MAPPINGS WITH DENSE DEFICIENCY SET

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1. INTRODUCTION

There are easily constructed examples of maps between compact, orientable manifolds which have deficient points; that is, there are points y in the image for which $\# f^{-1}(y) < |\deg(f)|$ ($\#$ = cardinality). If f is a d-to-1 ($d \geq 2$) covering map of the 1-sphere, then the suspension $\Sigma f: S^2 \to S^2$ has two deficient points while further suspensions $\Sigma^{q-1} f: S^q \to S^q$ yield a map whose deficient points comprise a $(q-2)$-sphere. Let Δ_f denote the set of deficient points of a map f between orientable manifolds. For maps between 1-manifolds $\Delta_f = \phi$, and it is a consequence of a result of Hopf [4] that for maps between 2-manifolds Δ_f is discrete. In dimensions $q \geq 3$, Honkapohja [2] showed that the non-deficient points are dense and, therefore, $\dim \Delta_f \leq q-1$; and Church and Timourian [1] showed that each compact subset of Δ_f has dimension at most $q-2$.

The question was posed to the author by P. T. Church whether the deficient points could be dense; the examples constructed in this paper have this property. Specifically, for each pair of integers $q \geq 3$ and $d \geq 2$ an example is constructed of a monotone map $f: S^q \to S^q$ such that $|\deg(f)| = d$, Δ_f is a $(q-3)$-dimensional dense subset, and $f^{-1}(\Delta_f)$ is a dense subset. Since each $f^{-1}(y)$ is connected, the restriction of f is a homeomorphism from $f^{-1}(\Delta_f)$ to Δ_f.

The above situation contrasts sharply with that which occurs for discrete maps, in which case $\dim \bar{\Delta}_f \leq q-2$ [1], and for light maps, in which case $\dim \bar{\Delta}_f \leq q-1$ [1].

The techniques used to produce the examples are taken from those developed in [8]. The techniques developed in the latter paper are more systematic and "controlled" that their predecessors used in [9], [10] and [6].

A map is *monotone* provided each point-inverse is compact and connected. We define $st(a,B) = a \cup \{b \in B : b \cap a \neq \emptyset\}$ and, recursively, $st^i(a,B) = st(st^{i-1}(a,B))$.

2. PRELIMINARIES

The basic approach which will be used to construct the examples is that developed in [8]. The machinery described there is more complicated than what is needed for our current purposes. In order to have a self contained description, the necessary components with proofs will be reproduced.

The barycentric subdivision of a triangulation L is denoted βL and the nth-barycentric subdivision is defined by the recursive formula $\beta^n L = \beta(\beta^{n-1} L)$. Geometric

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barycenters are used so that the diameters of the simplices of the $\beta^n L$'s are converging to zero.

A collection P is a *stratified partition* of a closed PL manifold M^q provided:

(1) P is a cover of M consisting of q-dimensional PL submanifolds;

(2) for $2 \le i \le q+1$, if $p[1], \ldots, p[i]$ are distinct elements of P with $p[1] \cap \ldots \cap p[i] \neq \emptyset$, then the intersection is a $(q-i+1)$-dimensional PL submanifold of the boundary of $p[1] \cap \ldots \cap p[j-1] \cap p[j+1] \cap \ldots \cap p[i]$ for $1 \le j \le i$.

For $1 \le i \le q+1$, let $S_{q-i+1} = \{p[1] \cap \ldots \cap p[i]$: the distinct elements $p[1], \ldots, p[i]$ of P have nonempty intersection$\}$; the elements of the collection S_{q-i+1} are called the *$(q-i+1)$-dimensional strata of P.*

Remark. Let L be a triangulation of a PL manifold M^q and, for $n \ge 0$, let $J_n = \{st(v,\beta^{n+1}L) : v \text{ is a vertex of } \beta^n L\}$ where $\beta^0 L = L$. Each collection J_n is a stratified partition of M whose i-dimensional strata consist of i-cells.

