# A TORUS THEOREM FOR REGULAR BRANCHED COVERS OF $S^3$

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A closed, orientable, smooth 3-manifold $M$ is said to be a *regular branched cover of* $S^3$ if there is a finite group $G$ of orientation-preserving diffeomorphisms of $M$ such that the orbit space $M/G$, regarded as a smooth manifold in the natural way, is diffeomorphic to $S^3$. The class of 3-manifolds which are regular branched covers of $S^3$ includes, for instance, the closed, orientable 3-manifolds of Heegaard genus $\leq 2$. (This follows from the Birman-Hilden theorem [2]. In this case the group $G$ may be taken to have order 2.)

The proof of the Smith conjecture [1] makes certain hard questions about general 3-manifolds accessible for the class of regular branched covers of $S^3$. To take a striking example, if the group $G$ in the above definition is cyclic, then it follows from the generalized Smith conjecture (see the Introduction to [1]) that $M$ cannot be simply connected unless it is diffeomorphic to $S^3$. The main result of this paper is that a strong analogue of the "torus theorem" is true for *all* regular branched covers of $S^3$.

Let us review the torus theorem in a language convenient for our purposes. For the moment let $M$ be a *prime* 3-manifold. (For the definition of this and other standard terms in 3-dimensional topology, we refer the reader to [6].) By an *essential singular torus* in $M$ we mean a map $f: T^2 \to M$ that induces a monomorphism of fundamental groups. We shall say that *the torus conjecture holds in* $M$ if for every essential singular torus $f$, $f: T^2 \to M$ is homotopic to a map $g: T^2 \to M$ such that $g(T^2) \subset \Sigma$, where $\Sigma \subset M$ is a compact Seifert fibered space whose boundary components are all incompressible in $M$. The strongest standard form of the torus theorem, which was proved by Johannson [7, p. 9]; and by Jaco and Shalen [8, p. 55] and which refines results proved by Waldhausen [17] for the bounded case and Feustel [4] in the closed case, asserts that the torus conjecture holds in every *Haken manifold,* i.e. in every compact, orientable, irreducible 3-manifold which contains an incompressible surface. In particular it holds in every bounded, orientable, irreducible 3-manifold.

Now let $M$ be any closed orientable 3-manifold. We shall say that *the torus conjecture holds in* $M$ if it holds in every prime factor of $M$. We may now state our

MAIN THEOREM. *The torus conjecture holds in every regular branched cover of* $S^3$.

In Section 1 we interpret results recently proved by Scott, Meeks, Yau and Simon [12], [13], [11], [10] in forms that are useful for our purposes. Scott's results turn out to imply that the torus conjecture holds in manifolds covered by Haken manifolds; we state this in a stronger form, as Theorem 1.2. The Meeks-Yau

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equivariant sphere theorem is used (Proposition 1.3) to reduce the study of all regular branched covers of $S^3$ to the study of those having trivial second homotopy group.

In Section 2 we apply the ideas in the proof of the Smith conjecture to obtain an interesting property of the fundamental group of a regular branched cover of $S^3$ (Theorem 2.1), to the effect that it has a good many subgroups of finite index. In Section 3, we show (Theorem 3.1) that in an irreducible 3-manifold whose fundamental group has this property, the torus conjecture holds. The proof of 3.1 involves a tower argument, and depends on an unpublished idea of Jaco's. The results described above are assembled at the end of Section 3 to prove the Main Theorem.

The standard reference on 3-dimensional topology is Hempel's book [6], and we shall take its contents for granted. However, we shall understand manifolds to have $C^\infty$ structures, and in general submanifolds and maps between manifolds will be understood to be smooth. (The only exception is in the proof of Theorem 3.1, in which we explicitly state that we are shifting to the PL category.) It follows that we are really using the smooth analogues of the PL results stated in [6]; but these are all easy corollaries of the PL versions.

We suppress base points whenever it creates no ambiguity to do so. Unlabeled homomorphisms are understood to be induced by inclusion maps.

I am very much indebted to William Jaco for the idea of a tower argument of the general type used in the proof of Theorem 3.1. This idea is about eight or ten years old, but to my knowledge it had not been written down before. I also want to thank Jaco and Marc Culler for preliminary discussions of the contents of this paper, and Peter Scott for promptly informing me of his result in [13].

## 1. BACKGROUND

We begin with a simple criterion for the torus conjecture to hold in a given closed irreducible 3-manifold.

PROPOSITION 1.1. *Let M be a closed orientable, irreducible 3-manifold. The torus conjecture holds in M if and only if at least one of the following conditions is satisfied:*

(i) *M is a Haken manifold;*

(ii) *M is Seifert-fibered;*

(iii) *M contains no essential singular torus.*

*Proof.* If (i) holds, then the torus conjecture holds in $M$; this is the standard torus theorem stated in the introduction. If (ii) holds, the torus conjecture trivially holds in $M$; we need only take $\Sigma = M$ in the statement, since then $\partial\Sigma = \emptyset$. If (iii) holds, the torus conjecture holds vacuously.

Conversely, if the torus conjecture holds in $M$, and if (iii) does not hold, then $M$ contains a (compact) Seifert-fibered manifold $\Sigma$ whose boundary components are all incompressible in $M$. Thus if $\partial\Sigma \neq \emptyset$, $M$ contains an incompressible torus and (i) holds; whereas if $\partial\Sigma = \emptyset$, then $\Sigma = M$ and so (ii) holds.

Our next result will follow from two recent results of G. P. Scott's, a new result due to Meeks, Yau, and Simon, and an older result of Waldhausen's.

