AN EIGHT-TERM EXACT SEQUENCE ASSOCIATED WITH A GROUP EXTENSION

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Let $1 \to N \to G \to Q \to 1$ be an extension of groups and let A be a G-module. We shall denote by A^N the submodule of A consisting of all N-invariant elements. Hochschild and Serre [4], using spectral sequences, proved that the sequence

$$0 \to H^n(Q,A^N) \to H^n(G,A) \to H^n(N,A)^Q \to H^{n+1}(Q,A^N) \to H^{n+1}(G,A)$$

is exact provided $H^i(N,A) = 0$ for $0 < i < n$. In case $n = 1$, the sequence was extended (to the right) to three more terms by Huebschmann [6] which includes the exact sequence

$$H^2(Q,A^N) \to H^2(G,A) \to \mathrm{XPext}(G,N;A) \to H^3(Q,A^N) \to H^3(G,A)$$

where $\mathrm{XPext}(G,N;A)$ denotes the abelian group of equivalence classes of "crossed pairs." In this paper we show that this can be done even when $n > 1$. More explicitly we prove that if $H^i(N,A) = 0$ for $0 < i < n$, then the following sequence is exact

$$\begin{aligned} 0 \to H^n(Q,A^N) \to H^n(G,A) \to \mathrm{Sext}_G^{n-1}(N,A) \to H^{n+1}(Q,A^N) \to H^{n+1}(G,A) \\ \to \mathrm{Sext}_G^{n}(N,A) \to H^{n+2}(Q,A^N) \to H^{n+2}(G,A) \end{aligned}$$

where $\mathrm{Sext}_G^n(N,A)$ denotes the abelian group of equivalence classes of pseudo n-fold extensions of A by N (see Section 2). Note that no spectral sequences are used in the proofs contained in this paper.

In Section 1 we recall the definition of pseudo modules and define pseudo extensions. In Section 2 we derive a long exact sequence of "Sext" in the second variable (which is natural in the first variable). In Section 3 we derive the sequence of Huebschmann to show that it is in fact natural in the variable A. In Section 4 we deduce the main result.

1. PSEUDO MODULES AND PSEUDO EXTENSIONS

Let E be a group with normal subgroup X. Let $\mathrm{Aut}_X E$ denote the group of automorphisms of E which map X onto itself. The subgroup of all automorphisms that "conjugate" by elements of X (i.e. $\phi : E \to E$ such that $\phi(e) = xex^{-1}$ for some x) will be denoted by $c(X)$. If G is a group then a *pseudo G-action* on E is a group homomorphism $\theta : G \to \mathrm{Aut}_X E/c(X)$ for some normal subgroup X of E. For convenience we write $g * e = \hat{\theta}(g)(e)$, $g \in G$, $e \in E$ where $\hat{\theta}(g)$ is a previously chosen element of the coset $\theta(g)$. If $\hat{\theta}_1(g)$ is another element of $\theta(g)$ then $g * e = x\hat{\theta}_1(g)(e)x^{-1}$

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for some $x \in X$. We shall abbreviate this by $g * e \underset{X}{\equiv} \hat{\theta}_1(g)(e)$. Suppose $\theta_i : G \to \mathrm{Aut}_{X_i} E_i / c(X_i)$ is a pseudo G-action on E_i, $i = 1, 2$. Then a group homomorphism $f : E_1 \to E_2$ is a *pseudo G-map* if $f(X_1) \subseteq X_2$ and

$$c(x) f \hat{\theta}_1(g) = \hat{\theta}_2(g) f \qquad \text{for some} \quad x \in X_2$$

A *pseudo module* is a pair (∂, θ) where $\partial : E \to G$ is a group homomorphism with kernel X such that the image N of ∂ is normal in G and where $\theta : G \to \mathrm{Aut}_X E / c(X)$ is a pseudo G-action on E such that the following diagram is commutative

(1)
$$\begin{array}{ccccc} E & \xrightarrow{\partial} & G & & \\ c_E \downarrow & & \theta \downarrow & \searrow^{c_N} & \\ \mathrm{Aut}_X E & \underset{\lambda}{\to} & \mathrm{Aut}_X E / c(X) & \underset{\rho}{\to} & \mathrm{Aut}\, N. \end{array}$$

Here $c_E(e)(e') = ee'e^{-1}$, $c_N(g)(n) = gng^{-1}$, $e, e' \in E$, $g \in G$, $n \in N$, and λ denotes the canonical projection. Furthermore if $f \in \mathrm{Aut}_X E$ then $\rho(\bar{f})$ is the automorphism that makes the following diagram commutative

$$\begin{array}{ccccccccc} 1 & \to & X & \to & E & \xrightarrow{\partial} & N & \to & 1 \\ & & \downarrow & & \downarrow f & & \downarrow \rho(\bar{f}) & & \\ 1 & \to & X & \to & E & \underset{\partial}{\to} & N & \to & 1. \end{array}$$

Using the abbreviation introduced in the first paragraph one sees that the commutative diagram (1) is equivalent to the following conditions:

(P1) $$\partial(e) * e' \underset{X}{\equiv} ee'e^{-1} \qquad \text{where } e, e' \in E.$$

(P2) $$\partial(g * e) = g\partial(e)g^{-1} \qquad e \in E, g \in G.$$

Note that (P2) implies that θ induces a pseudo G-action $G \to \mathrm{Aut}_X X / c(X)$. Thus if X is abelian then it has a G-module structure induced by θ. It is easy to see that if X is central in E then (∂, θ) is simply a crossed module. (In this case we shall denote it by ∂.)

We say that the pseudo module (∂, θ) is *extendible* if there exists a commutative diagram of groups with exact rows

(2)
$$\begin{array}{ccccccccc} & & 1 & \to & E & \to & H & \to & Q & \to 1 \\ & & & & \| & & \downarrow \gamma & & \| & \\ 1 & \to & X & \to & E & \xrightarrow{\partial} & G & \to & Q & \to 1 \end{array}$$

where $\gamma(h) * e \underset{X}{\equiv} heh^{-1}$, where $e \in E$ and $h \in H$. The group H will be called an *extension* of (∂, θ). Taylor [13, Section 6] reduced the problem in determining extendibility of a pseudo module to that of an associated crossed module.

PROPOSITION 1.1. *Suppose* (∂,θ) *is a pseudo module. Then there exists a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 1 \to & X & \to & E & \xrightarrow{\partial} & G & \to & Q & \to 1 \\ & \uparrow & & \| & & \uparrow\beta & & \| & \\ 1 \to & X \cap Z(E) & \to & E & \xrightarrow{\partial'} & \Gamma & \to & Q & \to 1 \end{array} \tag{3}$$

where $Z(E)$ *is the center of* E *and* Γ *is constructed from the pullback diagram*

$$\begin{array}{ccc} \Gamma & \xrightarrow{\beta} & G \\ \alpha\downarrow & & \downarrow\theta \\ \mathrm{Aut}_X E & \xrightarrow{\lambda} & \mathrm{Aut}_X E/c(X). \end{array}$$

Furthermore (∂',α) *is a crossed module and it is extendible if and only if* (∂,θ) *is. In this case the extensions are the same. We shall refer to* (∂',α) *as the associated crossed module of* (∂,θ).

