FINITENESS AND RIGIDITY THEOREMS FOR HOLOMORPHIC MAPPINGS

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In this paper we study the complex spaces $\mathrm{Hol}_k(X,Y)$ of holomorphic maps of rank $\geq k$ from a compact complex space X into a complex manifold Y. Our results are of the following type: If Y satisfies certain conditions, then for particular k, the space $\mathrm{Hol}_k(X,Y)$ is either discrete or finite, independent of X. Of particular interest is the case $k = \dim Y$, where $\mathrm{Hol}_k(X,Y)$ is the space of surjective holomorphic maps.

Our results are modelled on the classical result of de Franchis that for Y a compact Riemann surface of genus greater than 1, the number of surjective holomorphic maps is finite. Lang [9] raised the question whether finiteness holds for Y compact hyperbolic. Kobayashi and Ochiai [8] proved that the set of surjective meromorphic maps from a Moišezon space into a compact complex space of general type is finite. Recently Noguchi and Sunada [13] proved that if X is Moišezon and $\Lambda^k T_Y$ is Grauert negative, then the number of meromorphic maps of rank $\geq k$ from X to Y is finite. Borel and Narasimhan [1] have also proved discreteness results for holomorphic maps. Similar finiteness theorems for harmonic mappings are given by Lemaire [10].

Our results, which are valid only for holomorphic maps, complement the results of [8] and [13] mentioned above. Theorem 1 says that $\mathrm{Hol}_{k+1}(X,Y)$ is discrete if the holomorphic tangent bundle T_Y satisfies a k-pseudo-convexity condition. In Theorem 1, Y may be noncompact. A consequence of this result (Corollary 2) is that if Y is a compact hermitian manifold with negative holomorphic sectional curvature, then the set of surjective holomorphic maps is finite. We also prove that if Y is an n-dimensional compact Kähler manifold with $c_1(Y)$ represented by a negative semidefinite form and either $c_n(Y) \neq 0$ (Theorem 1) or $c_1^n(Y) \neq 0$ (Theorem 3), then $\mathrm{Hol}_n(X,Y)$ is discrete. In particular, if Y is compact Kähler with first Chern class zero and Euler class nonzero (for example, a Kähler K3 surface), then the space of surjective holomorphic maps onto Y is discrete.

Our method of proof is to consider a one-parameter family of holomorphic maps and to view the derivative with respect to the deformation parameter as a holomorphic mapping from X into the tangent bundle of Y. This method was used independently by Urata [14] to prove Corollary 2.

In the following we let X be a compact, connected complex space, and we let Y be a connected n-dimensional complex manifold. We denote by $\mathrm{Hol}(X,Y)$ the space of holomorphic maps from X to Y, equipped with the compact-open topology. By a well-known result of Douady, $\mathrm{Hol}(X,Y)$ is a complex space (see Lemma 3). If $f \in \mathrm{Hol}(X,Y)$ we define rank f to be the maximum rank of f on the regular points of X; thus $f(X)$ is an analytic space and $\dim f(X) = \operatorname{rank} f$. We let

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$$\mathrm{Hol}_k(X,Y) = \{ f \in \mathrm{Hol}(X,Y) | \mathrm{rank}\, f \geq k \},$$

for $1 \leq k \leq n$.

Definition. Let E be a holomorphic vector bundle of rank r over Y, and let $0 \leq k \leq n-1$. We say that E is G_k-negative if there exists a nonnegative function $\phi \in C^2(E)$ such that

(i) $\phi^{-1}(0)$ equals the zero section Z of E;

(ii) the Levi form of ϕ has at least $r+n-k$ positive eigenvalues at each point of $E-Z$.

Remark. If Y is compact and E is Grauert negative, then E is G_0-negative; i.e., ϕ is strictly plurisubharmonic on $E-Z$.

We shall prove the following general rigidity theorem.

THEOREM 1. *If T_Y is G_k-negative, then* $\mathrm{Hol}_{k+1}(X,Y)$ *is discrete.*

Note that Y may be noncompact, although X must of course be compact.