THEOREM 2.1. *Let M^q and N^q be closed* PL *manifolds, let L be a triangulation of N, and let $\{T_n: P_n \to J_n\}_{n=0}^{\infty}$ be a sequence of triples satisfying:*

(1) $J_n = \{st(v,\beta^{n+1}L) : v \text{ is a vertex of } \beta^n L\}$;

(2) *P_n is a stratified partition of M;*

(3) *T_n is a bijection and, given elements $p[i] \in P_n$, $\cap\, p[i] \neq \emptyset$ if and only if $\cap\, T_n(p[i]) \neq \emptyset$;*

(4) *for $p_n \in P_n$ and $p_{n+1} \in P_{n+1}$, $T_n(p_n) \cap T_{n+1}(p_{n+1}) \neq \emptyset$ if and only if* $\text{Int}(p_n \cap p_{n+1}) \neq \emptyset$.

Then there is a continuous function h from M onto N satisfying

$$h^{-1}(y) = \bigcap_{n=1}^{\infty} st(p_n,P_n)$$

for any choice of $p_n \in P_n$ with $y \in T_n(p_n)$.

Proof. It is worth noting that the symmetric version of condition (4) is valid since it is easy to verify that elements $j_n \in J_n$ and $j_{n+1} \in J_{n+1}$ meet if and only if $\text{Int}(j_n \cap j_{n+1}) \neq \emptyset$.

It is an easy exercise to show that $st^2(j_n,J_{n+1}) \subset \text{Int}(st(j_n,J_n))$ for $j_n \in J_n$ and, therefore, that, for $j_n \in J_n$ and $j_{n+1} \in J_{n+1}$,

$$(*) \qquad \text{if } j_n \cap j_{n+1} \neq \emptyset, \qquad \text{then } st(j_{n+1},J_{n+1}) \subset \text{Int}(st(j_n,J_n)).$$

Conditions (3) and (4) combine with (*) to yield the property that, for $p_n \in P_n$ and $p_{n+1} \in P_{n+1}$,

$$(\dagger) \qquad \text{if } p_n \cap p_{n+1} \neq \emptyset, \qquad \text{then } st(p_{n+1},P_{n+1}) \subset \text{Int}(st(p_n,P_n)).$$

Verifying that h is a function from M onto N proceeds as follows. First, using

(*), (†), and (3) we deduce that if $p_n \in P_n$ is chosen so that $y \in T_n(p_n)$, then both $\cap\, st(p_n, P_n)$ and $\cap\, st(T_n(p_n), J_n)$ are nested intersections and, in particular are nonempty. Second, an element $x \in M$ is contained in $h^{-1}(y)$ where y is determined by choosing $p_n \in P_n$ with $x \in p_n$ and letting $\{y\} = \cap\, st(T_n(p_n), J_n)$. Finally, if $y, y' \in N$ and $y \neq y'$, then, for some k, $st^2(y, J_k) \cap st^2(y', J_k) = \emptyset$ and, therefore, choosing $j, j' \in J_k$ with $y \in j$ and $y' \in j'$, we have the containment

$$h^{-1}(y) \cap h^{-1}(y') \subset st(T_k^{-1}(j), P_k) \cap st(T_k^{-1}(j'), P_k)$$

and the latter intersection is readily seen to be empty using condition (3).

A consequence of (*) is that if $x \in p \in P_n$, then $h(x) \in st^2(T_n(p), J_{n+1})$. In addition, if $p' \in P_n$ and $p \cap p' \neq \emptyset$, then

$$h(p') \subset st^2(T_n(p'), J_{n+1}) \subset st^2(st(T_n(p), J_n), J_{n+1})$$

and, therefore, we conclude that $h(st(p, P_n)) \subset st^2(st(T_n(p), J_n), J_{n+1})$. The continuity of h at x follows since the sets $st^2(st(T_n(p), J_n), J_{n+1})$ form a neighborhood basis for $h(x)$ and $x \in \mathrm{Int}(st(p, J_n))$.

Remark. If the elements of each stratified partition P_n are connected, then the map h is easily seen to be monotone. The examples construction in Section 5 are monotone for precisely this reason, however, in addition, the connectivity of the elements of P_n plays an important role in the inductive construction of P_{n+1} (see the Main Proposition).

3. DETECTING CELLULAR POINT-INVERSES

We find it convenient to call a sequence of triples $\{T_n: P_n \to J_n\}_{n=0}^{\infty}$ which satisfies conditions (1)-(4) of Theorem 2.1 a *defining sequence* and to call h the *associated map.* A first attempt at producing functions with many cellular point-inverses might be to produce defining sequences for which the P_n's contain many cells. However, since the point-inverses of the associated map are intersections of $st(p_n, P_n)$'s, it is not immediately evident that such an approach can work. This section consists of showing that such a program can be successful. For the remainder of this section, let $\{T_n: P_n \to J_n\}$ be a defining sequence with associated map h (the underlying spaces being PL manifolds M^q and N^q).

