ON THE RUSSO-DYE THEOREM

Sorin Popa

Let A be a unital C^*-algebra and $x \in A$, $\|x\| < 1$. Denote by $n(x,A)$ the least natural number n such that x is a convex combination of n unitary elements of A. The Russo-Dye theorem asserts that $n(x,A)$ is finite. Let $n(\rho;A)$ denote the least upper bound of the numbers $n(x,A)$, where $x \in A$, $\|x\| \leq \rho$, $0 < \rho < 1$. It is known that $n(2^{-1};A) \leq 4$ and it is shown (see [3]) that if A is the C^*-algebra of continuous functions on the unit disk and $f \in A$ is the identity function, then

$$(*) \qquad n(\rho f, A) \geq 2(1-\rho)^{-1}, \quad \text{for} \quad 0 < \rho < 1,$$

which shows that $\sup_{0<\rho<1} n(\rho;A)$ is infinite.

In a seminar on operator algebras at the Math. Dept. of INCREST, A. Ocneanu raised the question of whether $n(\rho;A)$ is finite for $\rho < 1$. In this paper we answer affirmatively this question, namely we prove that

$$(**) \qquad n(\rho;A) \leq 2\pi(1+\rho)(1-\rho)^{-1} + 2.$$

To do this we follow Harris' proof of the Russo-Dye theorem ([1]). We also exhibit another class of C^*-algebras for which the inequality $(*)$ holds, namely if a C^*-algebra A contains a nonunitary isometry v, then $n(\rho v, A) \geq 2(1-\rho)^{-1}$, $0 < \rho < 1$.

This shows that in certain C^*-algebras the estimate $(**)$ is best possible, in the sense that only the constant 2π may be improved.

First we recall some definitions.

Let H be a Hilbert space and $B(H)$ the space of bounded linear operators on H; consider a contraction $x \in B(H)$, $\|x\| < 1$; denote by $D_x = (1 - x^*x)^{1/2}$, $D_{x^*} = (1 - xx^*)^{1/2}$. For $\lambda \in C$, $|\lambda| < 1/\|x\|$, let

$$\theta_x(\lambda) = D_{x^*}(1 - \lambda x^*)^{-1}(\lambda - x)D_x^{-1} = -x + \sum_{n\geq 1} \lambda^n D_{x^*} x^{*n-1} D_x$$

be the characteristic function of the contraction x (see [2, Chapter VI]). Then $\theta_x(\lambda)$ is analytic for $|\lambda| < 1/\|x\|$ and it takes unitary values for $|\lambda| = 1$. Also by the Cauchy integral formula we have $-x = \theta_x(0) = \int_0^1 \theta_x(e^{2\pi i t})\,dt$.

Thus, to obtain x as a convex combination of n unitaries, with n as small as possible, we need a good estimate for the norm of $(d/d\lambda\,\theta_x)(\lambda)$. An easy computation shows that $(d/d\lambda\,\theta_x)(\lambda) = D_{x^*}(1 - \lambda x^*)^{-2} D_x$.

Received February 26, 1980. Revision received May 12, 1981.

Michigan Math. J. 28 (1981).

In the particular case of a constant operator $x = \rho \in C$, we obtain

$$\left(\frac{d}{d\lambda}\theta_\rho\right)(\lambda) = \frac{1-|\rho|^2}{(1-\lambda\bar\rho)^2} \quad \text{and} \quad \sup_{|\lambda|=1}\frac{1-|\rho|^2}{|1-\lambda\bar\rho|^2} = \frac{1+|\rho|}{1-|\rho|}.$$

In the following lemma we prove a similar fact for a general operator $x \in B(H)$, $\|x\| < 1$.

LEMMA. *For $|\lambda| = 1$ and $h \in H$ we have*

$$\left\|\left(\frac{d}{d\lambda}\theta_x\right)(\lambda)h\right\| = \|D_x(1-\bar\lambda x)^{-1}(1-\lambda x^*)^{-1}D_x h\|$$

and $\|(d/d\lambda\,\theta_x)(\lambda)\| \le (1+\|x\|)/(1-\|x\|)$.