Let G be a group with a normal subgroup N, and let A be a G-module. A *pseudo G-extension* of A by N is an exact sequence of groups $0 \to A \to E \xrightarrow{\pi} N \to 1$ together with a pseudo G-action $\theta: G \to \mathrm{Aut}_A E/c(A)$ such that $(i\pi,\theta)$ is a pseudo module and the given G-module structure on A coincides with the one induced by θ. (Here $i: N \to G$ denotes the inclusion map.) It is not hard to see, using (P1), that $A^N = A \cap Z(E)$. Two pseudo G-extensions $0 \to A \to E_i \to N \to 1$, $i = 1, 2$, are *equivalent* if there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & A & \to & E_1 & \to & N & \to 1 \\ & \| & & \downarrow f & & \| & \\ 0 \to & A & \to & E_2 & \to & N & \to 1 \end{array}$$

where f is a pseudo G-map. It is clear that this gives an equivalence relation on these pseudo G-extensions. We shall denote the set of equivalence classes by $\mathrm{Sext}^1_G(N,A)$ and show that it can be made into an abelian group in a natural way.

LEMMA 1.2. *Suppose* $0 \to A \to E \to N \to 1$ *is a pseudo G-extension and suppose* $\alpha: A \to B$ *is a G-module homomorphism.*

(a) *There exists a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 \to & A & \xrightarrow{\mu} & E & \xrightarrow{\pi} & N & \to 1 \\ & \alpha\downarrow & & \gamma\downarrow & & \| & \\ 0 \to & B & \xrightarrow{\beta} & P & \xrightarrow{\pi'} & N & \to 1 \end{array} \tag{4}$$

where the bottom row can be made into a pseudo G-extension such that γ *is a pseudo G-map.*

(b) *The left square of diagram* (4) *has the following "push-out" property: if there exists group homomorphisms* $\beta': B \to X$ *and* $\gamma': E \to X$ *with* $\beta'\alpha = \gamma'\mu$ *such* *that* $\beta'(\pi(e)\circ b) = \gamma'(e)\beta'(b)\gamma'(e)^{-1}$ *for* $e \in E$ *and* $b \in B$ *then there exists a unique group homomorphism* $\phi : P \to X$ *with* $\phi\beta = \beta'$ *and* $\phi\gamma = \gamma'$.

Proof. (a) It is easy to check that $S = \{(-\alpha a, \mu a) \mid a \in A\}$ is a normal subgroup of the semidirect product $B \rtimes E$ where the E-action on B is via π. Set $P = B \rtimes E/S$ and define $\pi' : P \to N$ by $\pi'[b,e] = \pi e$, $b \in B$, $e \in E$. Then diagram (4) is commutative with exact rows. Define $\theta_1 : G \to \mathrm{Aut}_B P/c(B)$ by $\theta_1(g) = [\hat\theta_1(g)]$ where $\hat\theta_1(g)[b,e] = [g\circ b, \hat\theta(g)(e)]$, $\theta(g) = [\hat\theta(g)]$, $g \in G$ and $[b,e] \in P$. We abbreviate this by $g * [b,e] \underset{B}{\equiv} [g\circ b, g * e]$. Using (P2) of θ together with the fact that $\hat\theta(g)$ is a homomorphism it is not hard to see that $\hat\theta_1(g) \in \mathrm{Aut}_B P$. Now

$$\begin{aligned}
\hat\theta_1(g_1g_2)[b,e] &= [g_1g_2\circ b, (g_1g_2) * e] \\
&= (g_1g_2\circ b, \mu a(g_1 * (g_2 * e))(\mu a)^{-1}) \\
&= (0,\mu a)(g_1g_2\circ b, g_1 * (g_2 * e))(0,(\mu a)^{-1}) \\
&= (\alpha a,1)(g_1g_2\circ b, g_1 * (g_2 * e))(-\alpha a,1) \\
&\underset{B}{\equiv} \hat\theta_1(g_1)\hat\theta_1(g_2)
\end{aligned}$$

and so θ_1 is a group homomorphism. It remains to show that the bottom sequence of (4) is a pseudo G-extension. It is easy to see that $(i\pi', \theta_1)$ satisfies (P2) since $(i\pi, \theta)$ does. Now

$$\begin{aligned}
\pi'[b,e] * [b',e'] &= \pi e * [b',e'] \\
&= [\pi e\circ b', \pi e * e'] \\
&= [e\circ b', (\mu a)ee'e^{-1}(\mu a)^{-1}] \\
&= [0,\mu a]\,[e\circ b', ee'e^{-1}]\,[0,\mu a^{-1}] \\
&= [\alpha a,1]\,[e\circ b', ee'e^{-1}]\,[-\alpha a,1] \\
&\underset{B}{\equiv} [e\circ b', ee'e^{-1}] \\
&\underset{B}{\equiv} [b,1]\,[e\circ b', ee'e^{-1}]\,[-b,1] \\
&= [b,e]\,[b',e']\,[-e^{-1}\circ b, e^{-1}] \\
&= [b,e]\,[b',e']\,[b,e]^{-1}
\end{aligned}$$

i.e. $(i\pi', \theta_1)$ satisfy (P1). Clearly the G-module structure on B is induced by θ_1 and that γ is a pseudo G-map.

(b) Define $\phi : P \to X$ by $\phi[b,e] = (\beta' b)(\gamma' e)$, $[b,e] \in P$, and the result follows.

LEMMA 1.3. *Given a pseudo* G*-extension* $0 \to A \to E \to N \to 1$ *and a group homomorphism* $\phi : G' \to G$ *with* $\phi(N') \subseteq N$ *where* N' *is normal in* G' *construct the commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
0 \to & A & \to & E' & \xrightarrow{\pi'} & N' & \to 1 \\
 & \| & & \phi'\downarrow & PB & \downarrow\phi & \\
0 \to & A & \xrightarrow{\mu} & E & \xrightarrow{\pi} & N & \to 1
\end{array}$$

where PB denotes pullback of groups. Then the top row can be made into a pseudo G'-extension with ϕ' a pseudo G'-map (where the pseudo G'-action on E is via ϕ).

Proof. We may take $E' = \{(e,n') \mid \pi e = \phi n'\}$, a subgroup of $E \times N'$. Define $\theta_1 : G' \to \mathrm{Aut}_A E'/c(A)$ by $\theta_1(g) = [\hat{\theta}_1(g)]$ where

$$\hat{\theta}_1(g)(e,n') = (\hat{\theta}(\phi(g))(e), gn'g^{-1}),$$

$g \in G'$ and $\theta(x) = [\hat{\theta}(x)]$. Since $\hat{\theta}(x) \in \mathrm{Aut}_A E$, one checks that $\hat{\theta}_1(g) \in \mathrm{Aut}_A E'$. Now

$$\begin{aligned} \hat{\theta}_1(g_1g_2)(e,n') &= (\hat{\theta}(\phi(g_1g_2))(e), g_1g_2n'(g_1g_2)^{-1}) \\ &= ((\mu a)(\hat{\theta}(\phi(g_1))\,\hat{\theta}(\phi(g_2))(e))(\mu a^{-1}), g_1g_2n'(g_1g_2)^{-1}) \\ &= (\mu a,1)(\hat{\theta}_1(g_1)\,\hat{\theta}(g_2)(e,n'))(\mu a^{-1},1) \end{aligned}$$

and so θ_1 is a group homomorphism. Finally $(i\pi',\theta_1)$ is a pseudo module since (P1) and (P2) follow from the following computations:

$$\begin{aligned} \pi'(e,n) * (e_1,n_1') &= n' * (e_1,n_1') \\ &= (\hat{\theta}(\phi(n'))(e_1), n'n_1'(n')^{-1}) \\ &= (\hat{\theta}(\pi e)(e_1), n'n_1'(n')^{-1}) \\ &= ((\mu a)ee_1e^{-1}(\mu a^{-1}), n'n_1'(n')^{-1}) \\ &= (\mu a,1)(ee_1e^{-1}, n'n_1'(n')^{-1})(\mu a^{-1},1) \\ &= (\mu a,1)(e,n')(e_1,n_1')(e,n')^{-1}(\mu a,1)^{-1} \\ \pi'(g * (e,n')) &= \pi'(\hat{\theta}(\phi(g))(e), gn'g^{-1}) \\ &= gn'g^{-1} = g\pi'(e,n')g^{-1}. \end{aligned}$$