THEOREM 1.2. *Let $M$ be a closed, orientable, irreducible 3-manifold, and suppose that some finite-sheeted covering space of $M$ contains an incompressible surface. Then the torus conjecture holds in $M$.*

*Proof.* By Proposition 1.1, we need only show that, under the additional assumption that $M$ contains an essential singular torus, $M$ is either a Seifert fibered space or a Haken manifold. The main result of [12] asserts (in the closed case) that if a closed, irreducible, orientable 3-manifold $M$ contains an essential singular torus, then either $M$ contains an incompressible torus, or $\pi_1(\tilde{M})$ has a non-trivial center for some finite-sheeted covering space $\tilde{M}$ of $M$. In the first case, $M$ is a Haken manifold. In the second case, we invoke the hypothesis that some other finite-sheeted covering space of $M$, say $\tilde{M}'$, is a Haken manifold. Let $\tilde{M}''$ be a finite-sheeted covering space of $M$ which covers both $\tilde{M}$ and $\tilde{M}'$.

Since $\tilde{M}'$ contains an incompressible surface, so does $\tilde{M}$. It is proved in [10] that any covering space of an orientable, irreducible 3-manifold is irreducible. Thus $\tilde{M}''$ is irreducible, and is therefore a Haken manifold. On the other hand, $\pi_1(M)$ is infinite since $M$ contains an essential singular torus; and since $M$ is irreducible it follows [6, Lemma 9.4] that $\pi_1(M)$ is torsion-free. Hence $\pi_1(\tilde{M}'')$, as well as $\pi_1(\tilde{M})$, has non-trivial center. The main theorem of [16] asserts that a Haken manifold whose fundamental group has non-trivial center is a Seifert fibered space. Thus $\tilde{M}''$ is Seifert-fibered. Finally, it is shown in [13] that any compact, irreducible, orientable 3-manifold which has infinite fundamental group and is covered by a Seifert fibered space is Seifert-fibered. Thus $M$ is a Seifert fibered space.

The final result of this section will be deduced from the Meeks-Yau equivariant sphere theorem.

PROPOSITION 1.3. *Let $M$ be a regular branched cover of $S^3$. Then $M$ may be expressed as a connected sum $M_1 \# \ldots \# M_r$, where each $M_i$ is a regular branched cover of $S^3$, and for each $i$ either $\pi_1(M_i) \approx \mathbf{Z}$ or $\pi_2(M_i) = 0$.*

(If the Poincaré conjecture is true, it is easy to see that the $M_i$ may be taken to be the prime factors of $M$.)

*Proof of 1.3.* If $\pi_2(M) = 0$ or $\pi_1(M) \approx \mathbf{Z}$ the assertion is trivial. If $\pi_2(M) \neq 0$ and $\pi_1(M) \not\approx \mathbf{Z}$, we shall show that $M$ is a connected sum $N_1 \# \ldots \# N_l$, where each $N_i$ is a regular branched cover of $S^3$ and at least two of the $N_i$ are non-simply-connected. Then $\pi_1(M) = \pi_1(N_1) * \ldots * \pi_1(N_l)$, and by a standard corollary of Grushko's theorem [9, p. 234, Example 6.1], each $\pi_1(N_i)$ has a smaller minimal number of generators than $\pi_1(M)$. The proposition will therefore follow by induction on the minimal number of generators of $\pi_1(M)$.

By hypothesis there is a finite group $G$ of orientation-preserving diffeomorphisms of $M$ such that $M/G \approx S^3$. According to the equivariant sphere theorem [11, Sections 7 and 8], if a finite group $G$ acts on a 3-manifold $M$ with $\pi_2(M) \neq 0$, there is a finite, non-empty collection of disjoint, smoothly embedded 2-spheres $S_1, \ldots, S_n$ in $M$, such that no $S_i$ is homotopically trivial, and such that the set $S_1 \cup \ldots \cup S_n$ is invariant under $G$.

We claim that the $S_i$ may be chosen so that for each $g \in G$ and each $S_i$ such that $gS_i = S_i$, $g|S_i$ is an orientation-preserving diffeomorphism of $S_i$. In fact, let $T$ be a tubular neighborhood of $S_1 \cup \ldots \cup S_n$ which is invariant under $G$. Then $\partial T$ is invariant under $G$ and the components of $\partial T$ are 2-spheres. But $g|T$ is an orientation-preserving diffeomorphism of $T$ (since $g$ preserves orientation on $M$) and hence $g$ preserves orientation on any component of $\partial T$ which is invariant under $g$. Thus the claim is established by replacing $S_1, \ldots, S_n$ by the components of $\partial T$.

Now set $S = S_1 \cup \ldots \cup S_n$, let $p: M \to M/G$ denote the orbit map, and set $\bar{S} = p(S)$. Each component of $\bar{S}$ is the quotient of a 2-sphere by a group of orientation-preserving diffeomorphisms, and is therefore a 2-sphere. Since $M/G \approx S^3$, each component of $S^3 - \bar{S}$ is the interior of a 3-sphere-with-holes.

Let $C$ be any component of the manifold obtained by splitting $M$ along $S$, and let $G_C$ be the largest subgroup of $G$ that leaves $C$ invariant. Then $\bar{C} = C/G_C$ is a 3-sphere-with-holes. Hence if we let $C^*$ denote the manifold obtained from $C$ by attaching a 3-ball to each component of $\partial C$, the action of $G_C$ on $C$ extends to a smooth action of $G_C$ on $C^*$ for which the orbit space is a 3-sphere. (The smoothness of the extended action depends on the fact that every finite group action on $S^2$ is conjugate to a linear action, and hence extends to a smooth action on the 3-ball.) This shows that $C^*$ is a regular branched covering of $S^3$.

Now let $C$ range over all components of the manifold obtained by splitting $M$ at $S$. We can describe $M$ as the connected sum of the corresponding 3-manifolds $C^*$, together with a finite number of copies of $S^2 \times S^1$. The manifold $S^2 \times S^1$ is easily seen to be a regular branched covering of $S^3$. This gives the decomposition $N_1 \# \ldots \# N_l$; it remains only to check that at least two of the $N_i$ are non-simply-connected.

