THE ENERGY 1 METRIC ON HARMONICALLY IMMERSED SURFACES

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1. INTRODUCTION

In this paper we show that two theorems about minimally immersed surfaces are just special cases of more general statements about harmonically immersed surfaces. Both results characterize the harmonic immersion of a surface in the sphere in terms of the behavior of an immersion into the containing Euclidean space.

At the heart of these extensions is use of the energy 1 metric Γ, which at many points plays the role which the induced metric does on a minimal surface. We highlight the metric Γ here since it appears to be of general use in studying harmonically mapped surfaces. (See [7] and [9].)

Suppose a Riemannian metric $g = g_{ij}\,dx^i dx^j$ is specified on a surface S which is immersed in a manifold M^n with Riemannian metric $G = G_{\alpha\beta}\,du^\alpha du^\beta$. (We assume C^∞ smoothness throughout.) Let $\mathrm{I} = h_{ij}\,dx^i dx^j$ be the metric induced on S by G. The energy function e of the immersion is given by

$$(1)\qquad e = e(g,\mathrm{I}) = \frac{g_{11}h_{22} + g_{22}h_{11} - 2g_{12}h_{12}}{2(g_{11}g_{22} - g_{12}^2)} = \frac{1}{2}\, tr_g \mathrm{I}.$$

Among all metrics on S proportional to g, only the choice $\Gamma = e(g,\mathrm{I})g$ on S yields energy function 1. Thus, we refer to Γ as the *energy 1 metric* of the immersion. Note that only the conformal class of g matters in the determination of Γ.

An immersion $X:(S,g) \to (M^n,G)$ is called *harmonic* in case X is extremal for the integral of $e(g,\mathrm{I})$ with respect to g. The Euler-Lagrange equation which must be satisfied is

$$(2)\qquad \Delta_g X^\alpha + \overset{G}{\Gamma}{}_{\beta\gamma}^{\alpha} \frac{\partial X^\beta}{\partial x^i} \frac{\partial X^\gamma}{\partial x^j} g^{ij} = 0.$$

Here Δ_g is the Laplace-Beltrami operator of g, $\overset{G}{\Gamma}{}_{\beta\gamma}^{\alpha}$ is the Christoffel symbol associated with G, and $X = (X^\alpha)$. One sums on i, j, β, γ with $i, j = 1, 2$ and $\alpha, \beta, \gamma = 1, 2, \ldots, n$. Again, only the conformal class of g matters. In particular, (2) holds if and only if $X(S,\Gamma) \to (M^n,G)$ is harmonic for $\Gamma = e(g,\mathrm{I})g$. (See [2].)

An immersion $X:(S,g) \to (M^n,G)$ is *minimal* if and only if it is harmonic with $g \alpha \mathrm{I}$. For a minimal immersion, the energy 1 metric $\Gamma = e(g,\mathrm{I})g$ is I itself. It is

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usual to study minimal immersions by taking $g = \mathrm{I}$ from the outset, so that the energy integral is just the ordinary area integral. For a general harmonic immersion, it seems natural to take $g = \Gamma$ from the outset, if only because the energy integral is then just the area integral with respect to $g = \Gamma$. A more compelling reason for the choice is that certain geometric properties of I and Γ are interrelated. In particular, we have the following.

Remark 1. (See [8].) If I is complete on S, so is $\Gamma = e(g,\mathrm{I})g$ for *any* choice of metric g on S. Suppose now that $X:(S,g) \to (M^n,G)$ is harmonic. Then I satisfies the Codazzi-Mainardi equations of classical surface theory with respect to Γ as metric, a property we denote by Cod(Γ,I). Moreover, the curvatures of Γ and I are related by

$$K(\Gamma) \leq \mu K(\mathrm{I}), \tag{3}$$

where $0 < \mu = (\det \mathrm{I}/\det \Gamma) \leq 1$. Here $\mu \equiv 1$ for a minimal immersion, while otherwise, $\mu = 1$ at isolated points on S. From (3) one gets $K(\mathrm{I}) \geq 0$ if $K(\Gamma) \geq 0$. Thus $K(\Gamma) \equiv 0$ if $K(\Gamma) \geq 0$ and $K(\mathrm{I}) \equiv 0$. Similarly, $K(\Gamma) \leq 0$ if $K(\mathrm{I}) \leq 0$. Thus $K(\mathrm{I}) \equiv 0$ if $K(\mathrm{I}) \leq 0$ and $K(\Gamma) \equiv 0$.

The following theorems about minimal immersions are generalized in this paper. Note that reference to G is omitted since standard metrics are used on the n-sphere and Euclidean space. We thank U. Simon for helpful comments related to Theorem 2.