Suppose $\epsilon_i : 0 \to A \to E_i \xrightarrow{\pi_i} N \to 1$, $i = 1, 2$, are pseudo G-extensions. Construct the commutative diagram

$$\begin{array}{ccccccccc} \epsilon_1 \times \epsilon_2 : 0 \to & A \times A & \to & E_1 \times E_2 & \to & N \times N & \to 1 \\ & \| & & \uparrow & PB & \uparrow\Delta & \\ 0 \to & A \times A & \xrightarrow{j} & P & \to & N & \to 1 \\ & \nabla\downarrow & PO & \downarrow & & \| & \\ \epsilon_1 + \epsilon_2 : 0 \to & A & \to & E & \to & N & \to 1 \end{array}$$

where PB denotes pullback of groups, Δ and ∇ denotes the diagonal and the addition map respectively, and where PO denotes the pushout construction of Lemma 1.2. Since $\epsilon_1 \times \epsilon_2$ is a pseudo $G \times G$-extension the middle row is a pseudo G-extension by Lemma 1.3 and so $\epsilon_1 + \epsilon_2$ is a pseudo G-extension by Lemma 1.2. Define addition on $\mathrm{Sext}^1_G(N,A)$ by $[\epsilon_1] + [\epsilon_2] = [\epsilon_1 + \epsilon_2]$. Note that we may let

$$P = \{(e_1,e_2) \mid \pi_1 e_1 = \pi_2 e_2\}$$

and $E = A \rtimes P/S$ where $S = \{(-\nabla(a,b), j(a,b)) \mid a,b \in A\}$. Hence the pseudo G-action on E is given by $g * [a,e_1,e_2] = [g \circ a, g * e_1, g * e_2]$. It is straight forward to show by the usual extension theory arguments that the addition is well-defined. The identity is represented by the split extension $0 \to A \to A \rtimes N \to N \to 1$ with the (pseudo) G-action defined by $g \circ (a,n) = (g \circ a, gng^{-1})$. The inverse of the element represented by $0 \to A \xrightarrow{\mu} E \xrightarrow{\pi} N \to 1$ is represented by $0 \to A \xrightarrow{-\mu} E \xrightarrow{\pi} N \to 1$ where $(-\mu)(a) = a^{-1}$, $a \in A$.

Remark. Pseudo G-extensions clearly generalize the notion of G-crossed extensions of [11]. The group $\mathrm{Sext}^1_G(N,A)$ is in one-to-one correspondence with $\mathrm{Xpext}(G,N;A)$ of [6]. For one could define $\varphi : \mathrm{Sext}^1_G(N,A) \to \mathrm{XPext}(G,N;A)$ as follows. Let $\epsilon : 0 \to A \to E \xrightarrow{\pi} N \to 1$ represent an element of $\mathrm{Sext}^1_G(N,A)$ with pseudo G-action θ. Note that (P1) and (P2) are equivalent to the fact that the image of θ lies in $\mathrm{Aut}^A_G E/c(A)$. Define $\tilde{\psi} : G \to \mathrm{Out}_G(e_N)$ by $\tilde{\psi}(g) = [(\hat{\theta}(g), g)]$ where $\theta(g) = [\hat{\theta}(g)]$. $\tilde{\psi}$ is well-defined and $\tilde{\psi}(n) = [(C_e, e)] = 0$ where $e \in E$ and C_e denotes conjugation by e. Thus $\tilde{\psi}$ induces $\psi_\epsilon : Q \to \mathrm{Out}_G(e_N)$. Define $\varphi[\epsilon] = [(\epsilon, \psi_\epsilon)]$ and φ is bijective.

2. A LONG EXACT SEQUENCE OF SEXT

Let N be a normal subgroup of G and let A be a G-module. An *n-fold pseudo G-extension* ($n > 1$) of A by N is an exact sequence of groups

$$0 \to A \to M_1 \to \dots \to M_{n-1} \to M_n \xrightarrow{\pi} N \to 1$$

such that $0 \to A \to M_1 \to \dots \to \ker \pi \to 0$ is exact as G-modules and that $0 \to \ker \pi \to M_n \to N \to 1$ is a pseudo G-extension. Two such extensions ϵ, ϵ' are related if there exists a commutative diagram

$$\begin{array}{rcccccccccc} \epsilon : 0 \to & A & \to & M_1 & \to \dots \to & M_{n-1} & \to & M_n & \to & N & \to 1 \\ & \| & & \downarrow \alpha_1 & & \downarrow \alpha_{n-1} & & \downarrow \alpha_n & & \| & \\ \epsilon' : 0 \to & A & \to & M'_1 & \to \dots \to & M'_{n-1} & \to & M'_n & \to & N & \to 1 \end{array}$$

where α_i is a G-module homomorphism, $1 \le i \le n-1$, and α_n is a pseudo G-map. The set of equivalence classes of these extensions determined by the above relation will be denoted by $\mathrm{Sext}^n_G(N,A)$. Define an addition on it by $[\epsilon] + [\epsilon'] = [\epsilon + \epsilon']$ where $\epsilon + \epsilon'$ is the bottom row of the commutative diagram

$$\begin{array}{rcccccccccc} \epsilon \times \epsilon' : 0 \to & A \times A & \to & M_1 \times M'_1 & \to \dots \to & M_{n-1} \times M'_{n-1} & \to & M_n \times M'_n & \to & N \times N & \to 1 \\ & \nabla \downarrow & PO & \downarrow & & \| & & \uparrow & PB & \uparrow \Delta & \\ \epsilon + \epsilon' : 0 \to & A & \to & M''_1 & \to \dots \to & M_{n-1} \times M'_{n-1} & \to & M''_n & \to & N & \to 1 \end{array}$$

Using Lemmas 1.2 and 1.3 it is easy to see that the addition is well-defined, associative and commutative. Note that the identity is represented by

$$0 \to A = A \to 0 \to \dots \to N = N \to 1$$

with the obvious (pseudo) G-action on N. Similar to the case $n = 1$ it can be checked that $0 \to A \overset{-\mu}{\to} M_1 \to \ldots \to M_n \to N \to 1$ represents the inverse of the element represented by $0 \to A \overset{\mu}{\to} M_1 \to \ldots \to M_n \to N \to 1$.