Consider first the case in which every component $S_i$ of $S$ separates $M$. In this case there are at least two components $C_1$, $C_2$ of $\overline{M - S}$ such that $\partial C_j$ is a single sphere $S_{i_j}$. Since by construction $S_{i_1}$ and $S_{i_2}$ are homotopically non-trivial, $C_1$ and $C_2$ are non-simply-connected. It follows that in this case two of the $N_i$ are non-simply-connected. Now consider the case in which some $S_i$ fails to separate $M$. In this case some $N_j$ is a copy of $S^1 \times S^2$. If all the remaining $N_j$ were simply connected, we should have $\pi_1(M) \approx \mathbf{Z}$; but we have assumed that this is not the case.

## 2. FUNDAMENTAL GROUPS OF REGULAR BRANCHED COVERS

Recall that a group $G$ is *residually finite* if the intersection of all finite-index subgroups of $G$ is the trivial subgroup. It is an unpublished result of W. Thurston's that the fundamental group of any Haken manifold is residually finite.

*Definition.* A group $G$ is *half-way residually finite* if either (a) $G$ is finite, or (b) for every positive integer $n$, $G$ has a subgroup of finite index $\geq n$.

The proof of the following result is the goal of this section.

THEOREM 2.1. *The fundamental group of any regular branched cover of $S^3$ is half-way residually finite.*

The heart of the proof of Theorem 2 is contained in the proof of Lemma 2.3 below, which closely follows the proof of the Smith conjecture [1]. The following result is a preliminary to Lemmas 2.3 and 2.4. (The assumption of prime period in 2.2 and 2.3 is only for convenience.)

LEMMA 2.2. *Let $M$ be a closed, orientable 3-manifold, and let $h$ be an orientation-preserving periodic diffeomorphism of $M$ whose period is a prime and whose fixed-point set is a smoothly embedded 1-sphere in $M$. Set $\bar{M} = M/\langle h \rangle$, and let $p: M \to \bar{M}$ denote the orbit map. Then $p_*: \pi_1(M) \to \pi_1(\bar{M})$ is an epimorphism.*

*Proof.* Let $k$ denote the fixed-point set of $h$ and set $\bar{k} = p(k)$. Since the period $l$ of $h$ is prime, $\langle h \rangle$ acts without fixed points on $M - k$. Hence $M - k$ is a regular (unbranched) covering space of $\bar{M} - \bar{k}$ with covering projection $p|M - k$ and covering group $\langle h|M - k \rangle \approx \mathbf{Z}_l$. Hence $N = \operatorname{im}((p|M - k)_*: \pi_1(M - k) \to \pi_1(\bar{M} - \bar{k}))$ is normal in $\pi_1(\bar{M} - \bar{k})$, and $\pi_1(\bar{M} - \bar{k})/N \approx \mathbf{Z}_l$. But it is easy to see that $k$ has a tubular neighborhood $T$ in $M$, invariant under $h$, which may be identified with $D^2 \times S^1$ in such a way that $h|T$ is $\alpha \times 1$, where $\alpha$ is a rotation of $D^2$ having order $l$. This implies that if $\mu$ is a meridian of $\bar{k}$ in $\bar{M}$, i.e. a simple closed curve bounding a disc that meets $\bar{k}$ transversally in a single point, then $[\mu] \in \pi_1(\bar{M} - \bar{k})$ determines a generator of $\pi_1(\bar{M} - \bar{k})/N$. Since $\pi_1(\bar{M})$ is obtained from $\pi_1(\bar{M} - \bar{k})$ by adding the relation $[\mu] = 1$, we may now conclude that

$$(p|M - k)_*: \pi_1(M - k) \to \pi_1(\bar{M})$$

is surjective, and hence that $p_*: \pi_1(M) \to \pi_1(\bar{M})$ is surjective.

The notation of the following lemma is chosen to be consistent with that of [1], not with that of the rest of this paper.

LEMMA 2.3. *Let $\tilde{M}$ be a closed 3-manifold which is not a 3-sphere. Let $h$ be an orientation-preserving periodic diffeomorphism of $M$ having prime period $l$, and suppose that the fixed-point set of $h$ is a smoothly embedded 1-sphere $\tilde{k} \subset \tilde{M}$. Suppose that $(\tilde{M} - \tilde{k})/\langle h \rangle$ is an irreducible 3-manifold. Then $\pi_1(\tilde{M})$ has a proper subgroup of finite index.*

*Proof.* Let $M = \tilde{M}/\langle h \rangle$, let $p: \tilde{M} \to M$ denote the orbit map, and set $k = p(\tilde{k})$. In the terminology of [5], we regard $k$ as a knot (=1-component link) in $M$. By the discussion in Section 2 of [5], $k$ is a connected sum of prime knots $k_1, \ldots, k_r$ in closed 3-manifolds $M_1, \ldots, M_r$. We shall prove the lemma by induction on $r$. We first suppose that $r = 1$, i.e., that $k$ is prime.

We may assume that $H_1(\tilde{M}) = 0$, since otherwise $\pi_1(\tilde{M})$ admits a homomorphism onto a finite cyclic group and the conclusion is immediate. By Lemma 2.2 with $M$, $M'$ replaced by $\tilde{M}$, $M$, it follows that $H_1(M) = 0$ also. Let $N$ be a tubular neighborhood of $k$ in $M$. We distinguish three cases.

*Case* I. *There is a closed, orientable, incompressible surface in $M - k$ which is not isotopic to $\partial N$ in $M - k$.* In this case the hypotheses of Theorem 1 of [5] are satisfied, and hence $\tilde{M}$ contains either an orientable, incompressible surface of genus >0 or a non-separating 2-sphere. In the latter case, $\pi_1(\tilde{M})$ admits a homomorphism on to $\mathbf{Z}$ and our assumption $H_1(\tilde{M}) = 0$ is contradicted. Now suppose that $\tilde{M}$ contains an orientable, incompressible surface of genus >0. Then some prime factor $\tilde{M}_0$ of $\tilde{M}$ is a Haken manifold. By the theorem of Thurston's quoted at the beginning of this section, $\pi_1(\tilde{M}_0)$ is residually finite. Moreover, $\tilde{M}_0$ cannot be simply connected, and so $\pi_1(\tilde{M}_0)$ has a proper subgroup of finite index. Since $\pi_1(\tilde{M}_0)$ is a free factor of $\pi_1(\tilde{M})$, the latter group also has a proper subgroup of finite index.