Proof. For $|\lambda|, |\mu| < 1/\|x\|$ and $h \in H$, we have the formula

$$\|h\|^2 - \langle\theta_x(\lambda)h, \theta_x(\mu)h\rangle = (1-\lambda\bar\mu)\langle(1-\lambda x^*)^{-1}D_x h, (1-\mu x^*)^{-1}D_x h\rangle$$

(Cf. Sz.-Nagy and Foiaş [2, Chapter VI, 1.4]).

Since $\theta_x(\alpha)$ is unitary for $|\alpha| = 1$, this gives for $|\lambda| = 1$, $\mu = 1$ the equalities

$$\begin{aligned}
\|\theta_x(\lambda)h - \theta_x(1)h\|^2 &= \|\theta_x(\lambda)h\|^2 + \|\theta_x(1)h\|^2 - 2\mathrm{Re}\,\langle\theta_x(\lambda)h, \theta_x(1)h\rangle \\
&= 2\mathrm{Re}\,(\|h\|^2 - \langle\theta_x(\lambda)h, \theta_x(1)h\rangle) \\
&= 2\mathrm{Re}\,(1-\lambda)\langle(1-\lambda x^*)^{-1}D_x h, (1-x^*)^{-1}D_x h\rangle \\
&= 2\mathrm{Re}\,(1-\lambda)\,\mathrm{Re}\,\langle(1-\lambda x^*)^{-1}D_x h, (1-x^*)^{-1}D_x h\rangle \\
&\quad - 2\mathrm{Im}\,(1-\lambda)\,\mathrm{Im}\,\langle(1-\lambda x^*)^{-1}D_x h, (1-x^*)^{-1}D_x h\rangle \\
&= |1-\lambda|^2\,\mathrm{Re}\,\langle D_x(1-x)^{-1}(1-\lambda x^*)^{-1}D_x h, h\rangle \\
&\quad - i(\lambda-\bar\lambda)\,\mathrm{Im}\,\langle D_x(1-x)^{-1}(1-\lambda x^*)^{-1}D_x h, h\rangle
\end{aligned}$$

$$= \frac{1}{2}|1-\lambda|^2\langle D_x[(1-x)^{-1}(1-\lambda x^*)^{-1} + (1-\bar\lambda x)^{-1}(1-x^*)^{-1}]D_x h, h\rangle$$

$$-\frac{1}{2}(\lambda-\bar\lambda)\langle D_x[(1-x)^{-1}(1-\lambda x^*)^{-1} - (1-\lambda x)^{-1}(1-x^*)^{-1}]D_x h, h\rangle.$$

Consequently we obtain for $\|(d/d\lambda\,\theta_x)(1)h\|$ the following formula

$$\begin{aligned}
\left\|\left(\frac{d}{d\lambda}\theta_x\right)(1)h\right\|^2 &= \lim_{\substack{\lambda\to 1\\ |\lambda|=1}}\frac{\|(\theta_x(\lambda)-\theta_x(1))h\|^2}{|\lambda-1|^2} \\
&= \langle D_x(1-x)^{-1}(1-x^*)^{-1}D_x h, h\rangle \\
&\quad - \langle D_x(1-x)^{-1}(1-x^*)^{-1}y(1-x)^{-1}(1-x^*)^{-1}D_x h, h\rangle,
\end{aligned}$$

where

$$y = \lim_{\substack{\lambda\to 1\\ |\lambda|=1}} \frac{\lambda-\bar\lambda}{2|\lambda-1|^2}((\bar\lambda-\lambda)x^*x-(1-\lambda)x^*+(1-\bar\lambda)x) = 2x^*x-x-x^*.$$

Finally we get

$$\begin{aligned}\left\|\left(\frac{d}{d\lambda}\theta_x\right)(1)h\right\|^2 &= \langle D_x(1-x)^{-1}(1-x^*)^{-1}(1+x^*x-x-x^*-y)\\ &\quad\cdot(1-x)^{-1}(1-x^*)^{-1}D_xh,h\rangle\\ &= \langle D_x(1-x)^{-1}(1-x^*)^{-1}D_x^2(1-x)^{-1}(1-x^*)^{-1}D_xh,h\rangle\\ &= \|D_x(1-x)^{-1}(1-x^*)^{-1}D_xh\|^2.\end{aligned}$$