THEOREM 1. (See [3] or [4].) =An immersion $X:(S,\mathrm{I}) \to S^n$ *is minimal if and only if the cone immersion* $\tilde{X}:(S \times \mathbf{R}^+,\tilde{\mathrm{I}}) \to \mathbf{R}^{n+1}$ (described in Section 2) *is minimal with respect to its induced metric* $\tilde{\mathrm{I}} = r^2\mathrm{I} + dr^2$.

THEOREM 2. (See [4], [11] and [12].) *An immersion* $X:(S,\mathrm{I}) \to \mathbf{R}^{n+1}$ *satisfies* $\Delta_{\mathrm{I}}X = \lambda X$ *for a function* $\lambda \neq 0$ *if and only if* $X(S) \subset S^n(r)$ *where* $r^2 = -2/\lambda > 0$ *is constant, and* $X:(S,\mathrm{I}) \to S^n(r)$ *is minimal.*

2. USING THE CONE OVER S TO CHARACTERIZE HARMONIC IMMERSION OF S IN S^n

Given an immersion $X:S \to S^n \subset \mathbf{R}^{n+1}$ with $n \geq 3$, the set

$$C(S) = \{rx/x \in X(S), 0 < r\}$$

is called the (infinite) cone over S. We think of $C(S)$ as the image under the immersion $\tilde{X}:S \times \mathbf{R}^+ \to \mathbf{R}^{n+1}$ where $\tilde{X}(p,r) = rX(p)$ for $r > 0$. The first fundamental form of $\tilde{X}$ is $\tilde{\mathrm{I}} = r^2\mathrm{I} + dr^2$. In fact, to any choice of metric g on S, one associates the metric $\tilde{g} = r^2 g + dr^2$ on $S \times \mathbf{R}^+$. With this understanding, we have the following generalization of Theorem 1.

PROPOSITION 1. *Let* Γ *be the energy* 1 *metric of the immersion*

$$X:(S,g) \to S^n \subset \mathbf{R}^{n+1}$$

and let $\tilde{X}:(S \times \mathbf{R}^+, \tilde{g}) \to \mathbf{R}^{n+1}$ be the associated cone immersion. Then these statements are equivalent.

(i) *X is harmonic if and only if $\tilde{X}$ is harmonic.*

(ii) $g = \Gamma$.

Proof. Assume that g is arbitrary, and that $\tilde{g} = r^2 g + dr^2$. Let II and $\tilde{\mathrm{II}}$ denote the second fundamental forms for X and $\tilde{X}$ respectively. Since II at $X(p)$ and $\tilde{\mathrm{II}}$ at $\tilde{X}(p,r)$ are virtually the same for all $r > 0$ (see [3]), it is easy to check that

$$(4) \qquad tr_g \mathrm{II} = 0 \quad \Leftrightarrow \quad tr_{\tilde{g}} \tilde{\mathrm{II}} = 0.$$

In terms of local coordinates on S and S^n, let β be the vector valued 2-form on S defined by

$$(5) \qquad \beta_{ij}^{\gamma} = \mathrm{II}_{ij}^{\gamma} + (\overset{\mathrm{I}}{\Gamma}{}_{ij}^{k} - \overset{g}{\Gamma}{}_{ij}^{k}) \frac{\partial X^{\gamma}}{\partial x^k}$$

where $i, j, k = 1, 2$ and $\gamma = 1, 2, \ldots, n$. By comparing the definition of II in formula (116.4) of [10] with the formula for ∇dX in (3.1) of [1], one sees that β is the form ∇dX studied in [1] and [12]. It follows (see [1]) that X is harmonic if and only if the tension field $tr_g \beta \equiv 0$. Similarly, $\tilde{X}$ is harmonic if and only if its tension field $tr_{\tilde{g}} \tilde{\beta} \equiv 0$, where $\tilde{\beta}$ is the vector valued 2-form on $S \times \mathbf{R}^+$ defined by

$$(6) \qquad \tilde{\beta}_{ij}^{\gamma} = \tilde{\mathrm{II}}_{ij}^{\gamma} + (\overset{\tilde{\mathrm{I}}}{\Gamma}{}_{ij}^{k} - \overset{\tilde{g}}{\Gamma}{}_{ij}^{k}) \frac{\partial \tilde{X}^{\gamma}}{\partial x^k}.$$

Here $i, j, k = 1, 2, 3$ and $\gamma = 1, 2, \ldots, n+1$.