For $n \geq 1$, each element $j \in J_n$ is assigned an *index,* denoted Index(j), which is equal to the dimension of σ where $j = st(v, \beta^{n+1}L)$ and v is the barycenter of $\sigma \in \beta^{n-1}L$. In practice (e.g. Section 5), L may not be a barycentric subdivision itself in which case the elements of J_0 cannot be assigned an index. The next lemma in concerned with relationships between the J_n's; the proof is left as an exercise for the reader.

LEMMA 3.1. *For $n \geq 1$, let $J_n^{(s)}$ be the collection of elements of J_n with index equal to s.*

(1) *For $n \geq 1$, the elements of $J_n^{(s)}$ are a maximal pairwise disjoint subcollection of J_n.*

(2) *For $n \geq 0$, $j \in J_{n+1}^{(s)}$ if and only if j intersects exactly $s + 1$ elements of J_n.*

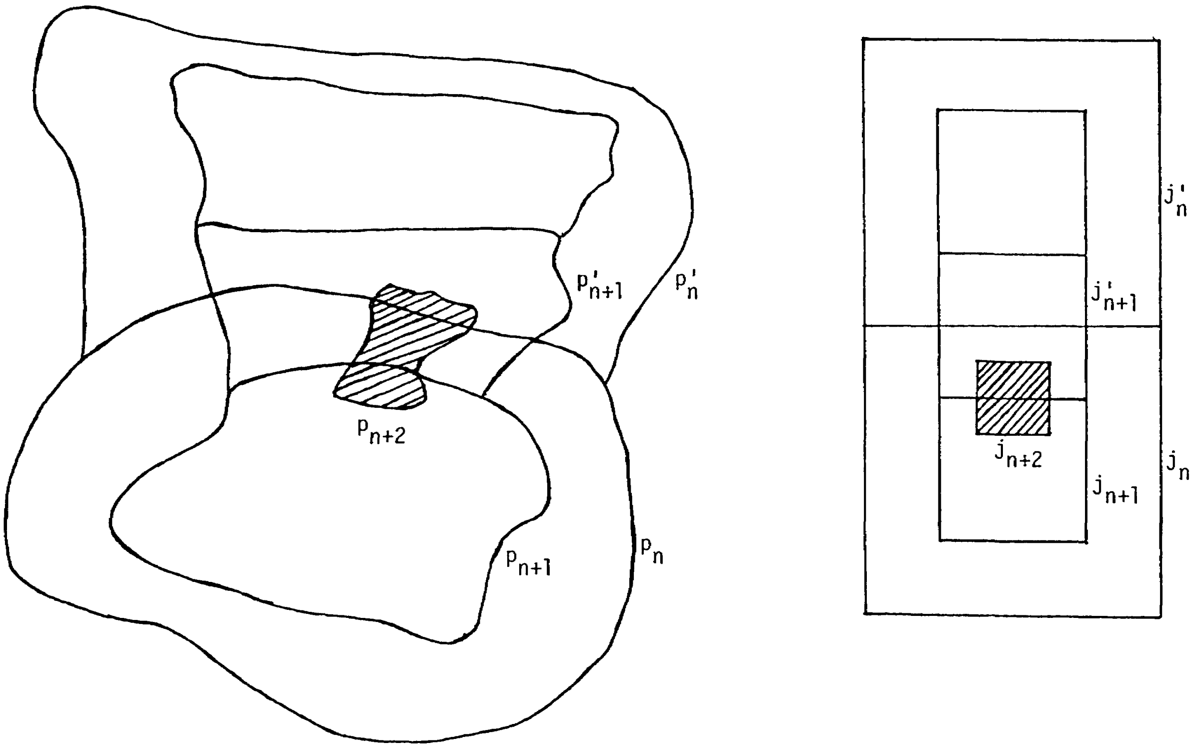

Figure 1

(3) *For* $n \geq 0, j \in J_n$, *and* $i \geq 1$, $st(j, J_{n+i}) \subset \operatorname{Int}(st(j, J_{n+i-1}))$.