LEMMA 2.1. *Suppose* $[\xi] \in \mathrm{Sext}_G^n(N,A)$, $n > 1$. *Then* $[\xi] = 0$ *if and only if there exists a commutative diagram*

$$
\begin{array}{cccccccccccc}
\xi: 0 \to & A & \to & M_1 & \to & M_2 & \to \ldots \to & M_{n-1} & \to & M_n & \to & N \to 1 \\
 & & & {\scriptstyle\alpha_1}\uparrow & & {\scriptstyle\alpha_2}\uparrow & & {\scriptstyle\alpha_{n-1}}\uparrow & & {\scriptstyle\alpha_n}\uparrow & & \| \\
 & 0 & \to & M_1' & \to & M_2' & \to \ldots \to & M_{n-1}' & \to & M_n' & \to & N \to 1
\end{array} \tag{5}
$$

where the bottom row is an $(n-1)$*-fold pseudo G-extension,* α_n *is a pseudo G-map and* α_i *is a G-map,* $1 \le i \le n-1$. *Furthermore* M_1' *may be taken to be* M_1 *and* α_1 *the identity.*

Proof. The sufficiency follows immediately from the following diagram

$$
\begin{array}{cccccccccc}
\xi: 0 \to A \to & M_1 & \to & M_2 & \to \ldots \to & M_{n-1} & \to & M_n & \to & N \to 1 \\
\| & {\scriptstyle\gamma}\uparrow & & \uparrow & & \uparrow & & \uparrow & & \| \\
0 \to A \to & A \times M_1' & \to & M_2' & \to \ldots \to & M_{n-1}' & \to & M_n' & \to & N \to 1 \\
\| & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| \\
0 \to A = & A & \to & 0 & \to \ldots \to & 0 & A & N & = & N \to 1
\end{array}
$$

where $\gamma(a,m) = a + \alpha_1(m)$, $a \in A$, $m \in M_1'$. Suppose $[\xi] = 0$. Then there exists a commutative diagram (see added in proof)

$$
\begin{array}{cccccccccccc}
\xi: 0 \to & A & \to & M_1 & \to & M_2 & \to \ldots \to & M_{n-1} & \to & M_n & \to & N \to 1 \\
 & \| & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \| \\
0 \to & A & \to & M_1'' & \to & M_2'' & \to \ldots \to & M_{n-1}'' & \to & M_n'' & \to & N \to 1 \\
 & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| \\
0 \to & A & = & A & \to & 0 & \to \ldots \to & 0 & \to & N & = & N \to 1
\end{array}
$$

and, therefore, $A \to M_1''$ splits. Hence $0 \to A \to M_1'' \to R \to 0$ is right split. The composition of $M_1'' \to M_1$ with such a splitting gives a G-map $\alpha_1 : R \to M_1$. Setting $M_1' = R$ and $M_i' = M_i''$, $1 < i \le n$, we have diagram (5). To see that M_1' may be taken to be M_1 and α_1 the identity construct the pushout diagram of G-modules if $n > 2$ and according to the pushout construction of Lemma 1.2 if $n = 2$

$$
\begin{array}{ccc}
M_1' & \to & M_2' \\
{\scriptstyle\alpha_1}\downarrow & & \downarrow \\
M_1 & \to & P
\end{array}
$$

Then the result follows by taking $M_1' = M_1$, $M_2' = P$.

PROPOSITION 2.2. *Suppose* $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ *is an exact sequence of G-modules. Then the following sequence is exact*

$$\mathrm{Sext}^n_G(N,A) \xrightarrow{i^*_n} \mathrm{Sext}^n_G(N,B) \xrightarrow{p^*_n} \mathrm{Sext}^n_G(N,C) \xrightarrow{\delta_n} \mathrm{Sext}^{n+1}_G(N,A) \xrightarrow{i^*_{n+1}} \mathrm{Sext}^{n+1}_G(N,B)$$

for all $n \geq 1$.

Proof. For $[\epsilon_A] \in \mathrm{Sext}^n_G(N,A)$ define $i^*_n[\epsilon_A] = [i(\epsilon_A)]$ where $i(\epsilon_A)$ is the bottom row of the commutative diagram

$$\begin{array}{ccccccccccccc} \epsilon_A: 0 \to & A & \to & M_1 & \to & M_2 \to \ldots \to & M_{n-1} & \to & M_n & \to & N \to 1 \\ & i\downarrow & PO & \downarrow & & \| & \| & & \| & & \| \\ i(\epsilon_A): 0 \to & B & \to & M'_1 & \to & M_2 \to \ldots \to & M_{n-1} & \to & M_n & \to & N \to 1 \end{array}$$

where PO denotes the construction in Lemma 1.2 if $n = 1$ and pushout of G-modules otherwise. The map p^*_n is defined similarly via p. For $[\epsilon_C] \in \mathrm{Sext}^n_G(N,C)$ define $\delta_n[\epsilon_C] = [\delta\epsilon_C]$ where $\delta\epsilon_C$ is the sequence $0 \to A \to B \to M_1 \to \ldots \to M_n \to N \to 1$ obtained by splicing together the sequences

$$0 \to A \to B \to C \to 0 \quad \text{and} \quad \epsilon_C: 0 \to C \to M_1 \to \ldots \to M_n \to N \to 1.$$

It is routine to check that these are well-defined homomorphisms using Lemmas 1.2 and 1.3.

Exactness at $\mathrm{Sext}^n_G(N,B)$. Since $pi = 0$ we have $p^*_n i^*_n = 0$ using the argument in the proof of Lemma 2.1. Suppose $p^*_n[\epsilon] = 0$. Then, by Lemma 2.1, there exists a commutative diagram

$$\begin{array}{ccccccccccc} \epsilon: 0 \to & B & \to & M_1 & \to & M_2 & \to \ldots \to & M_n & \to & N \to 1 \\ & p\downarrow & PO & \downarrow p' & & \| & & \| & & \| \\ p(\epsilon): 0 \to & C & \to & M'_1 & \to & M_2 & \to \ldots \to & M_n & \to & N \to 1 \\ & & & \| & & \uparrow & & \uparrow & & \| \\ & 0 & \to & M'_1 & \underset{d}{\to} & M'_2 & \to \ldots \to & M'_n & \to & N \to 1 \end{array}$$

unless $n = 1$, in which case we have a commutative diagram

$$\begin{array}{ccccccc} \epsilon: 0 \to & B & \to & M_1 & \to & N \to 1 \\ & p\downarrow & & \uparrow & & \| \\ p(\epsilon): 0 \to & C & \to & M'_1 & \underset{\mu}{\rightleftarrows} & N \to 1 \end{array}$$

where the bottom row has a right splitting pseudo G-map μ. In the first case it is easy to see that $[\epsilon] = [i\xi]$ where $\xi: 0 \to A \to M_1 \xrightarrow{dp'} M'_2 \to \ldots \to M'_n \to N \to 1$. As for the latter case ξ is the bottom row of the commutative diagram

$$\begin{array}{ccccccc} 0 \to A \to & M_1 & \to & M'_1 & \to 1 \\ \| & \uparrow & PB & \uparrow\mu & \\ 0 \to A \to & E & \to & N & \to 1. \end{array}$$