We must show $tr_g \beta \equiv 0$ equivalent to $tr_{\tilde{g}} \tilde{\beta} \equiv 0$ if and only if $g = \Gamma$. Keeping g arbitrary, introduce coordinates x^1, x^2 on S isothermal for g, and use the coordinates x^1, x^2, r on $S \times \mathbf{R}^+$. Then

$$g = \lambda\{(dx^1)^2 + (dx^2)^2\} \text{ and } \tilde{g} = r^2\{(dx^1)^2 + (dx^2)^2\} + dr^2,$$

so that

$$\overset{g}{\Gamma}{}_{11}^{1} + \overset{g}{\Gamma}{}_{22}^{1} = \overset{g}{\Gamma}{}_{11}^{2} + \overset{g}{\Gamma}{}_{22}^{2} = 0,$$

while

$$\overset{\tilde{g}}{\Gamma}{}_{11}^{1} + \overset{\tilde{g}}{\Gamma}{}_{22}^{1} = \overset{\tilde{g}}{\Gamma}{}_{11}^{2} + \overset{\tilde{g}}{\Gamma}{}_{22}^{2} = \overset{\tilde{g}}{\Gamma}{}_{33}^{1} = \overset{\tilde{g}}{\Gamma}{}_{33}^{2} = \overset{\tilde{g}}{\Gamma}{}_{33}^{3} = 0,$$
$$\overset{\tilde{g}}{\Gamma}{}_{11}^{3} = \overset{\tilde{g}}{\Gamma}{}_{22}^{3} = -r\lambda.$$

Because II_{ij} is the normal component of β_{ij}, (5) shows that $tr_g \beta \equiv 0$ if and only if $tr_g \mathrm{II} \equiv 0$ and

$$\overset{\mathrm{I}}{\Gamma}{}^{1}_{11} + \overset{\mathrm{I}}{\Gamma}{}^{1}_{22} \equiv \overset{\mathrm{I}}{\Gamma}{}^{2}_{11} + \overset{\mathrm{I}}{\Gamma}{}^{2}_{22} \equiv 0. \tag{7}$$

Similarly, since $\tilde{\mathrm{II}}_{ij}$ is the normal component of $\tilde{\beta}_{ij}$, (6) shows that $tr_{\tilde{g}}\tilde{\beta} \equiv 0$ if and only if $tr_{\tilde{g}}\tilde{\mathrm{II}} \equiv 0$ and

$$\begin{aligned} &\overset{\tilde{\mathrm{I}}}{\Gamma}{}^{1}_{11} + \overset{\tilde{\mathrm{I}}}{\Gamma}{}^{1}_{22} + r^2\lambda \overset{\tilde{\mathrm{I}}}{\Gamma}{}^{1}_{33} \equiv \overset{\tilde{\mathrm{I}}}{\Gamma}{}^{2}_{11} + \overset{\tilde{\mathrm{I}}}{\Gamma}{}^{2}_{22} + r^2\lambda \overset{\tilde{\mathrm{I}}}{\Gamma}{}^{2}_{33} \equiv 0, \\ &\overset{\tilde{\mathrm{I}}}{\Gamma}{}^{3}_{11} + \overset{\tilde{\mathrm{I}}}{\Gamma}{}^{3}_{22} - (\overset{\tilde{g}}{\Gamma}{}^{3}_{11} + \overset{\tilde{g}}{\Gamma}{}^{3}_{22}) + r^2\lambda \overset{\tilde{\mathrm{I}}}{\Gamma}{}^{3}_{33} \equiv 0. \end{aligned} \tag{8}$$

Since (4) holds for any choice of g, it is enough to check that (7) and (8) are equivalent if and only if $g = \Gamma$. If $\mathrm{I} = E(dx^1)^2 + 2F dx^1 dx^2 + G(dx^2)^2$, the fact that $\tilde{\mathrm{I}} = r^2\mathrm{I} + dr^2$ yields

$$\begin{aligned} &\overset{\tilde{\mathrm{I}}}{\Gamma}{}^{1}_{11} = \overset{\mathrm{I}}{\Gamma}{}^{1}_{11}, \quad \overset{\tilde{\mathrm{I}}}{\Gamma}{}^{1}_{22} = \overset{\mathrm{I}}{\Gamma}{}^{1}_{22}, \quad \overset{\tilde{\mathrm{I}}}{\Gamma}{}^{2}_{11} = \overset{\mathrm{I}}{\Gamma}{}^{2}_{11}, \quad \overset{\tilde{\mathrm{I}}}{\Gamma}{}^{2}_{22} = \overset{\mathrm{I}}{\Gamma}{}^{2}_{22}, \\ &\overset{\tilde{\mathrm{I}}}{\Gamma}{}^{1}_{33} = \overset{\tilde{\mathrm{I}}}{\Gamma}{}^{2}_{33} = \overset{\tilde{\mathrm{I}}}{\Gamma}{}^{3}_{33} = 0, \quad \overset{\tilde{\mathrm{I}}}{\Gamma}{}^{3}_{11} = -rE, \quad \overset{\tilde{\mathrm{I}}}{\Gamma}{}^{3}_{22} = -rG. \end{aligned}$$

Thus (7) and (8) are equivalent if and only if

$$2\lambda = E + G. \tag{9}$$

But by (1), (9) is the statement that $e(g,\mathrm{I}) = 1$, so the proof is complete.

