsilon) &= \text{closure of the component of } (\pi\circ i)^{-1}[U(z,\epsilon)] \text{ containing } g, \\
W(g,\epsilon) &= V(g,\epsilon)\setminus(\pi\circ i)^{-1}[U(z,\epsilon)].
\end{aligned}$$

Then $\pi\circ i:(V(g,\epsilon),W(g,\epsilon))\to(B(z,\epsilon),S(z,\epsilon))$. We say g is an inessential component of $(\pi\circ i)^{-1}(z)$ if there is $\epsilon_0>0$ such that for each $0<\epsilon<\epsilon_0$ the homomorphism

$$(\pi\circ i)^*:H^{n-1}(B(z,\epsilon),S(z,\epsilon))\to H^{n-1}(V(g,\epsilon),W(g,\epsilon))$$

is trivial. The stable multiplicity of $\pi\circ i$ at z is $S(\pi\circ i,z)=$ the number of essential components of $(\pi\circ i)^{-1}(z)$. It should be noted that if there is, for each $0<\epsilon<\epsilon_0$, a continuous map $F:(V(g,\epsilon),W(g,\epsilon))\to(B(z,\epsilon),S(z,\epsilon))$ such that

$$F|W(g,\epsilon)=(\pi\circ i)|W(g,\epsilon)$$

and $F^{-1}(z)=\emptyset$ then g is inessential.

LEMMA 3.2. *Let $f:C\times[0,1]\to A$ be a collaring of C, $z\in\mathbf{R}^{n-1}$ and g be a component of $(\pi\circ i)^{-1}(z)$. Suppose s_1 and s_2 are such that the line segment $K=\{(z,s):s_1\le s\le s_2\}$ contains $i(g)$ in its interior and $K\subset f[C\times[0,1]]$. Then g is an essential component of $(\pi\circ i)^{-1}(z)$.*

Proof. Since $i(g)$ is a compact set contained in K, $(z,s_1)\notin i(g)$, $(z,s_2)\notin i(g)$ and $K\subset f[C\times[0,1]]$ there are s_1' and s_2' such that $s_1<s_1'<s_2'<s_2$,

$$(z,s_1')\in f[C\times(0,1)],$$

$(z,s_2')\in f[C\times(0,1)]$ and $i(g)\subset\{(z,s):s_1'<s<s_2'\}=f[L]$.There is $\epsilon_0>0$ so that for $0<\epsilon<\epsilon_0$ we have

$$V(g,\epsilon)\subset B(z,\epsilon)\times(s_1',s_2').$$

For each such ϵ there is $\delta>0$ such that the δ-neighborhoods in $\mathbf{R}^n$ of (z,s_1') and (z,s_2') are contained in $f[C\times(0,1)]$ and $\delta<\epsilon$. By the continuity of f there is a t such that $0<t<1$ and $f[\bar{L}+t]$ is a simple arc joining the δ-neighborhoods of (z,s_1') and (z,s_2') and contained in $U(z,\epsilon)\times\mathbf{R}^1$. Clearly, the arc $f[\bar{L}+t]$ is disjoint from $V(g,\epsilon)$. Hence a straight forward application of Lemma 3.1 yields for each $0<\epsilon<\epsilon_0$ the homomorphism

$$(\pi\circ i)^*:H^{n-1}(B(z,\epsilon),S(z,\epsilon))\to H^{n-1}(V(g,\epsilon),W(g,\epsilon))$$

is trivial. Thereby g is inessential and the lemma is proved.

LEMMA 3.3. *Let $f:C\times[0,1]\to A$ be a collaring of C, $z\in\mathbf{R}^{n-1}$, and g be a component of $(\pi\circ i)^{-1}(z)$. Suppose s_1 and s_2 are such that the line segment $K=\{(z,s):s_1\le s\le s_2\}$ contains $i(g)$ in its interior and $K\cap f[C\times(0,1]]=\emptyset$. Then there exists $\epsilon_0>0$ such that for each $0<\epsilon<\epsilon_0$ there is a continuous map $F:(V(g,\epsilon),W(g,\epsilon))\to(B(z,\epsilon),S(z,\epsilon))$ such that*

$$F|W(g,\epsilon)=(\pi\circ i)|W(g,\epsilon)\qquad\text{and}\qquad F^{-1}(z)=\emptyset.$$

Consequently, g is an inessential component of $(\pi\circ i)^{-1}(z)$.