If $y \in j \in J_n$, then $h^{-1}(y)$ may not be contained in $T_n^{-1}(j)$; in fact, even if $y \in \operatorname{Int}(j)$, $h^{-1}(y)$ may not be contained in $T_n^{-1}(j)$. The problem is that although if $j_n \in J_n$ and $j_{n+1} \in J_{n+1}$ with $j_{n+1} \subset \operatorname{Int}(j_n)$ and $j_{n+2} \in J_{n+2}$ with $j_{n+2} \cap j_{n+1} \neq \emptyset$, then $j_{n+2} \subset \operatorname{Int}(j_n)$, the analogous statement for the P_n's need not be true. For example, if $j_{n+2} \subset \operatorname{Int}(j_{n+1} \cup j'_{n+1}) \subset \operatorname{Int}(j_n \cup j'_n)$, then in general the best we can expect is that $T_{n+2}^{-1}(j_{n+2}) \subset \operatorname{Int}(T_n^{-1}(j_n) \cup T_n^{-1}(j'_n))$; see Figure 1.

LEMMA 3.2. *If* $j \in J_n$ *and* $y \in \operatorname{Int}(j)$, *then* $h^{-1}(y) \subset \operatorname{Int}(st(T_n^{-1}(j), P_{n+1}))$.

Proof. We need the following consequence of condition (4) (of Theorem 2.1) which defining sequences satisfy: if $A \subset J_i$ and $B \subset J_{i+1}$ with

$$\cup\{b \in B\} \subset \operatorname{Int}(\cup\{a \in A\}),$$

then $\cup\{T_{i+1}^{-1}(b) : b \in B\} \subset \operatorname{Int}(\cup\{T_i^{-1}(a) : a \in A\})$.

Since $y \in \operatorname{Int}(j)$, there is an integer k and a $j_{n+k} \in J_{n+k}$ with $y \in j_{n+k}$ and $st(j_{n+k}, J_{n+k}) \subset st(j, J_{n+k})$. In turn we have that

$$h^{-1}(y) \subset \operatorname{Int}(st(T_{n+k}^{-1}(j_{n+k}), P_{n+k})) \subset \cup\{T_{n+k}^{-1}(a) : a \in J_{n+k} \quad \text{and} \quad a \cap j \neq \emptyset\}.$$

The observation made in the preceding paragraph combines with condition (3) of Lemma 3.1 to show that, for $2 \leq i \leq k$,

$$\cup \{T^{-1}_{n+i}(a) : a \in \mathcal{J}_{n+i} \quad \text{and} \quad a \cap j \neq \emptyset\}$$
$$\subset \cup \{T^{-1}_{n+i-1}(a) : a \in \mathcal{J}_{n+i-1} \quad \text{and} \quad a \cap j \neq \emptyset\}.$$

Finally, these containments yield that

$$h^{-1}(y) \subset \text{Int}(\cup \{T^{-1}_{n+1}(a) : a \in \mathcal{J}_{n+1} \qquad \text{and} \qquad a \cap j \neq \emptyset\}$$

Condition (3) (of Theorem 2.1) which defining sequences satisfy shows that this is in fact the sought after containment.

We now determine those points of N which are "purely of index q" (with respect to the $\mathcal{J}_n$'s) in the sense described in part (3) of the next lemma. For each $n \geq 1$, define a sequence of closed subsets of N by $F_n = \bigcap_{i \geq n} [N - \cup \{\text{Int}(j) : j \in \mathcal{J}_i^{(q)}\}]$.

LEMMA 3.3. *Adopting the preceding notation,*

(1) $\dim F_n \leq q - 1$;

(2) $N - \cup F_n$ *is dense in* N;

(3) *if* $y \in N - \cup F_n$, *then for infinitely many* i *there is a* $j \in \mathcal{J}_i^{(q)}$ *with* $y \in \text{Int}(j)$.

Proof. Condition (3) follows easily from the definitions of the F_n's and (2) follows from (1) since the Sum theorem [5; p. 30] yields that $\dim \cup F_n \leq q - 1$. In order to verify condition (1), observe that for each $i \geq 1$,

$$N - \cup \{\text{Int}(j) : j \in \mathcal{J}_i^{(q)}\} = st(D_i, \beta^{i+1} L)$$

where D_i is the dual $(q-1)$-skeleton of $\beta^{i-1} L$ ($\beta^0 L = L$). Given $\epsilon > 0$ and integer n, if $i \geq n$ is chosen sufficiently large, then $N - \cup \{\text{Int}(j) : J \in \mathcal{J}_i^{(q)}\}$ ϵ-maps to D_i and the map restricts to produce an ϵ-map of F_n to D_i; therefore, $\dim F_n \leq q - 1$.