Exactness at $\mathrm{Sext}^n_G(N,C)$. Let $[\epsilon] \in \mathrm{Sext}^n_G(N,B)$. Then construct the commutative diagram

$$\begin{array}{lccccccccccc}
\epsilon:0\to & B & \to & M_1 & \to & M_2 \to & \ldots \to & M_n & \to & N \to 1 \\
 & p\downarrow & PO & \downarrow & & \| & & \| & & \| \\
p(\epsilon):0\to & C & \to & M_1' & \to & M_2 \to & \ldots \to & M_n & \to & N \to 1.
\end{array}$$

Hence $\delta_n p_n^* [\epsilon] = 0$ follows from Lemma 2.1 and the following diagram

$$\begin{array}{lccccccccc}
\delta p(\epsilon):0\to A\to & B \to & M_1' & \to & M_2 \to & \ldots \to & M_n & \to & N \to 1 \\
 & \| & \uparrow & & \| & & \| & & \| \\
\epsilon:0 \quad 0\to & B \to & M_1 & \to & M_2 \to & \ldots \to & M_n & \to & N \to 1.
\end{array}$$

Suppose $\xi: 0 \to C \to M_1 \to M_2 \to \ldots \to M_n \to N \to 1$ represents an element of $\mathrm{Sext}^n_G(N,C)$ with $\delta_n [\xi] = 0$. Then by Lemma 2.1 we have the commutative diagram

$$\begin{array}{lccccccccc}
\delta(\xi):0\to A\to & B \to & M_1 & \to & M_2 & \to \ldots \to & M_n & \to & N \to 1 \\
 & \| & \uparrow & & \uparrow & & \uparrow & & \| \\
\epsilon: \quad 0\to & B \to & M_1' & \to & M_2' & \to \ldots \to & M_n' & \to & N \to 1
\end{array}$$

which shows that $[p(\epsilon)] = [\xi]$.

Exactness at $\mathrm{Sext}^{n+1}_G(N,A)$. If $\epsilon: 0 \to C \to M_1 \to \ldots \to M_n \to N \to 1$ represents an element of $\mathrm{Sext}^n_G(N,C)$ then $i^*_{n+1}\delta_n [\epsilon] = 0$ Lemma 2.1 and the following commutative diagram

$$\begin{array}{lccccccccc}
\delta(\epsilon):0\to & A & \overset{i}{\to} & B & \to M_1 \to & \ldots \to & M_n & \to & N \to 1 \\
 & i\downarrow & PO & \downarrow & \| & & \| & & \| \\
i\delta(\epsilon):0\to & B & \to & B\times C & \to M_1 \to & \ldots \to & M_n & \to & N \to 1 \\
 & & & \uparrow & \| & & \| & & \| \\
 & 0 & \to & C & \to M_1 \to & \ldots \to & M_n & \to & N \to 1.
\end{array}$$

Suppose $[\xi] \in \mathrm{Sext}^{n+1}_G(N,A)$ with $i^*_{n+1} [\xi] = 0$. Then, using Lemma 2.1, we have a commutative diagram

$$\begin{array}{llccccccccc}
 & \xi:0\to & A & \to & M_1 & \to & M_2 & \to \ldots \to & M_{n+1} & \to & N \to 1 \\
 & & i\downarrow & PO & \downarrow & & \| & & \| & & \| \\
(6) & i(\xi):0\to & B & \to & M_1' & \overset{\gamma}{\to} & M_2 & \to \ldots \to & M_{n+1} & \to & N \to 1 \\
 & & & & \| & & \uparrow & & \uparrow & & \| \\
 & & 0 & \to & M_1' & \to & M_2' & \to \ldots \to & M_{n+1}' & \to & N \to 1.
\end{array}$$

Since the first three terms in the first two rows are G-modules it follows from the extension theory of modules (see, for example, page 179 of [8]) that there is induced a homomorphism φ such that the following diagram commutes

$$\begin{array}{ccccccc} 0 \to & A & \to & M_1 & \to & M_2 \\ & \downarrow i & PO & \downarrow & & \| \\ 0 \to & B & \to & M_1' & \to & M_2 \\ & \downarrow & & \downarrow \varphi & & \\ & C & = & C & & \end{array}$$

Let η be the n-fold pseudo G-extension $0 \to M_1' \to M_2' \to \ldots M_{n+1}' \to N \to 1$. Then $\delta[\varphi(\eta)] = [\xi]$.

3. AN EXACT SEQUENCE OF HUEBSCHMANN

Let G be a group and let A be a G-module. Then $H^2(G,A)$ is isomorphic to the group of extensions of A by G (see, for example, [3] and [9]). A *two-fold crossed extension* of A by G is an exact sequence of groups

$$0 \to A \to M \xrightarrow{\partial} E \to G \to 1 \tag{7}$$

where ∂ is a crossed module and the G-module structure induced on A coincides with the given one. If one defines an equivalence relation on these extensions similar to that of module extensions then the set of equivalence classes becomes an abelian group which is isomorphic to $H^3(G,A)$ (see, for example, [5], [7], [11], [14]). Note that the definition of this in [14] is incorrectly stated. In the following we use extensions to represent elements of $H^2(G,A)$ and $H^3(G,A)$. Recall that (7) represents zero element of $H^3(G,A)$ if and only if ∂ is extendible (see, for example, [4], [13], [14], or [15]).

THEOREM 3.1 *(Huebschmann). Suppose* $1 \to N \xrightarrow{i} G \xrightarrow{p} Q \to 1$ *is an exact sequence of groups and suppose A is a G-module. Then the following sequence is exact*

$$H^2(Q,A^N) \xrightarrow{p_2^*} H^2(G,A) \xrightarrow{i_2^*} \mathrm{Sext}_G^1(N,A) \xrightarrow{\delta} H^3(Q,A^N) \xrightarrow{p_3^*} H^3(G,A).$$

Proof. Define $p_2^*[\epsilon] = [p_2(\epsilon)]$ where $p_2(\epsilon)$ is the bottom row of

$$\begin{array}{rccccccc} \epsilon: 0 \to & A^N & \to & E & \to & Q & \to 1 \\ & \| & & \uparrow & PB & \uparrow & \\ 0 \to & A^N & \to & P & \to & G & \to 1 \\ & \downarrow & PO & \downarrow & & \| & \\ p_2(\epsilon): 0 \to & A & \to & E' & \to & G & \to 1 \end{array}$$

Define $i_2^*[\epsilon] = [i(\epsilon)]$ by the following diagram

$$\begin{array}{rcccccccc}
\epsilon: 0 \rightarrow & A & \rightarrow & E & \overset{\pi}{\rightarrow} & G & \rightarrow 1 \\
 & \| & & \uparrow & {\scriptstyle PB} & \uparrow i & \\
i(\epsilon): 0 \rightarrow & A & \rightarrow & E' & \rightarrow & N & \rightarrow 1
\end{array}$$

where the pseudo G-action θ on E' is defined by $\theta(g) = [\hat{\theta}(g)]$, and $\hat{\theta}(g)(e') = ee'e^{-1}$, $\pi(e) = g$. For $\xi: 0 \rightarrow A \rightarrow E \overset{\partial}{\rightarrow} N \rightarrow 1$, representing an element of $\mathrm{Sext}_G^1(N,A)$, define $\delta[\xi] = [\delta(\xi)]$ where $\delta(\xi)$ is the associated crossed module of $0 \rightarrow A \rightarrow E \overset{i\partial}{\rightarrow} G \rightarrow Q \rightarrow 1$ (see Section 1). Define $p_3^*([\epsilon]) = [p_3(\epsilon)]$ by the diagram

$$\begin{array}{rcccccccc}
\epsilon: 0 \rightarrow & A^N & \rightarrow & M \rightarrow & E & \rightarrow & Q & \rightarrow 1 \\
 & \| & & \| & \uparrow & {\scriptstyle PB} & \uparrow p & \\
0 \rightarrow & A^N & \rightarrow & M \rightarrow & L & \rightarrow & G & \rightarrow 1 \\
 & \downarrow & {\scriptstyle PO} & \downarrow & \| & & \| & \\
p_3(\epsilon): 0 \rightarrow & A & \rightarrow & P \rightarrow & L & \rightarrow & G & \rightarrow 1.
\end{array}$$

In the above definition PB denotes pullback of groups and PO the pushout construction of Lemma 1.2. It is routine to check that these maps are well-defined homomorphisms.