*Case* II. *$M - \mathring{N}$ is not Seifert-fibered, and every incompressible surface in $M - k$ is isotopic to $\partial N$ in $M - k$.* Then we apply Proposition 5 of [14] to the compact manifold $M - N$. This gives a diffeomorphism of $M - k$ with $H^3/\Gamma$, where $\Gamma$ is a torsion-free subgroup of $\mathrm{PSL}_2(A)$, $A \subset \mathbf{C}$ is the ring of algebraic integers in some number field, and $\Gamma$ is discrete as a subgroup of $\mathrm{PSL}_2(\mathbf{C})$. (It is understood that $\mathrm{PSL}_2(\mathbf{C})$ acts in the standard way on the hyperbolic 3-space $H^3$, cf. [14, Section 2]. The proof of Proposition 1 of [14] shows that $\pi_1(\tilde{M})$ will have a non-trivial representation in $\mathrm{PSL}_2(A/\mathscr{M})$, for some maximal ideal $\mathscr{M}$ of $A$, provided that (a) $\partial N$ is incompressible in $M - k$, and (b) $\pi_1(M - k)$ is not solvable. But the negation of (a) would imply, with the irreducibility of $M - k$, that $M - \mathring{N}$ is a solid torus, a contradiction since $M - \mathring{N}$ is not Seifert-fibered. We get a similar contradiction from the negation of (b) by the virtue of the result due to Evans and Moser [3] that every compact irreducible 3-manifold having non-empty boundary and solvable fundamental group is Seifert-fibered.

Thus $\pi_1(M)$ has a non-trivial representation in $\mathrm{PSL}_2(A/\mathscr{M})$ for some prime ideal $\mathscr{M}$ of $A$. We claim that the field $A/\mathscr{M}$ is *finite.* Note that $\mathscr{M} \neq 0$, since $A$ cannot be a field, and recall that $A$ is a finitely generated abelian group. Choose $x \in \mathscr{M}$, $x \neq 0$. Since $\mathscr{M}$ is a prime ideal, the map from $A$ to $\mathscr{M}$ defined by $a \to xa$ is injective; hence $\mathscr{M} \subset A$ is a free abelian group having at least (and hence exactly) the same rank as $A$. This proves the claim. It now follows that the kernel of the nontrivial representation of $\pi_1(\tilde{M})$ in $\mathrm{PSL}_2(A/\mathscr{M})$ is a proper subgroup of finite index in $\pi_1(\tilde{M})$.

*Case* III. *$M - \mathring{N}$ is Seifert-fibered.* Since $H_1(M) = 0$, the proof of Proposition 2 of [14] shows that $M - \mathring{N}$ must be Seifert-fibered over the disc; and that if $M - \mathring{N}$ has more than one singular fiber than $\pi_1(\tilde{M})$ has a non-trivial representation in $\mathrm{PSL}_2(A/\mathscr{M})$, where $A$ is the ring of integers in some number field and $\mathscr{M}$ is a maximal ideal of $A$. It then follows precisely as in Case II that $\pi_1(\tilde{M})$ has a proper subgroup of finite index. There remains the possibility that $M - \mathring{N}$ is Seifert-fibered over a disc with one singular fiber. Then $M - \mathring{N}$ is a solid torus, and hence so is its finite-sheeted covering $\tilde{M} - p^{-1}(\mathring{N})$. But $p^{-1}(N)$ is also a solid torus, since it is a tubular neighborhood of $\tilde{k}$ in $\tilde{M}$. Thus $\tilde{M}$ is a union of two solid tori with a common boundary, and is therefore either a lens space or a copy of $S^2 \times S^1$. By hypothesis $\tilde{M}$ is not a 3-sphere. Hence $\pi_1(\tilde{M})$ is a non-trivial cyclic group, and the conclusion of the lemma is immediate. This completes the proof for $r = 1$.

Finally, suppose that $r > 1$. Then $k$ is a connected sum of knots $k_1$, $k_2$ in 3-manifolds $M_1$, $M_2$, where $k_i$ $(i = 1,2)$ is a connected sum of fewer than $r$ prime knots. There is a 2-sphere $S \subset M$ meeting $k$ transversally in two points, and dividing

$M$ into compact submanifolds $A_1$ and $A_2$, such that $M_i$ is obtained from $A_i$ by attaching a 3-ball $B_i$ to its boundary, and $k_i$ is the union of $k \cap A_i$ with an unknotted arc in $B_i$. It is clear that $\tilde{S} = p^{-1}(S)$ is a 2-sphere in $\tilde{M}$, separating $\tilde{M}$ into the compact submanifolds $\tilde{A}_i = p^{-1}(A_i)$ $(i = 1,2)$, each invariant under $h$. If we construct a closed manifold $\tilde{M}_i$ by attaching a ball $\tilde{B}_i$ to $\partial \tilde{A}_i$, it is easy to extend $h|\tilde{A}_i$ to a diffeomorphism $h_i$ of $\tilde{M}_i$ which is periodic with period $l$. By the induction hypothesis, for each $i(=1,2)$, either $\tilde{M}_i$ is a 3-sphere or $\pi_1(\tilde{M}_i)$ has a proper subgroup of finite index. But $\tilde{M} \approx M_1 \# M_2$; hence one of the $M_i$ fails to be a 3-sphere, and so $\pi_1(\tilde{M}) = G_1 * G_2$, where at least one of the $G_i$ has a proper subgroup of finite index. Hence so does $\pi_1(\tilde{M})$.

LEMMA 2.4. *Let $M$ be a closed, orientable 3-manifold such that $\pi_2(M) = 0$ but $\pi_1(M) \neq 1$. Suppose that $M$ admits a non-trivial orientation-preserving periodic diffeomorphism with nonempty fixed-point set. Then $\pi_1(M)$ has a proper subgroup of finite index.*

*Proof.* we may assume that $H_1(M) = 0$, since otherwise $\pi_1(M)$ admits a homomorphism on to a finite cyclic group. Thus $M$ is a homology 3-sphere.