PROPOSITION 3.4. *Let* $\{T_n : P_n \to \mathcal{J}_n\}_{n=0}^{\infty}$ *be a defining sequence with associated map* $h : M^q \to N^q$ *and suppose that for each* $j \in \mathcal{J}_n^{(q)}$ $(n \geq 1)$, *there is a* q*-cell* Q *such that* $st(T_n^{-1}(j), P_{n+1}) \subset \text{Int}\, Q \subset st(T_n^{-1}(j), P_n)$. *Then the points* $y \in N$ *for which* $h^{-1}(y)$ *is cellular form a dense subset of* N.

Proof. A closed subset of M is *cellular* provided it is the nested intersection of open q-cells. Let the F_n's be the closed subsets of N from Lemma 3.3; we show that, for each $y \in N - \cup F_n$, $h^{-1}(y)$ is cellular. Let $j_n \in \mathcal{J}_n$ with $y \in j_n$; condition (3) of Lemma 3.3 implies that, for infinitely many n, $y \in \text{Int}(j_n)$ and $j_n \in \mathcal{J}_n^{(q)}$. On the one hand

$$st(T^{-1}_{n+1}(j_{n+1}), P_{n+1}) \subset \text{Int}(st(T_n^{-1}(j_n), P_n)) \quad \text{and} \quad h^{-1}(y) = \cap\, st(T_n^{-1}(j_n), P_n);$$

on the other hand, for infinitely many n, there is a q-cell Q with

$$st(T^{-1}_{n+1}(j_{n+1}), P_{n+1}) \subset st(T_n^{-1}(j_n), P_{n+1}) \subset \text{Int}\, Q \subset st(T_n^{-1}(j_n), P_n).$$

Therefore, $h^{-1}(y)$ is cellular.

4. SHRINKING COUNTABLY MANY CELLULAR POINT-INVERSES

The material of this section involves an elementary application of standard techniques used in the study of cellular decompositions.

PROPOSITION 4.1. *Let $h: M^q \to Y$ be a map from a compact q-manifold to a compact metric space and let $y_1, y_2, \ldots$ be a sequence of points in Y such that $h^{-1}(y_i)$ is a cellular subset of M^q. Then h can be approximated arbitrarily closely by a map g with each $g^{-1}(y_i)$ a point.*

Proof. The map g is constructed as the limit of maps θ_k where $\theta_k^{-1}(y_i)$ is a point for $i = 1, \ldots, k$.

Since $h^{-1}(y_1)$ is cellular, the quotient map $\pi: M \to M/h^{-1}(y_1)$ can be uniformly approximated by homeomorphisms. Let α be such a homeomorphism and let $\theta_1 = h \circ \pi^{-1} \circ \alpha$. Then $\theta_1^{-1}(y_1)$ is a point and, if $h^{-1}(y)$ is a point for some y, then $\theta_1^{-1}(y)$ remains a point. Furthermore, the restriction of $\alpha^{-1} \circ \pi$ is a homeomorphism from $M - h^{-1}(h_1)$ to $M - \theta_1^{-1}(y_1)$, and, therefore, each $\theta_1^{-1}(y_i) = \alpha^{-1} \circ \pi(h^{-1}(y_i))$ is cellular.

The process is repeated starting with θ_1 and a map θ_2 is obtained which approximates θ_1 with $\theta_2^{-1}(y_i)$ a point for $i = 1, 2$. Continuing in this manner, the sequence of θ_k's is obtained and, if they are chosen to converge fast enough, then $g = \lim \theta_k$ approximates h and each $g^{-1}(y_i)$ is a point.

Remark. A map g constructed as above has the property that, for each $y \in Y$, $h^{-1}(y)$ and $g^{-1}(y)$ have the same shape; a discussion of this can be found in [7]. In particular, if h is monotone, then g is also monotone.

5. THE EXAMPLES

The next proposition presents an inductive procedure which is combined with the material of Sections 2, 3, and 4 in order to produce the promised examples.