Exactness at $H^2(G,A)$. For $[\epsilon] \in H^2(Q,A^N)$ construct the commutative diagrams with exact rows

$$\begin{array}{rcccccc}
\epsilon: 0 \rightarrow & A^N & \rightarrow & E & \rightarrow & Q & \rightarrow 1 \\
 & \| & & \uparrow & {\scriptstyle PB} & \uparrow p & \\
0 \rightarrow & A^N & \rightarrow & P & \rightarrow & G & \rightarrow 1 \\
 & \downarrow & {\scriptstyle PO} & \downarrow & & \| & \\
p_2(\epsilon): 0 \rightarrow & A & \rightarrow & E' & \rightarrow & G & \rightarrow 1 \\
 & \| & & \uparrow & {\scriptstyle PB} & \uparrow i & \\
ip_2(\epsilon): 0 \rightarrow & A & \rightarrow & \bar{E} & \rightarrow & N & \rightarrow 1,
\end{array}
\qquad
\begin{array}{rcccccc}
0 \rightarrow & A^N & \rightarrow & P & \rightarrow & G & \rightarrow 1 \\
 & \| & & \uparrow & {\scriptstyle PB} & \uparrow i & \\
\epsilon': 0 \rightarrow & A^N & \rightarrow & E'' & \rightarrow & N & \rightarrow 1 \\
 & \downarrow & {\scriptstyle PO} & \downarrow & & \| & \\
0 \rightarrow & A & \rightarrow & \hat{E} & \rightarrow & N & \rightarrow 1.
\end{array}$$

We claim that the bottom rows of the above diagrams are equivalent pseudo G-extensions. Since $[\epsilon'] = 0$ this implies that $[ip_2(\epsilon)] = 0$. The claim follows from routine diagram chasing involving PO and PB. Suppose $i_2^*[\xi] = 0$ where $[\xi] \in H^2(G,A)$. Then we have the following commutative diagram

$$\begin{array}{rcccccc}
\xi: 0 \rightarrow & A & \rightarrow & E_G & \overset{\pi}{\rightarrow} & G & \rightarrow 1 \\
 & \| & & \uparrow & & \uparrow i & \\
i(\xi): 0 \rightarrow & A & \rightarrow & A \rtimes N & \rightleftarrows & N & \rightarrow 1.
\end{array}$$

Let M_G be the normalizer of the image of $N \to A \rtimes N \to E_G$. Note that the (pseudo) G-action on $A \rtimes N$ is defined by $g * (a,n) = e(a,n)e^{-1}$, where $\pi(e) = g$, $e \in E_G$, $g \in G$. Since $[i(\xi)] = 0$ it is also defined by pointwise actions. Hence $(0, g \circ n) \underset{A}{\equiv} e(0,n)e^{-1}$ where $\pi(e) = g$, i.e. $(0, g \circ n) = ae(0,n)(ae)^{-1}$ for some $a \in A$. This shows that for every $g \in G$ there exists $ae \in M_G$ with $\pi(ae) = g$. Hence $\pi | M_G$ is surjective and it is easy to see that its kernel is $A \cap M_G = A^N$. Thus we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} \xi : 0 & \to & A & \to & E_G & \to & G & \to & 1 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \to & A^N & \to & M_G & \to & G & \to & 1 \\ & & & & \uparrow & & \uparrow & & \\ & & & & N & = & N. & & \end{array}$$

Construct the commutative diagram

$$\begin{array}{ccccccccc} \epsilon_Q : 0 & \to & A^N & \to & E_Q & \to & Q & \to & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \to & A^N & \to & M_G & \to & G & \to & 1 \\ & & & & \uparrow & & \uparrow & & \\ & & & & N & = & N & & \end{array}$$

where $E_Q \doteq M_G/N$. Then $[\epsilon_Q] \in H^2(Q, A^N)$ and $p_2^* [\epsilon_Q] = [\xi]$.

Exactness at $\mathrm{Sext}_G^1(N,A)$. For each $[\epsilon] \in H^2(G,A)$ construct the commutative diagrams

$$\begin{array}{ccccccccc} \epsilon : 0 & \to & A & \to & E_G & \to & G & \to & 1 \\ & & \| & & \uparrow & PB & \uparrow i & & \\ i(\epsilon) : 0 & \to & A & \to & E_N & \to & N & \to & 1 \end{array} \qquad \begin{array}{ccccccccccc} \epsilon' : 0 & \to & A & \to & E_N & \to & G & \to & Q & \to & 1 \\ & & \uparrow & & \| & & \uparrow & & \| & & \\ \delta(i(\epsilon)) : 0 & \to & A^N & \to & E_N & \to & \Gamma & \to & Q & \to & 1. \end{array}$$

Then we see that ϵ' is extendible from the commutative diagram

$$\begin{array}{ccccccccccc} & & 1 & \to & E_N & \to & E_G & \to & Q & \to & 1 \\ & & & & \| & & \downarrow & & \| & & \\ 0 & \to & A & \to & E_N & \to & G & \to & Q & \to & 1. \end{array}$$

Hence, by proposition 1.1, $\delta(i(\epsilon))$ is also extendible and hence it represents zero in $H^3(Q, A^N)$. Suppose $[\xi] \in \mathrm{Sext}_G^1(N,A)$ with $\delta[\xi] = 0$. Then there exists an exact sequence $0 \to E \to M \to Q \to 0$ such that the following diagram is commutative

$$\begin{array}{rcccccccc}
\xi:0\to & A & \to & E & \to & N & \to 1 & & \\
 & \| & & \| & & \downarrow i & & & \\
\xi':0\to & A & \to & E & \to & G & \to & Q & \to 1 \\
 & \uparrow & & \| & & \uparrow & & \| & \\
\delta(\xi):0\to & A^N & \to & E & \to & \Gamma & \to & Q & \to 1 \\
 & & & \| & & \uparrow & & \| & \\
 & 0 & \to & E & \to & M & \to & Q & \to 1
\end{array}$$

since $\delta(\xi)$ is extendible. Hence we have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & A & = & A & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \to & E & \to & M & \to & Q & \to & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \to & N & \xrightarrow{i} & G & \to & Q & \to & 1
\end{array}$$

and hence $i(\epsilon) = \xi$ where $\epsilon : 0 \to A \to M \to G \to 1$ represents an element of $H^2(G,A)$.