Let $h$ be a non-trivial periodic diffeomorphism of $M$ with fixed point $k \neq \emptyset$. We may assume that the period $l$ of $h$ is prime. The fixed-point set of an orientation-preserving periodic diffeomorphism of a smooth manifold is always a smooth submanifold of even codimension. Since $k$ is non-empty and $\neq M$, it must be 1-dimensional. But a theorem of P. A. Smith's [15, p. 366, Theorem 12.1] asserts that the fixed-point set of a periodic homeomorphism of a homology sphere is a homology sphere. Hence $k$ is a smoothly embedded 1-sphere in $M$. Let $\bar{M} = M/\langle h \rangle$, and let $p: M \to \bar{M}$ denote the orbit map. Since $M$ is a homology 3-sphere, Lemma 2.2 implies that $\bar{M}$ is also a homology 3-sphere.

We claim that $\pi_2(\bar{M} - \bar{k}) = 0$. If this is not the case, then by the Sphere Theorem there is a non-contractible embedded 2-sphere $\bar{S} \subset \bar{M} - \bar{k}$. Since $\bar{M}$ is a homology 3-sphere, $\bar{S}$ bounds a compact, acyclic 3-manifold $\bar{A} \subset \bar{M} - \bar{k}$. Since $l$ is prime, $\langle h \rangle$ acts without fixed points on $M - k$; hence $M - k$ is a regular covering space of $\bar{M} - \bar{k}$. Let $N = \operatorname{im}((p|M - k)_* : \pi_1(M - k) \to \pi_1(\bar{M} - \bar{k}))$. Then $N$ is normal and $\pi_1(\bar{M} - \bar{k})/N \approx \mathbf{Z}_l$. Since $\bar{A}$ is acyclic and $\pi_1(\bar{M} - \bar{k})/N$ is abelian, we must have $\operatorname{im}(\pi_1(\bar{A}) \to \pi_1(\bar{M} - \bar{k})) \subset N$; hence there are disjoint submanifolds $A_1, \ldots, A_l$ of $M - k$, each of which is projected homeomorphically onto $\bar{A}$ by $p$. If $\pi_1(\bar{A}) \neq 1$, then both $A_1$ and $M - \mathring{A}_1 \supset A_2$ are non-simply-connected; hence the 2-sphere $S_1 = \partial A_1$ is non-contractible in $M$, contradicting the hypothesis $\pi_2(M) = 0$. On the other hand, if $\pi_1(\bar{A}) = 1$, then $\bar{S}$ is contractible in $\bar{M} - \bar{k}$, and the choice of $\bar{S}$ is contradicted. This proves our claim.

It now follows by Kneser's theorem [6, Lemma 3.14], that $\bar{M} - \bar{k}$ is the connected sum of an irreducible 3-manifold with a homotopy 3-sphere. In other words, there is a compact, contractible 3-manifold $B \subset \bar{M} - \bar{k}$ such that if $\bar{M}'$ denotes the manifold obtained from $\bar{M}$ by removing $\mathring{B}$ and attaching a 3-ball along $\partial B$, $\bar{M}' - \bar{k}$ is irreducible. Now let $M'$ denote the manifold obtained from $M$ by replacing each component of $p^{-1}(B)$ by a 3-ball. There is a diffeomorphism $h'$ of $M'$, having order $l$ and fixed-point set $k$, such that $(M' - k)/h' \approx \bar{M}' - \bar{k}$. Applying Lemma 2.3, with $M'$, $k$, and $h'$ in place of $\tilde{M}$, $\tilde{k}$ and $h$, we conclude that $\pi_1(M) \approx \pi_1(M')$ has a proper subgroup of finite index.

COROLLARY 2.5. *Let $M$ be a regular branched covering of $S^3$ with $\pi_1(M) \neq 1$. Then $\pi_1(M)$ has a proper subgroup of finite index.*

*Proof.* By Proposition 1.3, $\pi_1(M)$ is a free product $H_1 * \ldots * H_l$, where each $H_i$ is either infinite cyclic or is the fundamental group of a regular branched cover of $S^3$ with non-trivial second homotopy group. Since $\pi_1(M) \neq 1$ we may assume that $H_1 \neq 1$. It is enough to show that $H_1$ has a proper subgroup of finite index. If $H_1 \approx \mathbf{Z}$, this is obvious. How. suppose that $H_1 = \pi_1(M_1)$, where $M_1$ is a regular branched cover of $S^3$ and $\pi_2(M_1) \neq 0$. There is a finite group $G$ of orientation-preserving diffeomorphisms of $M_1$ such that $M_1/G \approx S^3$. Since $\pi_1(M) \neq 1$, we have $M \not\approx S^3$ and hence $G \neq 1$. If $G$ acted on $M_1$ without fixed points, we would have $G \approx \pi_1(S^3) = 1$, a contradiction. Hence some non-trivial element of $G$ has non-empty fixed-point set, and our assertion follows by applying Lemma 2.4 to $M_1$.

Recall that a subgroup of a group $G$ is said to be *characteristic* if it is invariant under all automorphisms of $G$.

LEMMA 2.6. *Let $M$ be a regular branched cover of $S^3$. Then any (unbranched) covering space of $M$ corresponding to a characteristic subgroup of finite index in $\pi_1(M)$ is also a regular branched cover of $S^3$.*

*Proof.* Suppose that $M/G \approx S^3$, where $G$ is a finite group of diffeomorphisms of $M$. Let $m \in M$ be a basepoint, let $X \subset \pi_1(M,p)$ be a characteristic subgroup of finite index, and let $(\tilde{M},\tilde{m})$ denote the corresponding based covering space. Let $p:\tilde{M} \to M$ denote the covering projection, and let $q:M \to M/G$ denote the orbit map. Let $\tilde{G}$ denote the group of all diffeomorphisms $\tilde{g}$ of $\tilde{M}$ such that $p\tilde{g} = gp$ for some $g \in G$. Since for a given $g \in G$, $gp:\tilde{M} \to M$ has at most finitely many lifts to the finite-sheeted covering space $\tilde{M}$, $\tilde{G}$ is a finite group. Clearly $qp\tilde{g} = qp$ for every $\tilde{g} \in \tilde{G}$; i.e., $\tilde{G}$ acts on each fiber of the map $qp$.