MAIN PROPOSITION. *Let M^q and N^q be* PL *manifolds with $q \geq 3$, let L be a triangulation of N, and let $T: P \to J$ be a triple satisfying:*

(a) *$J = \{st(v, \beta^1 L) : v$ is a vertex of $L\}$;*

(b) *P is a stratified partition of M with each element of P connected;*

(c) *T is a bijection and, for elements $p[i] \in P$, $\cap\, p[i] \neq \emptyset$ if and only if $\cap\, T(p[i]) \neq \emptyset$.*

Then there is a triple $\tilde{T}: \tilde{P} \to \tilde{J}$ satisfying:

(1) *$\tilde{J} = \{st(v, \beta^2 L)$: v is a vertex of $\beta^1 L\}$;*

(2) *$\tilde{P}$ is a stratified partition of M with each element of $\tilde{P}$ connected and with $\tilde{T}^{-1}(j)$ a q-cell for $j \in \tilde{J}^{(q)}$;*

(3) *$\tilde{T}$ is a bijection and, for elements $p[i] \in \tilde{P}$, $\cap\, p[i] \neq \emptyset$ if and only if $\tilde{T}(p[i]) \neq \emptyset$;*

(4) *for $p \in P$ and $\tilde{p} \in \tilde{P}$, $T(p) \cap \tilde{T}(\tilde{p}) \neq \emptyset$ if and only if* $\text{Int}(p \cap \tilde{p}) \neq \emptyset$;

(5) *if L itself is a barycentric subdivision, then given neighborhoods*

$$U[j] \quad \textit{of} \quad T^{-1}(j)$$

for $j \in J^{(q)}$, *it can be arranged that* $st(T^{-1}(j), \tilde{P}) \subset U[j]$.

Proof. The collection $\tilde{J}$ is a stratified partition of N; in fact, $\tilde{J}$ is a standard handlebody decomposition of M associated with the triangulation L. There is a "natural" bijection between the elements of $\tilde{J}$ and the strata of J; each i-dimensional stratum of J is the dual cell of a $(q-i)$-dimensional simplex σ of L and the associated element of $\tilde{J}$ is $st(\hat{\sigma}, \beta^2 L)$ where $\hat{\sigma}$ is the barycenter of σ. The collection $\tilde{P}$ is obtained in two steps with the first producing a collection $\hat{P}$ in a manner analogous to the preceding description of $\tilde{J}$.

The collection $\hat{P}$. Let K be a triangulation of M such that each stratum of P is a full subcomplex. For each stratum s of P, let

$$c(s) = \cup\{\sigma : \sigma \text{ is a simplex of } \beta K \text{ and } \sigma \subset s - \partial s\}.$$

The set $c(s)$ can be thought of as the core of s and plays the same role that the barycenter $\hat{\sigma}$ played in the description of $\tilde{J}$. Let $H = \{st(v, \beta^2 K) : v \text{ is a vertex of } \beta K\}$ be the handlebody decomposition of M associated with K. The elements of $\hat{P}$ are in one-to-one correspondence with the strata of P with a stratum s determining the element

$$p_s = \cup\{st(v, \beta^2 K) \in H : v \in c(s)\}.$$

Since the cores of the strata of P are pairwise disjoint, each element of H is contained in at most one element of $\hat{P}$ and it is easy to check that each element of H is contained in at least one element of $\hat{P}$. It follows from the fact that any "amalgamation" of a handlebody decomposition is a stratified partition that $\hat{P}$ is a stratified partition; details of an inductive argument can be found in [8]. A function $\hat{T}: \hat{P} \to \tilde{J}$ is determined as follows: given a stratum s of P, say $s = \cap\, p[i]$ for elements $p[i] \in P$, $\hat{T}(p_s) = j$ where $j \in \tilde{J}$ is associated with the stratum $\cap\, T(p[i])$ of J. It is clear that the triple $\hat{T}: \hat{P} \to \tilde{J}$ satisfies condition (1) and condition (4) follows from the definition of $\hat{T}$ together with the fact that an element $h \in H$ meets an element $p \in P$ if and only if $\operatorname{Int}(h \cap p) \neq \emptyset$. If K is chosen so that the diameters of the elements of H are small, then condition (5) holds. It remains to verify condition (3). Condition (c) and the fact that each collection of elements $p[i] \in P$ with $\cap\, p[i] \neq \emptyset$ determines a unique stratum $s = \cap\, p[i]$ of P allow T to be extended to a bijection of the strata of P and the strata of J by letting $T(s) = \cap\, T(p[i])$. Since $\hat{T}(p_s) = j$ where j is determined by the stratum $T(s)$, $\hat{T}$ is a bijection and the second part of condition (3) follows since, given elements $p_{s[i]} \in \hat{P}$,

$$\cap\, p_{s[i]} \neq \emptyset \Leftrightarrow \cap\, s[i] \neq \emptyset \Leftrightarrow \cap\, T(s[i]) \neq \emptyset \Leftrightarrow \hat{T}(p_{s[i]}) \neq \emptyset$$