Before proving this result we give some applications. If E is a hermitian holomorphic vector bundle on Y, we let $R \in E^* \otimes \bar{E}^* \otimes T_Y^* \otimes \bar{T}_Y^*$ denote the curvature tensor with respect to the hermitian connection on E. For $v \in E$ we let $R_v \in T_Y^* \otimes \bar{T}_Y^*$ be given by $R_v(\sigma,\bar{\tau}) = R(v,\bar{v},\sigma,\bar{\tau})$, for $\sigma,\tau \in T_Y$.

LEMMA 1. *Let E be a hermitian holomorphic vector bundle over Y. If R_v has at least $n-k$ negative eigenvalues for all nonzero $v \in E$, then E is G_k-negative.*

Proof. Let $y_0 \in Y$, $v_0 \in E_{y_0} - \{0\}$ be arbitrary. Choose a local frame $\{e_1,\ldots,e_r\}$ for E in a neighborhood of y_0 such that, writing $h_{ij} = H(e_i,\bar{e}_j)$, we have

$$(h_{ij}(y_0)) = I, \qquad dh_{ij}|_{y_0} = 0$$

where I is the identity matrix. For $v \in E$ and $w \in T_Y$ write

$$v = \sum_{i=1}^{r} v^i e_i, \qquad w = \sum_{\alpha=1}^{n} w^\alpha \frac{\partial}{\partial z^\alpha}$$

where $\{z^1,\ldots,z^n\}$ are local coordinates on Y. We have

$$R_{v_0}(w,\bar{w}) = -\sum_{\alpha,\beta} \frac{\partial^2 h_{ij}}{\partial z^\alpha \partial \bar{z}^\beta} v_0^i \bar{v}_0^j w^\alpha \bar{w}^\beta.$$

Let $\phi \in C^2(E)$ be given by $\phi(v) = \|v\|^2 = \sum h_{ij} v^i \bar{v}^j$. The matrix of the Levi form of ϕ at v_0 with respect to the coordinates $(z^1,\ldots,z^n,v^1,\ldots,v^r)$ is

$$\begin{bmatrix} -R_{v_0} & 0 \\ 0 & I \end{bmatrix}$$

which verifies condition (ii) of the definition.

From Theorem 1 and Lemma 1, we obtain an immediate consequence.

COROLLARY 1. *If R_v has at least $n - k + 1$ negative eigenvalues for each nonzero $v \in T_Y$, then* $\mathrm{Hol}_k(X,Y)$ *is discrete.*

COROLLARY 2. *If Y is a compact hermitian manifold with negative holomorphic sectional curvature then* $\mathrm{Hol}_n(X,Y)$ *is finite.*

Proof. This follows from Corollary 1 with $k = n$ together with the fact that, for Y compact hyperbolic, $\mathrm{Hol}(X,Y)$ is compact. (See Kobayashi [6])

Remark. In Corollary 2, the curvature is computed with respect to the hermitian connection. If Y is Kähler, then the holomorphic sectional curvature corresponds to the Riemannian sectional curvature with respect to complex lines in T_Y. Note also that if Y is not compact, then $\mathrm{Hol}_n(X,Y)$ is empty since X is assumed to be compact.

COROLLARY 3. *If Y has negative holomorphic bisectional curvature, then* $\mathrm{Hol}_1(X,Y)$ *is discrete.*

Proof. This is just the case $k = 1$ of Corollary 1.

COROLLARY 4. *If Y is compact and T_Y is Grauert negative, then* $\mathrm{Hol}_1(X,Y)$ *is finite.*

Proof. This follows from Theorem 1 with $k = 0$, together with the compactness of $\mathrm{Hol}(X,Y)$.

Remark. Recently, Noguchi and Sunada [13] proved the following finiteness theorem, which also implies Corollary 4 (for X Moisĕzon): If Y is compact, $\Lambda^k T_Y$ is Grauert negative and X is Moisĕzon, then the set $\mathrm{Mer}_k(X,Y)$ (meromorphic maps of rank $\geq k$) is finite. Corollaries 2 and 4 were also proved independently by Urata [14].