Exactness at $H^3(Q,A^N)$. Let $\epsilon : 0 \to A \to E_N \to N \to 1$ represent an element of $\mathrm{Sext}^1_G(N,A)$. Construct the commutative diagram

$$\begin{array}{rccccccccc}
0\to & A & \to & E_N & \to & G & \to & Q & \to 1 \\
 & \uparrow & & \| & & \uparrow & & \| & \\
\delta(\epsilon):0\to & A^N & \to & E_N & \to & P & \to & Q & \to 1 \\
 & \| & & \| & & \uparrow & PB & \uparrow & \\
\epsilon':0\to & A^N & \to & E_N & \to & P' & \to & G & \to 1 \\
 & \| & & \uparrow & & \uparrow & & \| & \\
0\to & A^N & \to & A & \to & P & \to & G & \to 1 \\
 & \downarrow & PO & \downarrow & & \| & & \| & \\
p_3\delta(\epsilon):0\to & A & \to & E & \to & P & \to & G & \to 1 \\
 & \| & & \downarrow & & \downarrow & & \| & \\
0\to & A & \to & A & \xrightarrow{0} & G & \to & G & \to 1
\end{array}$$

where $P \to P'$ is induced by PB and $E \to A$ by PO. This shows that $p_3^*\delta\,[(\epsilon)] = 0$.

Before we show that $\mathrm{Ker}\, p_3^* \subseteq \mathrm{Im}\,\delta$, we note that if F is a free group on G, we have two exact sequences

$$1 \to R_Q \to F \to Q \to 1$$
$$1 \to R_G \to F \to G \to 1$$

and that R_G is normal in R_Q with $R_Q/R_G \cong N$. Moreover, every element in $H^3(G,A)$ has a representative of the form $0 \to A \to E \to F \to G \to 1$. It is also easy to see that if $\epsilon: 0 \to A^N \to E \to F \to Q \to 1$ represents an element of $H^3(Q,A^N)$, the homomorphism $p: H^3(Q,A^N) \to H^3(G,A^N)$ sends $[\epsilon]$ to $[p(\epsilon)]$ where $p(\epsilon)$ is the last row of the commutative diagram

$$\begin{array}{rcccccccccc}
\epsilon: 0 \to & A^N & \to & E & \to & F & \to & Q & \to 1 \\
& \| & & \| & & \uparrow & PB & \uparrow & \\
0 \to & A^N & \to & E & \to & P & \to & G & \to 1 \\
& \| & & \uparrow & PB & \uparrow \pi & & \| & \\
p(\epsilon): 0 \to & A^N & \to & E_1 & \to & F & \to & G & \to 1
\end{array}$$

where the map π is induced by the pullback on the upper right hand corner. Thus the homomorphism $p_3^*: H^3(Q,A^N) \to H^3(G,A)$ sends $[\epsilon]$ to $[p_3(\epsilon)]$ where $p_3(\epsilon)$ is given by the diagram

$$\begin{array}{rcccccccc}
p(\epsilon): 0 \to & A^N & \to & E_1 & \to & F & \to & G & \to 1 \\
& \downarrow & PO & \downarrow k & & \| & & \| & \\
p_3(\epsilon): 0 \to & A & \underset{g}{\to} & E_2 & \to & F & \to & G & \to 1
\end{array}$$

Moreover, if $e \in E_1$, $a \in A$, $g(e \cdot a) = k(e) \cdot g(a)$. In the following, all PO squares will have these properties. With these preparations, we now will prove that $\operatorname{Ker} p_3^* \subset \operatorname{Im} \delta$.

Suppose $[p_3(\epsilon)] = 0$. Then in view of F, the sequence $0 \to A \to E_2 \to R_G \to 1$ must be a split F-sequence, i.e., R_G is a normal subgroup of E_2. Consider the following commutative diagram

$$\begin{array}{rcccccc}
0 \to & A^N & \to & E & \to & R_Q & \to 1 \\
& \downarrow & PO & \downarrow & & \| & \\
0 \to & A & \to & \bar{E}_1 & \to & R_Q & \to 1 \\
& \| & & \uparrow & PB & \uparrow & \\
0 \to & A & \to & \bar{E}_2 & \to & R_G & \to 1
\end{array} \tag{8}$$

where $\bar{E}_1 = A \rtimes E/\{(-a,a)\}$, E operating on A via $F \to G$, (A^N is thus a trivial E-module). The homomorphisms $A \to \bar{E}_1$, $E_1 \to E \to \bar{E}_1$ give a homomorphism $E_2 \to \bar{E}_1$ which together with $E_2 \to R_G$ yields a homomorphism $E_2 \to \bar{E}_2$. Thus the following two extensions are equivalent

$$\begin{array}{rcccccc}
0 \to & A & \to & E_2 & \to & R_G & \to 1 \\
& \| & & \downarrow & & \| & \\
0 \to & A & \to & \bar{E}_2 & \to & R_G & \to 1
\end{array}$$

and we may identify $\bar{E}_2$ by E_2. Clearly, $R_G \to E_2 \to \bar{E}_1$ is normal. Let $E_N = \bar{E}_1/R_G$.

We have the following diagram

$$\begin{array}{ccccccccc} \epsilon': 1 \to & A & \to & E_N & \to & N & \to 1 \\ & \| & & \uparrow & & \uparrow & \\ \epsilon'': 1 \to & A & \to & \bar{E}_1 & \to & R_Q & \to 1 \\ & & & \uparrow & & \uparrow & \\ & & 1 \to & R_G & = & R_G & \to 1. \end{array}$$

The top row ϵ' represents an element in $\mathrm{Sext}_G^1(N,A)$. Diagram (8) says that $0 \to A \to \bar{E}_1 \to F \to Q \to 1$ has ϵ as an associated crossed module. Since

$$\begin{array}{ccccccccc} 0 \to & A & \to & E_N & \to & G & \to & Q & \to 1 \\ & \| & & \uparrow & & \uparrow & & \| & \\ 0 \to & A & \to & \bar{E}_1 & \to & F & \to & Q & \to 1 \end{array}$$

is commutative, we have $\delta\,[\epsilon'] = [\epsilon]$ and the proof is complete.

4. THE MAIN THEOREM

Throughout let $1 \to N \xrightarrow{i} G \xrightarrow{p} Q \to 1$ be an extension of groups and let A be a G-module.

PROPOSITION 4.1. $\mathrm{Sext}_G^n(N,J) = 0$ *for* $n > 1$ *where* J *is an injective G-module.*

Proof. Let $\epsilon: 0 \to J \to M_1 \to \ldots \to M_n \to N \to 1$ represent an element of $\mathrm{Sext}_G^n(N,J)$. Since $n > 1$, M_1 is a G-module and so the sequence $0 \to J \to M_1 \to R \to 0$ splits as G-modules thus giving the commutative diagram

$$\begin{array}{ccccccccccc} 0 \to & J & \to & M_1 & \to & M_2 & \to \ldots \to & M_n & \to & N & \to 1 \\ & & & \mu\uparrow & & \| & & \| & & \| & \\ & & 0 \to & R & \to & M_2 & \to \ldots \to & M_n & \to & N & \to 1 \end{array}$$

where μ is a splitting map. By Lemma 2.1 $[\epsilon] = 0$.