Proof. Since $i(g)$ is contained in the interior of K and $K\cap f[C\times(0,1]]=\emptyset$, there are s_1', s_2' such that

$$s_1 < s'_1 < s'_2 < s_2, \quad (z,s'_1) \notin f[C \times [0,1]], \quad (z,s'_2) \notin f[C \times [0,1]]$$

and $i(g) \subset \{(z,s) : s'_1 < s < s'_2\}$. Let ϵ_0 be such that the $2\epsilon_0$-neighborhoods in $\mathbf{R}^n$ of (z,s'_1) and (z,s'_2) are disjoint from $f[C \times [0,1]]$. Clearly, for $0 < \epsilon < \epsilon_0$, $i(V(g,\epsilon)) \subset B(z,\epsilon) \times (s'_1,s'_2)$. Let $N = V(g,\epsilon) \cap (\pi \circ i)^{-1}[U(z,\epsilon/2)]$ and let $\delta : V(g,\epsilon) \to [0,1]$ be given by

$$\delta(x) = \frac{d[x, V(g,\epsilon) \setminus N]}{d[x, V(g,\epsilon) \setminus N] + d[x, V(g,\epsilon) \cap (\pi \circ i)^{-1}(z)]},$$

where $d[x,S]$ is the distance from x to S. Then for $1 > \eta > 0$, let F_η be the continuous map on $V(g,\epsilon)$ given by $F_\eta(x) = f(x, \eta\delta(x))$. Now, $F_\eta | W(g,\epsilon) = i | W(g,\epsilon)$. Choose η small enough so that $F_\eta(x) \in B(z,\epsilon) \times (s'_1,s'_2)$. Then $F_\eta^{-1}(\pi^{-1}(z)) = \emptyset$ since $K \cap f[C \times (0,1]] = \emptyset$. Let $F = \pi \circ F_\eta$ and the lemma is proved.

Let A, B and C be as in the introduction and $z \in \mathbf{R}^{n-1}$. We denote by A_z, B_z and C_z the sets $A \cap \pi^{-1}(z)$, $B \cap \pi^{-1}(z)$ and $C \cap \pi^{-1}(z)$. We use A_z, B_z and C_z to denote the characteristics functions of the respective one dimensional sets.

LEMMA 3.4. *Let $f : C \times [0,1] \to A$ be a collaring of C and $z \in \mathbf{R}^{n-1}$. Then $S(\pi \circ i, z) \leq v(A_z)$ and $S(\pi \circ i, z) \leq v(B_z)$.*

Proof. Suppose $v(A_z) < \infty$. Then there is a finite family $\{I_k\}$ of mutually disjoint closed intervals such that A_z is Lebesgue equivalent to the characteristic function of $Z = \cup \{I_k\}$ and hence $v(A_z) = v(Z)$. Since A_z is closed, $Z \subset A_z$ and, for each I_k, the end points of I_k are members of C_z, $I_k \setminus B_z = I_k \cap C_z$ and each component of $I_k \cap C_k$ is a component of $\pi^{-1}(z)$. Next, let J be a component of $\mathbf{R} \setminus Z$. Then $\mu_1[J \cap A_z] = 0$, $J \cap A_z = J \cap C_z$ and each component of $J \cap C_z$ is a component of $\pi^{-1}(z)$. By Lemmas 3.2 and 3.3, we have that the essential components of $(\pi \circ i)^{-1}(z)$ must contain some end point of the intervals in $\{I_k\}$. Consequently, $S(\pi \circ i, z) \leq v(Z) = v(A_z)$.

Suppose $v(B_z) < \infty$. We have $v(\mathbf{R} \setminus B_z) = v(B_z)$ and $\mathbf{R} \setminus B_z$ is closed. The proof for B_z reduces to one analogous to the above case for A_z.

Proof of Theorem 3.2. From Lemma 3.4 we have for each $P \in O(n)$

$$\int_{\mathbf{R}^{n-1}} S(\pi \circ P \circ i, z)\, d\mu_{n-1}(z) \leq \int_{\mathbf{R}^{n-1}} v((P[A])_z)\, d\mu_{n-1}(z)$$

and

$$\int_{\mathbf{R}^{n-1}} S(\pi \circ P \circ i, z)\, d\mu_{n-1}(z) \leq \int_{\mathbf{R}^{n-1}} v((P[B])_z)\, d\mu_{n-1}(z).$$

We infer from the proof of [12] Theorem 4.5.11, page 506, that

$$\int_{O(n)} \int_{\mathbf{R}^{n-1}} S(\pi \circ P \circ i, z)\, d\mu_{n-1}(z)\, d\theta(P) < \infty$$

and hence the integral geometric stable area of i finite.

THEOREM 3.3. *Let $n = 3$ and A, B and C be as in the introduction. Suppose C is collared and finitely triangulable. Then $P(A)$ or $P(B)$ is finite only if $L_2(i)$ is finite.*

Proof. In [3], [4], Cesari proved that the essential multiplicity and the stable multiplicity of $\pi \circ i$ coincide except for a countable set of $z \in \mathbf{R}^2$. (Of course, Cesari proved this fact for all continuous mappings into $\mathbf{R}^2$.) Hence, as shown in [10], the integral geometric stable area and Lebesgue area coincide.

COROLLARY. *Assume the hypothesis of Theorem 3.3 above. If $\mu_3[C] = 0$ then $P(A) < \infty$ if and only if $L_2(i) < \infty$.*

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