We shall use the following well-known facts to prove Theorem 1.

LEMMA 2. *Let M be a complex manifold, and let $\phi \in C^2(M)$. Suppose $\phi \geq 0$ on M, and the Levi form of ϕ has at least p positive eigenvalues at each point of $M - \phi^{-1}(0)$. If A is an irreducible compact analytic subset of M with*

$$\dim A > \dim M - p,$$

then $A \subset \phi^{-1}(0)$.

Proof. Suppose $A \not\subset \phi^{-1}(0)$. Choose a point a such that $\phi(a) = \sup_A \phi > 0$.

Choose a p-dimensional local complex analytic submanifold N of M such that $a \in N$ and the Levi form of $\phi | N$ is positive definite on N. Let $B = A \cap N$; thus $\dim B \geq 1$. Choose a nonconstant holomorphic map, f, from the unit disc D into B, with $f(0) = a$. Since $\phi \circ f$ is subharmonic on D and attains its maximum at 0, $\phi \circ f$ is constant. Choose $t_0 \in D$ such that $f_{*t_0}(\partial / \partial t) \neq 0$. Then $\partial^2(\phi \circ f)/\partial t \partial \bar{t}\,(t_0) > 0$ which is a contradiction.

LEMMA 3. (Douady [3, pp. 87,90]). *There exists a complex space H and a holomorphic map $h: X \times H \to Y$ such that the induced map $\tilde{h}: H \to \mathrm{Hol}(X,Y)$ is a homeomorphism.*

We now prove Theorem 1. Let T_Y be G_k-negative and suppose $\mathrm{Hol}_{k+1}(X,Y)$ is not discrete. Since $\mathrm{Hol}_{k+1}(X,Y)$ is an open subset of $\mathrm{Hol}(X,Y)$, it follows from Lemma 3 that there exists a nonconstant holomorphic map $f: D \to \mathrm{Hol}_{k+1}(X,Y)$, where D is the unit disc in $\mathbf{C}$. Write $f(t) = f_t \in \mathrm{Hol}_{k+1}(X,Y)$, for $t \in D$. Let $F \in \mathrm{Hol}(X \times D, Y)$ be given by $F(x,t) = f_t(x) = h(x, f_t)$, where h is as in Lemma 3. Define $s \in \mathrm{Hol}(X,T_Y)$ by $s(x) = F_{*(x,t_0)}(\partial/\partial t)$ where $t_0 \in D$ is chosen so that $s(X)$ is not contained in the zero section Z of T_Y. If $\pi: T_Y \to Y$ is the projection, it follows from $\pi \circ s = f_{t_0}$ that $\dim s(X) \geq \dim f_{t_0}(X) \geq k+1$. By Lemma 2 with $M = T_Y$, $A = s(X)$ and $p = r + n - k = \dim T_Y - k$, it follows that

$$s(X) \subset \phi^{-1}(0) = Z,$$

which is a contradiction.

Our next results use the following well-known identity, for which we give a short proof.

LEMMA 4. *Let E be a hermitian holomorphic vector bundle on Y. Let s be a holomorphic section of E and let u be a holomorphic vector field on Y. Then* $u\bar{u}(\|s\|^2) = \|\nabla_u s\|^2 - R(s,\bar{s},u,\bar{u})$.