PROPOSITION 4.2. *The following sequence is exact*

$$0 \to H^1(Q,H^1(N,A)) \xrightarrow{v} \mathrm{Sext}_G^1(N,A) \xrightarrow{u} H^2(N,A).$$

In case A is a Q-module this is done in [11] and [15]. Note that Huebschmann [6] has established this sequence with $\mathrm{Sext}_G^1(N,A)$ replaced by $\mathrm{XPext}(G,N;A)$. However, we only need a special case of the above proposition. Since [6] contains no proofs we supply a proof for it in the following.

COROLLARY 4.3. $\mathrm{Sext}_G^1(N,J) = 0$ *for any injective G-module J.*

Proof. Let $\epsilon: 0 \to J \to E \xrightarrow{\partial} N \to 1$ represent an element of $\mathrm{Sext}^1_G(N,J)$. Then it also represents an element of $H^2(N,J) = 0$ and so must be a split extension, i.e. $E \cong J \times_S N$. It remains to show that the pseudo G-action $\tau: G \to \mathrm{Aut}_A E/c(A)$ is the same as the G-action $\theta: G \to \mathrm{Aut}\, E$ given by

$$\theta(g)(a,n) = (g \cdot a, gng^{-1}), \qquad a \in J, \quad g \in G.$$

By (P2) of τ and θ one sees that

$$\hat{\tau}(g)(e)[\theta(g)(e)]^{-1} = f(g,e) \in J$$

where $\tau(g) = [\hat{\tau}(g)]$, $g \in G$, $e \in E$. It is easy to check that $f(g)(n) = f(g,e)$, where $\partial e = n$, gives a homomorphism $f(g) \in \mathrm{Hom}(N,J)$. Define $d_f(g) \in \mathrm{Der}\,(N,J)$ by $d_f(g)(n) = f(g)(g^{-1} \circ n)$. Then, as $H^1(N,J) \cong \mathrm{Der}\,(N,J)/\mathrm{Ider}\,(N,J) = 0$, $d_f(g) \in \mathrm{Ider}\,(N,J)$. This, in turn, proves that $[\hat{\tau}(g)] = \theta(g)$ for $g \in G$, i.e. $[\epsilon] = 0$ as an element of $\mathrm{Sext}^1_G(N,J) = 0$.

THEOREM 4.4. *Suppose $H^i(N,A) = 0$ for $0 < i < n$, $n > 1$. Then the following sequence is exact*

$$\begin{aligned} 0 \to H^n(Q,A^N) \to H^n(G,A) &\to \mathrm{Sext}^{n-1}_G(N,A) \to H^{n+1}(Q,A^N) \\ \to H^{n+1}(G,A) \to \mathrm{Sext}^n_G(N,A) &\to H^{n+2}(Q,A^N) \to H^{n+2}(G,A). \end{aligned}$$

Proof. Construct an exact sequence $0 \to A \to J \to K \to 0$ of G-modules where J is injective. Then, for $k > 0$, there is an isomorphism

$$H^k(G,K) \cong H^{k+1}(G,A). \tag{9}$$

Similarly, by using Proposition 2.2, 4.1 and Corollary 4.3, one obtains

$$\mathrm{Sext}^k_G(N,K) \cong \mathrm{Sext}^{k+1}_G(N,A) \qquad \text{for } k > 0. \tag{10}$$

If $H^1(N,A) = 0$ then $0 \to A^N \to J^N \to K^N \to 0$ is exact as Q-modules where J^N is injective. In this case we have

$$H^k(Q,K^N) \cong H^{k+1}(Q,A^N) \qquad \text{for } k > 0. \tag{11}$$

Suppose $H^i(N,A) = 0$ for $0 < i < k+1$. Then $H^i(N,K) = 0$ for $0 < i < k$. Hence, by using (9), (10), and (11), one sees that the theorem is true for $n = k+1$ if it is true for $n = k$. When $n = 2$ the first part of the sequence is simply that of Theorem 3.1 and the second part follows from it by the above argument.

Remark 1. Since the sequence of Theorem 3.1 is natural in A it is not hard to prove that the sequence in Theorem 4.4 is also natural in A.

Remark 2. Using (9) and (10) one may restate proposition 4.2 to say that the sequence

$$0 \to H^1(Q,H^n(N,A)) \to \mathrm{Sext}^n_G(N,A) \to H^{n+1}(N,A)$$

is exact for $n \geq 1$.

Remark 3. Some assumption on A is necessary in Theorem 4.4 For, given any group extension $1 \to N \to F \to Q \to 1$ where F is free, one obtains an exact sequence from theorem 4.4

$$\begin{array}{ccccccccc} H^3(Q,A) & \to & H^3(F,A) & \to & \mathrm{Sext}_F^2(N,A) & \to & H^4(Q,A) & \to & H^4(F,A) \\ & & \| & & & & & & \| \\ & & 0 & & & & & & 0 \end{array}$$

where A is a Q-module. By remark 2, $\mathrm{Sext}_F^2(N,A) = 0$ since N, as a subgroup of a free, is free. Therefore if there are no restrictions on A then $H^4(Q,A) = 0$ for all A, i.e. $cd\, Q \leq 3$ which is not true in general.

Remark 4. It can be shown that $H^{n+1}(N,A)^Q \simeq \mathrm{Sext}_G^n(N,A)$ if $H^i(N,A) = 0$ for $0 < i \leq n$. Thus Theorem 4.4 extends the result of Hochschild and Serre [4].

Added in Proof. The authors would like to thank Dr. Heubschmann for indicating the need to clarify the proof of Lemma 2.1 at this point. Note that it suffices to show that the top row exists whenever the rest of the commutative diagram exists

$$\begin{array}{cccccccccccccccc} & & & & 0 & \to & M_1 & \to & E_2 & \to \dots \to & E_{n-1} & \to & E_n & \xrightarrow{p} & N & \to 0 \\ & & & & & & \| & & \downarrow & & \downarrow & & \downarrow & & \| & \\ \xi: & 0 & \to & A & \to & & M_1 & \to & M_2 & \to \dots \to & M_{n-1} & \to & M_n & \xrightarrow{\pi} & N & \to 0 \\ & & & \| & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| & \\ \xi': & 0 & \to & A & \to & & M_1' & \to & M_2' & \to \dots \to & M_{n-1}' & \to & M_n' & \xrightarrow{\pi'} & N & \to 0 \\ & & & & & & \| & & \downarrow & & \downarrow & & \downarrow & & \| & \\ & & & & 0 & \to & M_1' & \to & E_2' & \to \dots \to & E_{n-1}' & \to & E_n' & \xrightarrow{\pi'} & N & \to 0 \end{array}$$

where the top and the bottom rows are $(n-1)$-fold pseudo G-extensions and the middle two are n-fold pseudo G-extensions. To do this one takes E_n to be the pullback of $E_n' \to M_n \leftarrow M_n$. In case $n = 2$, $\mathrm{Ker}\, p = M_1$ and we are done. Otherwise take E_{n-1} to be the pullback of $M_{n-1} \to \mathrm{Ker}\, \pi \leftarrow \mathrm{Ker}\, p$ and $E_i = M_i$ with the obvious maps $E_i \to M_i$, $2\ i\ n - 2$.

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