Since $(d/d\lambda\,\theta_x)(\lambda) = D_x\cdot\left(\sum_{n\ge1} n(\lambda x^*)^{n-1}\right)D_x$, we have

$$\left(\frac{d}{d\lambda}\theta_x\right)(\lambda\mu) = D_x\cdot\sum_{n\ge1} n(\lambda(\bar\mu x)^*)^{n-1}D_x = \left(\frac{d}{d\lambda}\theta_{\bar\mu x}\right)(\lambda),$$

so that $(d/d\lambda\,\theta_x)(\mu) = (d/d\lambda\,\theta_{\bar\mu x})(1)$ and we obtain that for $|\lambda|=1$,

$$\left\|\left(\frac{d}{d\lambda}\theta_x\right)(\lambda)h\right\| = \left\|\left(\frac{d}{d\lambda}\theta_{\bar\lambda x}\right)(1)h\right\| = \left\|D_x(1-\bar\lambda x)^{-1}(1-\lambda x^*)^{-1}D_xh\right\|.$$

To prove the second part of the lemma we have to show that

$$D_x(1-\bar\lambda x)^{-1}(1-\lambda x^*)^{-1}D_x \le \frac{1+\|x\|}{1-\|x\|}.$$

This inequality is equivalent to $(1-\lambda x^*)(1-\bar\lambda x) \ge 1-\|x\|/1+\|x\|\,D_x^2$, that is $\langle(1-\lambda x^*)(1-\bar\lambda x)h,h\rangle \ge (1-\|x\|)/(1+\|x\|)\langle D_x^2h,h\rangle$, for all $h\in H$, $\|h\|=1$, or equivalently

$$\|(1-\bar\lambda x)h\|^2 \ge \frac{1-\|x\|}{1+\|x\|}(\|h\|^2-\|xh\|^2) = \frac{\|h\|+\|xh\|}{1+\|x\|}(1-\|x\|)(\|h\|-\|xh\|),\quad \|h\|=1.$$

This last inequality holds, since

$$\|(1-\bar\lambda x)h\|^2 \ge (\|h\|-\|\bar\lambda xh\|)^2 \ge (1-\|x\|)(\|h\|-\|xh\|),$$

and $1 \ge (\|h\|+\|xh\|)/(1+\|x\|)$, for $\|h\|=1$.

THEOREM. *If A is an arbitrary unital C^*-algebra and $x\in A$, $\|x\|<1$, then* $n(x,A) \le 2\pi(1+\|x\|)/(1-\|x\|)+2$.

Proof. If we denote by $y_k = -\int_0^{1/n}\theta_x(e^{2\pi i(t+((k-1)/n))})\,dt$, $1\le k\le n$, then by the Cauchy integral formula we have $x=\sum_{k=1}^n y_k$. By the preceding lemma, for

$n > \pi (1 + \|x\|)/(1 - \|x\|)$ we get

$$\left\| y_k + \frac{1}{n} \theta_x (e^{2\pi i (k-(1/2))/n}) \right\| = \left\| \int_0^{1/n} (\theta_x (e^{2\pi i (t+((k-1)/n))}) - \theta_x (e^{2\pi i (k-(1/2))/n})) \, dt \right\|$$
$$\leq \frac{1}{n} \frac{\pi}{n} \sup_{0 \leq t \leq 1} \left\| \left(\frac{d}{d\lambda} \theta_x \right) (2^{2\pi i t}) \right\| \leq \frac{\pi}{n^2} \frac{1 + \|x\|}{1 - \|x\|} < \frac{1}{n}.$$

Since $-\theta_x (e^{2\pi i (k-(1/2))/n})$ are all unitaries, this implies that y_k are invertible, $1 \leq k \leq n$. Consequently if we let a_k be the modulus of y_k, $a_k = (y_k^* y_k)^{1/2}$, then a_k are also invertible, $1 \leq k \leq n$, and $u_k = y_k a_k^{-1}$ are unitary elements of A, $1 \leq k \leq n$. Moreover we have $\|a_k\| = \|y_k\| \leq \int_0^{1/n} \|\theta_x (e^{2\pi i (t+((k-1)/n))})\| dt = 1/n$, so that $\|n a_k\| \leq 1$.