3. USING THE LAPLACIAN TO CHARACTERIZE HARMONIC IMMERSION OF S IN S^n

It is known (see [12]) that $X\colon (S,g) \to S^n \subset \mathbf{R}^{n+1}$ is harmonic if and only if $\Delta_g X = \lambda X$ for a function $\lambda \neq 0$. Setting $S^n(r) = \{x/|x| = r\}$, this same statement holds with $S^n(r)$ in place of S^n. A more exact result is found in 4.14 of [1]. It states that $X\colon (S,g) \to S^n \subset \mathbf{R}^{n+1}$ is harmonic if and only if $\Delta_g X = -2e(g,\mathrm{I})X$. This means that $X\colon (S,g) \to S^n(r) \subset \mathbf{R}^{n+1}$ is harmonic if and only if

$$\Delta_g X = -2e(g,\mathrm{I})X/r^2. \tag{10}$$

To complete the picture established by these facts, we have the following generalization of Theorem 2.

PROPOSITION 2. *An immersion $X\colon (S,g) \to \mathbf{R}^{n+1}$ with energy 1 metric Γ satisfies*

(a) $\Delta_\Gamma X = \lambda X$ *for a function* $\lambda \neq 0$, *and*

(b) $\mathrm{Cod}(\Gamma,\mathrm{I})$

if and only if

(i) $X(S) \subset S^n(r)$ *where* $r^2 = -2/\lambda > 0$ *is constant, and*

(ii) $X:(S,g) \to S^n(r)$ *is harmonic.*

Remark 2. Here is an example to show that condition (b) cannot be deleted in Proposition 2. Let S be the square $0 < \sqrt{2}x < \pi/2, 0 < \sqrt{2}y < \pi/2$ in $\mathbf{R}^2$, and define X by

$$\sqrt{2}\,X(x,y) = (\sin\sqrt{2}x, \cos\sqrt{2}x + \sin\sqrt{2}y, \cos\sqrt{2}y),$$

so that $|X| = 1 + \cos\sqrt{2}x \sin\sqrt{2}y$, which is not constant. Take $g = dx^2 + dy^2 = \Gamma$. Then $e(\Gamma,\mathrm{I}) = 1$ and $\Delta_\Gamma X = -2X$, even though $X(S)$ lies in no 2-sphere. Condition (b) is implicit in Theorem 2, since $\Gamma = \mathrm{I}$ there, and Cod(I,I) is automatic.

Remark 3. Let $\mathrm{I} = Edx^1 + 2Fdxdy + Gdy^2$ for any conformal parameter $z = x + iy$ on the Riemann surface R_g determined by S on g. Let Γ be the energy 1 metric of some immersion $X:(S,g) \to (M^n,G)$. In [8] we show that condition (b) holds if and only if the quadratic differential $\Omega = (E - G - 2iF)dz^2$ on R_g is holomorphic. On the other hand, Ω is holomorphic on R_g (see [5] or [6]) if and only if the vector field on S given by the left side of (2) has no component tangent to $X(S)$. In Proposition 1 we have an immersion $X:(S,g) \to \mathbf{R}^{n+1}$, so that $\Delta_g X$ is the left side of (2), and (b) is equivalent to the assumption that $\Delta_g X$ or $\Delta_\Gamma X$ is normal to $X(S)$.

Proof of Proposition 2. If (i) and (ii) hold, (10) yields (a), and (b) follows by Remark 3. if (a) and (b) hold, Remark 3 shows that $\Delta_\Gamma X$, and therefore X, must be normal to $X(S)$. Thus, for each vector field Y tangent to $X(S)$,

$$Y\langle X,X\rangle = 2\langle Y(X),X\rangle = 0,$$

where $\langle\ ,\ \rangle$ is the Euclidean inner product, and $\langle X,X\rangle$ is a constant which we call $r^2 \neq 0$, so that $X(S) \subset S^n(r)$. Now (a) implies that $X:(S,g) \to S^n(r)$ is harmonic, and (10) gives the fact that $\lambda = -2/r^2$ as required.

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