The Collection $\tilde{P}$. Let $S_1, S_2, \ldots, S_q$ denote the strata of P and notice that $S_q = P$ (a fact which is used several times in the following). An element $p_s \in \hat{P}$

is connected if and only if the associated stratum s is connected. Each stratum in S_q is connected (condition (b)) and, hence, for each $s \in S_q$, p_s is connected (p_s being equal to s with an open regular neighborhood of ∂s removed).

If $p \in \hat{P}$ and p is associated with the stratum $\cap p[i]$ of P, then each component of p intersects the connected element $p_s \in P$ where $s \in S_q$ is equal to any one of the $p[i]$'s. For an element $p \in \hat{P}$ which is not connected, choose an element $s \in S_q$ as just described with the additional provision that, if possible, choose $s \in J^{(q)}$. (In particular, if L is not a barycentric subdivision, then the last provision is to be ignored.) Let X_p be a set consisting of a single point from the intersection of each component of p with p_s chosen so that $X_p \subset \text{Int}(p \cup p_s)$. Since $q \geq 3$, the cone on X_p, say $C(X_p)$, PL embeds in p_s with $C(X_p) \cap \partial(p_s) = X_p$. Let $N_p \subset p_s$ be a "tubular" neighborhood of $C(X_p)$. The choices for different elements of $\hat{P}$ are to be made so that the N_p's are pairwise disjoint. The collection $\tilde{P}$ is obtained by replacing each $p \in \hat{P}$ which is not connected by $p \cup N_p$ and each $p' \in P$ which is connected by $\text{Cl}(p' - \cup\{N_p \subset p'\})$. Finally, $\tilde{T}$ is the function induced by $\hat{T}$. The elements of $\tilde{P}$ are connected and if $j \in \tilde{J}^{(q)}$, then $p = \hat{T}^{-1}(j)$ is associated with a stratum $s \in S_0$; that is, p is a regular neighborhood of a finite set of points. The element p is a finite union of pairwise disjoint q-cells and $T^{-1}(j) = p \cup N_p$ is a q-cell.

If $p \in \hat{P}$, then there is at most one element $s \in S_q \cap \{T^{-1}(j) : j \in J^{(q)}\}$ with $p \cap p_s \neq \emptyset$ and, in this case, if p is not connected, then $p \cup N_p \subset p \cup p_s$. It follows that if $j \in J^{(q)}$, then $st(T^{-1}(j),\hat{P}) = st(T^{-1}(j),\tilde{P})$ and that condition (5) continues to hold.

MAIN THEOREM. *For each pair of integers $q \geq 3$ and $d \geq 2$, there is a monotone map $g: S^q \to S^q$ and there are closed sets $A_1, A_2, \ldots$ such that $|\deg(g)| = d$, $\dim A_i = q - 3$, $\cup A_i$ is dense, and g is one-to-one over $\cup A_i$.*

Proof. First, the task is reduced to handling the case $q = 3$: suppose that $g: S^3 \to S^3$ is the sought after map where the collection of A_i's form a countable dense subset of S^3; then suspending $(q-3)$ times produces the desired map $\Sigma_q^{q-3}: S^q \to S^q$ and the desired closed sets $\Sigma^{q-3} A_i$.

Using Proposition 4.1, the problem further reduces to constructing a map $h: S^3 \to S^3$ with $|\deg(h)| = d$ for which there is a dense set of points $y \in S^3$ with $h^{-1}(y)$ being cellular.

The mapping h arises as the map associated with a defining sequence satisfying the additional hypothesis of Proposition 3.4.

Suppose that $T: P \to J$ is a triple satisfying the hypothesis of the Main Proposition. Repeated applications of the proposition produce an appropriate defining sequence provided that, for $n \geq 1$, each $U[j] \subset st(T_n^{-1}(j),P_j)$ is chosen to be a regular neighborhood of $T_n^{-1}(j)$ for $j \in J_n^{(q)}$ (and the triangulation L in the hypothesis is replaced by the appropriate barycentric subdivision of L for each application).