We claim that the action of $\tilde{G}$ on each fiber of $qp$ is *transitive.* Let $\tilde{a}, \tilde{b} \in \tilde{M}$ be such that $qp(a) = qp(b)$. Then some $g \in G$ maps $a = p(\tilde{a})$ onto $b = p(\tilde{b})$. Let $h:M \to M$ be a diffeomorphism homotopic to $g$ and such that $h(m) = m$. The automorphism $h$ of $\pi_1(M,m)$ must leave the characteristic subgroup $X$ invariant; hence there is a diffeomorphism $\tilde{h}$ of $\tilde{M}$ such that $p\tilde{h} = hp$. By the covering homotopy property, the lift $\tilde{h}:\tilde{M} \to \tilde{M}$ of $hp:\tilde{M} \to M$ is homotopic to a lift $\tilde{g}_1:\tilde{M} \to \tilde{M}$ of $gp:\tilde{M} \to M$. Clearly $\tilde{g}_1 \in \tilde{G}$ and $\tilde{g}_1(\tilde{a}) \in p^{-1}(b)$. But since $X$ is in particular normal in $\pi_1(M)$, $\tilde{M}$ is a regular covering, and so some covering transformation $\tilde{g}_2$ maps $\tilde{g}_1(\tilde{a})$ to $\tilde{b}$. But the covering transformations of $\tilde{M}$ clearly belong to $\tilde{G}$. Hence $\tilde{g} = \tilde{g}_2\tilde{g}_1 \in \tilde{G}$; and since $\tilde{g}(\tilde{a}) = \tilde{b}$, transitivity is established.

Thus $qp$ maps $\tilde{M}$ onto $M/G \approx S^3$, and two points of $\tilde{M}$ have the same image under $qp$ if and only if they lie in the same orbit of $\tilde{G}$. It follows that $\tilde{M}/\tilde{G} \approx S^3$, and hence that $\tilde{M}$ is a regular branched cover of $S^3$.

*Proof of Theorem 2.1.* Let $M$ be a regular branched cover of $S^3$. We must show that if the indices of finite-index subgroups of $\pi_1(M)$ are bounded above, then $\pi_1(M)$ is finite. Since $\pi_1(M)$ is finitely generated, it has only finitely many subgroups of a given index $n$. (It has only finitely many representations in the symmetric group on $n$ letters.) Hence our boundedness assumption implies that $\pi_1(M)$ has only finitely many subgroups of finite index; the intersection of all of them is a characteristic subgroup $X$ of finite index in $\pi_1(M)$. Obviously $X$ has no proper subgroups of finite index.

By Lemma 2.6, $X$ is the fundamental group of a regular branched cover $\tilde{M}$ of $S^3$. If $X$ were non-trivial, then by Corollary 2.5, $X$ would have a proper subgroup of finite index. Hence $X = 1$, and $\pi_1(M)$ is finite, as required.

## 3. TOWERS

We shall prove the following result; the proof of the Main Theorem stated in the introduction will then be very easy.

THEOREM 3.1. *Let $M$ be a closed, orientable, irreducible 3-manifold whose fundamental group is half-way residually finite. Then the torus conjecture holds in $M$.*

If $K$ and $L$ are finite simplicial complexes, $K$ finite, and $f: K \to L$ is a simplicial map, we define the *complexity* of $f$, denoted $c(f)$, to be the number of unordered pairs of (distinct) simplices $\sigma$, $\sigma'$ of $K$ such that $f(\sigma) = f(\sigma')$.

LEMMA 3.2. (Stallings). *Let $f: K \to L$ be a simplicial map of complexes ($K$ finite). Let $\tilde{L}$ be a non-trivial covering space of $L$ (with the induced triangulation) and $f: K \to L$ a lifting. If $\pi_1(f(K)) \to \pi_1(L)$ is surjective, then $c(\tilde{f}) < c(f)$.*

*Proof.* If $p: \tilde{L} \to L$ denotes the covering projection then $p$ is simplicial and $p\tilde{f} = f$. From the definition of complexity it is clear that $c(\tilde{f}) \leq c(f)$ and that equality can hold only if $p \mid \tilde{f}(K)$ is $1-1$. But this implies that

$$(p \mid f(\tilde{K}))_*: \tilde{f}(K) \to f(K)$$

is an isomorphism, and hence, by the surjectivity of $\pi_1(f(K)) \to \pi_1(=\text{L})$, that $p_*: \pi_1(\tilde{L}) \to \pi_1(L)$ is surjective; and this contradicts the hypothesis that $\tilde{L}$ is a non-trivial covering space.

LEMMA 3.3. *Let $G$ be a half-way residually finite group whose commutator quotient is finite. Then every proper abelian subgroup of $G$ is contained in a proper subgroup of $G$ which has finite index.*

*Proof.* If $G$ is finite the assertion is trivial. There remains the case in which $G$ has subgroups of arbitrarily large finite index. Then if $N$ denotes the intersection of all finite-index subgroups of $G$, $N$ has infinite index in $G$. But $N$ is clearly normal, and so $G/N$ is an infinite group; it is non-abelian since $G$ is assumed to have finite commutator quotient.