Proof. Since s and u and holomorphic, $\nabla_{\bar{u}} s = 0$ and $[u,\bar{u}] = 0$. Thus

$$\begin{aligned} u\bar{u}(\|s\|^2) &= \bar{u}u\langle s,\bar{s}\rangle = \bar{u}\langle \nabla_u s,\bar{s}\rangle \\ &= \|\nabla_u s\|^2 + \langle \nabla_{\bar{u}}\nabla_u s,\bar{s}\rangle \\ &= \|\nabla_u s\|^2 - \langle [\nabla_u,\nabla_{\bar{u}}]\, s,\bar{s}\rangle \\ &= \|\nabla_u s\|^2 - R(s,\bar{s},u,\bar{u}). \end{aligned}$$

THEOREM 2. *Let Y be a compact Kähler manifold (of dimension n) with nonzero Euler characteristic. If $c_1(Y)$ is represented by a negative semidefinite* (1,1) *form, then* $\mathrm{Hol}_n(X,Y)$ *is discrete.*

Proof. By Yau's solution to the Calabi conjecture [15], Y carries a Kähler metric with the given semidefinite (1,1) form as Ricci form. Suppose $\mathrm{Hol}_n(X,Y)$ is not discrete, and let $f_t \in \mathrm{Hol}_n(X,Y)$, $t_0 \in D$, and $s \in \mathrm{Hol}(X,T_Y)$ be given as in the proof of Theorem 1. Let $g: X \to X'$, $f': X' \to Y$ be the Stein factorization of f_{t_0}; i.e., X' is a complex space, $f_{t_0} = f' \circ g$, g is surjective with connected fibres, and f' has finite fibres. We note that s is constant on the fibres of g: Let $B = g^{-1}(x')$, where $x' \in X'$. Since B is a connected compact analytic set and $s(B) \subset T_{Y,f'(x')} \approx \mathbf{C}^n$, it follows that s is constant on B. Thus we have a holomorphic map $s' \in \mathrm{Hol}(X',T_Y)$ such that $s = s' \circ g$.

We now show that $\|s'\|$ is constant on X'. Let $a = \sup_{X'} \|s'\| > 0$, and let $x_0' \in X'$ such that $\|s'(x_0')\| = a$. It suffices to show that $\|s'\| \equiv a$ on a neighborhood of x_0'. Let $y_0 = f'(x_0')$. Choose connected neighborhoods X_0' of x_0' and Y_0' of y_0 such that the map $f_0' = f' | X_0' : X_0' \to Y_0$ is a λ_0-sheeted (branched) analytic cover, for some $\lambda_0 \geq 1$, and $f_0'^{-1}(y_0) = \{x_0'\}$, (When $\lambda_0 = 1$, f_0' is a biholomorphism. In the case of $\lambda_0 > 1$, it is a λ_0-sheeted cover branched at y_0.) We define the holomorphic tangent vector field σ on Y_0 by $\sigma(y) = \sum_{\nu=1}^{\lambda_0} s'(x^\nu)$ where $x^1, \ldots, x^{\lambda_0}$ are the points of ${f'_0}^{-1}(y)$ counted with multiplicities. In particular $\sigma(y_0) = \lambda_0 s'(x'_0)$. For arbitrary $y \in Y_0$,

$$\|\sigma(y)\| \leq \sum \|s'(x^\nu)\| \leq \lambda_0 a = \|\sigma(y_0)\|.$$

Thus $\|\sigma\|^2$ attains its maximum at y_0. We now compute the Laplacian of $\|\sigma\|^2$. Let $y \in Y_0$ be arbitrary, and let $u_1, \ldots, u_n$ be holomorphic tangent vector fields in a neighborhood of y that are orthonormal at y. By Lemma 4 we have

$$u_j \bar{u}_j \|\sigma\|^2 \geq -R(\sigma,\bar{\sigma},u_j,\bar{u}_j) = -R(u_j,\bar{u}_j,\sigma,\bar{\sigma}).$$

Thus, at the point y,

$$\Delta\|\sigma\|^2 = 2\sum u_j \bar{u}_j \|\sigma\|^2 \geq -2\sum R(u_j,\bar{u}_j,\sigma,\bar{\sigma}) = -2\mathrm{Ric}(\sigma,\bar{\sigma}) \geq 0.$$

Hence $\|\sigma\|^2$ is subharmonic on Y_0. Since $\|\sigma\|^2$ attains its maximum at an interior point y_0, $\|\sigma\|^2$ must be constant on Y_0. Since, for $y \in Y_0$,

$$\sum_{\nu=1}^{\lambda_0} \|s'(x^\nu)\| \geq \|\sigma(y)\| = \lambda_0 a$$

and $\|s'(x^\nu)\| \leq a$, it follows that $\|s'\| \equiv a$ on X'_0. It follows that $\|s'\| \equiv a$ on X'; hence s' has no zeroes.