If we denote by

$$u_k^1 = n a_k + i\sqrt{1 - n^2 a_k^2}, \quad 1 \leq k \leq n,$$
$$u_k^2 = n a_k - i\sqrt{1 - n^2 a_k^2}, \quad 1 \leq k \leq n,$$

then u_k^1, u_k^2 are unitary elements of A and

$$x = \sum_{k=1}^n y_k = \sum_{k=1}^n u_k a_k = \sum_{k=1}^n \frac{1}{2n} u_k (u_k^1 + u_k^2).$$

Now suppose A is a von Neumann algebra. If $x \in A$ and $\|x\| \leq 1$, then by the polar decomposition we have $x = ua$, with $a = (x^* x)^{1/2}$ and u a partial isometry; moreover if A is finite, u may be chosen to be unitary, so that

$$x = \frac{1}{2} u((a + i\sqrt{1 - a^2}) + (a - i\sqrt{1 - a^2})),$$

which means that $n(x, A) = 2$. In the case of an infinite von Neumann algebra this is no longer true, the obstruction being the existence of nonunitary isometries. More precisely we have the following.

PROPOSITION. *Let A be a unital C^*-algebra and v a nonunitary isometry or coisometry in A. Then $n(\rho v, A) \geq 2(1 - \rho)^{-1}$, for $0 < \rho < 1$.*

Moreover, if A is a von Neumann algebra and $(1 - \rho)^{-1}$ is integer, then $n(\rho v, A) = 2(1 - \rho)^{-1}$.

Proof. Since $n(\rho v, A) = n(\rho v^*, A)$ we may suppose v is an isometry. Let $\rho v = \sum_{i=1}^n \lambda_i u_i$ for some unitary elements $u_i \in A$ and positive scalars λ_i, $\sum_{i=1}^n \lambda_i = 1$. Remarking that $v u_j^*$ is still a nonunitary isometry, we have

$$\rho + \lambda_j = \|\rho v u_j^* - \lambda_j\| = \|\rho v - \lambda_j u_j\| = \left\| \sum_{i \neq j} \lambda_i u_i \right\| \leq \sum_{i \neq j} \lambda_i = 1 - \lambda_j, \quad 1 \leq j \leq n.$$

Thus $1/n \leq \max_j \lambda_j \leq (1-\rho)/2$, so that $n \geq 2(1-\rho)^{-1}$.

Suppose now that A is a von Neumann algebra and denote $e_0 = 1 - vv^*$, $e_n = v^n e_0 v^{*n}$, $n \geq 1$. It will be sufficient to consider only the case when

$$\sum_{n \geq 0} e_n = 1.$$

Thus there exists a subalgebra $B \subset A$, $v \in B$, such that B is isomorphic to some $B(H)$ and such that identifying B with $B(H)$ there exists an orthonormal basis $h_0, h_1, h_2, \ldots$ in H for which v is the unilateral shift, i.e. $vh_n = h_{n+1}$, $n \geq 0$. If $p = (1-\rho)^{-1} \in N$, define the elements $u_1, u_2, \ldots, u_{2p} \in B(H)$ by

$$u_{2k-1} h_m = \begin{cases} h_1, & \text{for} \quad m = k \\ h_{(n-1)p+k+1}, & \text{for } m = np + k, \quad n \geq 1 \\ h_{m+1}, & \text{for} \quad m \neq np + k \end{cases}$$

$$u_{2k} h_m = \begin{cases} -h_1, & \text{for} \quad m = k \\ -h_{(n-1)p+k+1}, & \text{for} \quad m = np + k, \quad n \geq 1 \\ h_{m+1}, & \text{for} \quad m \neq np + k. \end{cases}$$

By inspecting the above formulae we see that $u_1, u_2, \ldots, u_{2p}$ are unitary elements of $B(H)$ and that $(1 - 1/p)v = (1/2p) \sum_{j=1}^{2p} u_j$.

REFERENCES

1. L. A. Harris, *Banach algebras with involution and Möbius transforms.* J. Funct. Anal. 11 (1972), 1–16.

2. B. Sz-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert spaces,* North Holland, Amsterdam, 1970.

3. A. G. Robertson, *A note on the unit ball in C*-algebras.* Bull. London Math. Soc. 6 (1974), 335–335.

4. B. Russo and H. Dye, *A note on unitary operators in C*-algebras.* Duke Math. J. 33 (1966), 413–416.

Department of Mathematics
INCREST
Bd.Păcii 220, 79622 Bucharest
Romania