Let $\bar{a}$ and $\bar{b}$ be elements of $G/N$ such that $[\bar{a},\bar{b}] \equiv \bar{a}\bar{b}\bar{a}^{-1}\bar{b}^{-1} \neq 1$. Choose representatives $a$ and $b$ of $\bar{a}$ and $\bar{b}$. Then $[a,b] \in G$ does not lie in $N$, and so there is a subgroup $H$ of finite index in $G$ such that $[a,b] \notin H$. But any finite-index subgroup $H$ of a group $G$ contains a normal subgroup $H^*$ of $G$ which is also of finite index in $G$. (We may define $H^*$ to be the kernel of the natural permutation representation of $G$ on the right cosets of $H$ in $G$.) Now $G/H^*$ is clearly non-abelian, but $p(A) \subset G/H^*$, where $p: G \to G/H^*$ is the natural homorphism, is abelian.

Hence $p(A)$ is a proper subgroup of the finite group $G/H^*$, and so $p^{-1}(p(A))$ is a proper subgroup of finite index in $G$. It obviously contains $A$.

LEMMA 3.4. *A finite-index subgroup of a half-way residually finite group is half-way residually finite.*

*Proof.* Let $H$ have finite index in $G$. If $G$ is finite, so is $H$. Now suppose that $G$ has subgroups of arbitrarily large finite index. Then given $n > 0$, $G$ has a subgroup $G_n$ of index $\geq n\,|G:H|$; and $H_n = G_n \cap H$ has finite index $\geq n\,|G:H|$ in $G$, and hence has finite index $\geq n$ in $H$.

*Proof of Theorem 3.1.* If there is no essential singular torus in $M$, then the torus theorem automatically holds in $M$. If there is an essential singular torus $f$ in $M$, we shall show that some finite-sheeted (unbranched) covering space of $M$ contains an incompressible surface; the assertion will then follow from Theorem 1.2.

After modifying $f$ by a homotopy, we may assume that it is simplicial with respect to some triangulations of $T^2$ and $M$. The rest of this proof is to be interpreted in the PL category. If $\tilde{M}$ is a finite-sheeted covering space of $M$, and $\tilde{f}: T^2 \to \tilde{M}$ is a lifting of $f$, then $\tilde{f}$ is simplicial with respect to the induced triangulation of $\tilde{M}$, and thus has a well-defined complexity. Among all pairs $(\tilde{M}, \tilde{f})$, where $\tilde{M}$ is a finite-sheeted covering space of $M$ and $\tilde{f}$ is a lifting of $f$ to $\tilde{M}$, choose one for which $c(\tilde{f})$ takes its smallest possible value; call this minimizing pair $(\tilde{M}_0, \tilde{f}_0)$. We shall complete the proof by showing that $\tilde{M}_0$ contains an incompressible surface.

If $H_1(\tilde{M}_0)$ is infinite, then $\tilde{M}_0$ contains an incompressible surface by [6, Lemma 6.6]. We may therefore assume that $H_1(\tilde{M}_0)$ is finite. On the other hand, $\pi_1(\tilde{M}_0)$ is half-way residually finite by Lemma 3.4. Thus $G = \pi_1(\tilde{M}_0)$ satisfies the hypothesis of Lemma 3.3. The abelian subgroup $\tilde{f}_{0*}(\pi_1(T^2))$ of $\pi_1(\tilde{M}_0)$ is proper, since otherwise we would have $\pi_1(\tilde{M}_0) \approx \mathbf{Z} \times \mathbf{Z}$, and $H_1(\tilde{M}_0)$ would be infinite. Hence by 3.3, $\tilde{f}_{0*}(\pi_1(T^2))$ is contained in a finite-index subgroup of $\pi_1(\tilde{M}_0)$„ i.e., $\tilde{f}_0$ lifts to a map $\tilde{f}_1: T^2 \to \tilde{M}_1$, where $\tilde{M}_1$ is finite-sheeted covering space of $\tilde{M}_0$. If $\pi_1(\tilde{f}_0(T^2)) \to \pi_1(\tilde{M}_0)$ were surjective, then by Lemma 3.2, $\tilde{f}_1$ would have smaller complexity than $\tilde{f}_1$, contradicting the minimality of $c(\tilde{f}_1)$. This shows that $\tilde{f}_0(T^2)$ does not carry $\pi_1(\tilde{M}_0)$ (i.e., $\pi_1(\tilde{f}_0(T^2)) \to \pi_1(\tilde{M}_0)$ is not surjective).

Thus if $R$ is a regular neighborhood of $\tilde{f}_0(T^2)$ in $\tilde{M}_0$, $R$ does not carry $\pi_1(\tilde{M}_0)$. If $g: T^2 \to \tilde{M}_0$ is any map homotopic to $\tilde{f}_0$, and $N$ is any compact, connected 3-manifold such that $g(T^2) \subset \mathring{N}$ and such that $N$ does not carry $\pi_1(\tilde{M}_0)$, let $q(N)$ denote the sum of the squares of the genera of the components of $\partial N$. Among all pairs $(g,N)$ with these properties, choose one for which $q(N)$ is as small as possible; call it $(g_0,N_0)$. We claim that some component of $\partial N_0$ is an incompressible surface in $\tilde{M}_0$. This will show that $\tilde{M}_0$ is a Haken manifold and will therefore prove the theorem.

First we show that not every component of $\partial N_0$ is a 2-sphere. Indeed, since $\tilde{M}_0$ is irreducible, each 2-sphere in $\partial N_0$ bounds a ball $B$ which either contains $N_0$ or is disjoint from $\mathring{N}_0$. If some $B$ contains $N_0$, then $g_0(T^2) \subset B$, a contradiction since $g_0$ is an essential singular torus. If every component of $\partial N_0$ bounds a ball disjoint from $\mathring{N}_0$, then $\pi_1(N_0) \to \pi_1(M)$ is surjective, and again we have a contradiction.