Let $p: \tilde{X} \to X'$ be a resolution of the singularities of X'. Then $\tilde{X}$ is a connected n-dimensional complex manifold. Let $\tilde{s} = s' \circ p$, $\tilde{f} = f' \circ p$. Since $\tilde{s}(\tilde{x}) \in T_{Y,\tilde{f}(\tilde{x})}$ for $\tilde{x} \in \tilde{X}$, $\tilde{s}$ defines a section over $\tilde{X}$ of the topological pull-back bundle $\tilde{f}^* T_Y$. Since $\tilde{s}$ has no zeroes, it follows that $c_n(\tilde{f}^* T_Y) = 0$. (See, for example [12].) Thus

$$0 = \langle c_n(\tilde{f}^* T_Y), [\tilde{X}] \rangle = \langle \tilde{f}^* c_n(T_Y), [\tilde{X}] \rangle = \lambda \langle c_n(T_Y), [Y] \rangle = \lambda \chi(Y),$$

where λ is the number of points in the generic fibre of $\tilde{f}$ and χ stands for the Euler characteristic. This contradicts the hypothesis that $\chi(Y) \neq 0$.

COROLLARY 5. *If Y is compact Kähler with $c_1(Y) = 0$, $c_n(Y) \neq 0$, then* $\mathrm{Hol}_n(X,Y)$ *is discrete.*

COROLLARY 6. *If Y is a Kähler K3 surface, then* $\mathrm{Hol}_2(X,Y)$ *is discrete.*

THEOREM 3. *If Y is compact Kähler, $c_1(Y)$ is negative semidefinite and $c_1^n(Y) \neq 0$, then* $\mathrm{Hol}_n(X,Y)$ *is discrete.*

Remark. The hypothesis of Theorem 3 on $c_1(Y)$ is equivalent to the existence of a negative semidefinite (1,1) form η in $c_1(Y)$ such that η is negative definite somewhere.

Proof. By Yau [15], we can give Y the Kähler metric whose Ricci form is the form η given in the remark above. Suppose $\mathrm{Hol}_n(X,Y)$ is not discrete and let $s \in \mathrm{Hol}(X,T_Y)$ be as before. By the proof of Theorem 2, $\|s\|$ is constant on

X. Choose a point $y_0 \in Y$ with $\mathrm{Ric}_{y_0} < 0$. By again repeating the proof of Theorem 2 with $x_0' \in f'^{-1}(y_0)$, we obtain a holomorphic vector field σ on a neighborhood Y_0 of y_0 such that $\sup_{Y_0} \|\sigma\| = \|\sigma(y_0)\| > 0$. However $\Delta \|\sigma\|^2|_{y_0} \geq -2\mathrm{Ric}(\sigma,\sigma)|_{y_0} > 0$, which contradicts the maximality of $\|\sigma(y_0)\|^2$.

Theorem 3 generalizes a result of Lichnerowicz [11] (see also Kobayashi [5,p.104]) that if Y is as in Theorem 3, then the automorphism group of Y is discrete. In fact, if Y is a compact Kähler manifold such that $c_1(Y)$ is negative semidefinite and the Chern numbers of Y are not all zero, then the automorphism group of Y is discrete. (Theorems 2 and 3 consider the Chern numbers c_n and c_1^n, respectively.) This well-known result is shown as follows: Suppose on the contrary that Y carries a nontrivial holomorphic tangent vector field s. Then by Lemma 4, $\|s\|^2$ is subharmonic with respect to the Yau metric. Thus $\|s\|^2$ is constant and hence s has no zeroes. By a result of Bott [2] and Illusie [4], the Chern numbers of Y vanish, contrary to the hypothesis.

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