Thus some component $T$ of $\partial N_0$ has positive genus. We shall show that any such $T$ is incompressible. If some $T$ were compressible we could find a disc $D \subset \tilde{M}_0$ with $\partial D$ non-contractible in a component $T'$ of $\partial N_0$, such that $\mathring{D} \cap \partial N_0 = \emptyset$ or $D \subset N_0$. In the first case, if $N_1$ is a regular neighborhood of $N_0 \cup D$, then $N_1$ is a compact 3-manifold neighborhood of $g_0(T^2)$, and $\pi_1(N_1) \to \pi_1(\tilde{M}_0)$ cannot be surjective since $\pi_1(N_0) \to \pi_1(\tilde{M}_0)$ is not; but we have $q(N_1) < q(N_0)$, since $\partial N_1$ is obtained from $\partial N_0$ by replacing $T'$ by a surface of smaller genus or two surfaces whose positive genera add up to the genus of $T'$. This contradicts the minimality of $q(N_0)$.

Now suppose that $D \subset N_0$. Then using the fact that $g_0$ is an essential singular torus, it is easy to homotope $g_0$ within $N_0$ to a map $g$, such that $g_1(T^2) \cap D = \emptyset$. (In fact, we may assume that $g_0 : T^2 \to N_0$ is tranversal to $D$; and then each component of $g_0^{-1}(D)$ must bound a disc in $T^2$ since $g_{0*} : \pi_1(T^2) \to \pi_1(N_0)$ is a monomorphism. This makes it easy to replace $g_0$ by an essential singular torus $g_0'$ such that $g_0' : T^2 \to N_0$ is transversal to $D$ and $g_0'^{-1}(D)$ has fewer components than $g_0^{-1}(D)$. By repeating this process we obtain the map $g_1$.) Let $H$ be a regular neighborhood of $D$ in $N_0$, disjoint from $g_1(T^2)$, and let $N_1$ denote the component of $\overline{N_0 - H}$ containing $g_1(T^2)$. Then $N_1 \subset N$ cannot carry $\pi_1(\tilde{M}_0)$; and as above one sees that $q(N_1) < q(N_0)$. Again we have a contradiction to the minimality of $q(N_0)$.

*Proof of the Main Theorem.* Let $M$ be a regular branched covering of $S^3$. By Proposition 1.3, we may write $M = M_1 \# \dots \# M_r$, where each $M_i$ is a regular branched cover of $S^3$ and for each $i$ either $\pi_1(M_i) = \mathbf{Z}$ or $\pi_2(M_i) = 0$. By the Kneser-Milnor prime decomposition theorem [6, Theorem 3.21], each prime factor of $M$ is a prime factor of some $M_i$. Hence by definition (see introduction) the torus conjecture will hold in $M$ if it holds in each $M_i$; thus we may assume that either $\pi_1(M) \approx \mathbf{Z}$ or $\pi_2(M) = 0$.

If $\pi_1(M) \approx \mathbf{Z}$, then each prime factor of $M$ has cyclic fundamental group. Hence there is no singular torus in any prime factor of $M$, and the torus conjecture holds trivially. Now suppose that $\pi_2(M) = 0$. Then every 2-sphere in $M$ bounds a compact, simply-connected submanifold, and so $M$ has at most one non-simply connected prime factor. Again, in the simply-connected prime factors, the torus conjecture holds trivially. If $M$ has a non-simply-connected prime factor $M_0$, then $M_0$ is homotopy equivalent to $M$. In particular $\pi_1(M_0) \approx \pi_1(M)$. On the other hand, $\pi_2(M_0) \approx \pi_2(M) = 0$, so that $M_0 \not\approx S^2 \times S^1$, and hence $M_0$ is irreducible. But since $M$ is a regular branched covering of $S^3$, Theorem 2.1 implies that $\pi_1(M_0) \approx \pi_1(M)$ is half-way residually finite. By Theorem 3.1, the torus conjecture holds in $M_0$.

## REFERENCES


1. H. Bass, J. Birman, J. Morgan, Eds.: Symposium on the Smith conjecture, Columbia University, *1979*, to appear.
2. J. Birman and H. M. Hilden, *The homeomorphism problem for $S^3$*. Bull. Amer. Math. Soc. 79 (1973), no. 5, 1006–1010.
3. B. Evans and L. Moser, *Solvable fundamental groups of compact 3-manifolds.* Trans. Amer. Math. Soc. 168 (1972), 189–210.

4. C. D. Feustel, *On the torus theorem for closed 3-manifolds.* Trans. Amer. Math. Soc. 217 (1976), 45–57.

5. C. Mac A. Gordon and R. Litherland, *Incompressible surfaces in branched coverings,* Symp. on the Smith conjecture, to appear.

6. J. Hempel, *3-manifolds.* Ann. of Math. Studies, No. 86, Princeton Univ. Press, Princeton N.J., 1976.

7. K. Johannson, *Homotopy equivalences of 3-manifolds with boundaries.* Lecture Notes in Math., Vol. 761. Springer, Berlin, 1979.

8. W. Jaco and P. Shalen, *Seifert fibered spaces in 3-manifolds.* Mem. Amer. Math. Soc. 21 (1979), no. 220, viii + 192 pp.

9. W. S. Massey, *Algebraic topology: an introduction,* Springer, New York, 1977.

10. W. Meeks, L. Simon and S.-T. Yau, to appear.

11. W. Meeks and S.-T. Yau, *Topology of three-dimensional manifolds and the embedding problems in minimal surface theory,* Symp. on the Smith conjecture, to appear,

12. G. P. Scott, *A new proof of the annulus and torus theorems.* Amer. J. Math., to appear.

13. ———, *There are no fake Seifert fibre spaces with infinite* $\pi_1$, to appear.

14. P. Shalen, *The case of no incompressible surface,* Symp. on the Smith conjecture, to appear.

15. P. A. Smith, *Fixed points of periodic transformations.* Appendix B to Algebraic Topology by S. Lefschetz, Amer. Math. Soc. Colloquium Publications, Vol. XXVII, 1942.

16. F. Waldhausen, *Gruppen mit Zentrum und 3-dimensionale Mannigfaltigkeiten.* Topology 6 (1967), 505–517.

17. ———, *On the determination of some bounded 3-manifolds by their fundamental groups alone.* Proc. of Inter. Symp. Topology, Hercy-Novi, Yugoslavia, 1968. Belgrade (1969), 331–332.

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