It remains to produce the triple $T: P \to J$ and to do it in such a way that the mapping h ultimately produced has degree d.

The Triple $T: P \to J$. Let $\alpha: S^1 \to S^1$ be a d-to-1 covering map and let K^* and K be triangulations of S^1 so that α is a simplicial map from K^* to K. Let $\Sigma^2 K^*$ and $\Sigma^2 K$ be triangulations of S^3 $(= \Sigma^2 S^1)$ where each is obtained by introducing four additional vertices—these being the suspension points. The map $A = \Sigma^2 \alpha$ is simplicial and has degree d. Choose barycenters for $\beta(\Sigma^2 K^*)$ and $\beta(\Sigma^2 K)$ so that A remains simplicial. Let $J = \{st(v,\beta(\Sigma^2 K)) : v$ is a vertex of $\Sigma^2 K\}$, let $\hat{P} = \{A^{-1}(j) : j \in J\}$, and let $\hat{T}$ be defined by $\hat{T}(A^{-1}(j)) = j$. An alternate description of $\hat{P}$ and $\hat{T}$ is that

$$\hat{T}^{-1}(st(v,\beta(\Sigma^2 K))) = \cup \{st(w,\beta(\Sigma K^*)) : w \in A^{-1}(v)\}.$$

The triple $\hat{T}:\hat{P} \to J$ only has one defect—the elements of P are not connected. The four elements of $\hat{P}$ corresponding to the suspension points are q-cells but each of the other elements is the disjoint union of d q-cells. However, each component of each of the latter elements intersects each of the four connected elements. The triple $T:P \to J$ is obtained by modifying the elements of $\hat{P}$ as was done in the proof of the Main Proposition in order to obtain P.

It remains to show that the map h obtained has degree d; this is done in the next section by showing that h and A are homotopic.

6. INDUCED MAPS AND HOMOTOPIES

Let P be a stratified partition of a PL manifold M^q, let

$$J = \{st(v,\beta^1 L) : v \quad \text{is a vertex of} \quad L\}$$

for some triangulation L of a PL manifold N^q, and let $T:P \to J$ be a bijection such that for elements $p[i] \in P$, $\cap\, p[i] \neq \emptyset \Leftrightarrow \cap\, T(p[i]) \neq \emptyset$. In the discussion at the beginning of the proof of the Main Proposition, it was observed that T extends to a bijection, also denoted by T, between the strata of P and the strata of J and that each stratum of J is the dual cell of a simplex of L. Also, given a triangulation K of M with each stratum of P a full subcomplex, a core of each stratum s was defined by

$$c(s) = \cup \{\sigma : \sigma \quad \text{is a simplex of} \quad \beta K \quad \text{and} \quad \sigma \subset s - \partial s\}.$$

Define $f:M \to N$ by defining $f(c(s))$ to be equal to the barycenter of σ where $T(s)$ is the dual cell of σ and by extending linearly. The map f is simplicial from K to $\beta^1 L$ and is called an *induced map* (for $T:P \to J$). Although the map f depends on the triangulation K, $f(p) = T(p)$ for each $p \in P$ and it is easily deduced from (**), which is stated in the next paragraph, that any two induced maps are homotopic.

The following is well known; the proof is not difficult and is omitted.

(**) Let $\mathscr{U}$ be a finite closed cover of a space X such that each element of $\mathscr{U}$ is an absolute retract and such that each nonempty intersection of elements of $\mathscr{U}$ is an absolute retract. Then for any space Y, each pair of maps $f, g: Y \to X$ which are $\mathscr{U}$-close are homotopic.

LEMMA 6.1. *If $\{T_n : P_n \to J_n\}$ is a defining sequence with induced maps $\{f_n\}$, then each f_n is homotopic to f_{n+1} and $h = \lim f_n$ is the associated map.*

Proof. It is easy to verify that f_n and f_{n+1} are $\mathscr{U}$-close where

$$\mathscr{U} = \{st(j,J_{n+1}) : j \in J_n\}$$

and (**) applies to produce the homotopy. The second conclusion follows easily by comparing definitions and its verification is left to the reader.

The final step in the proof of the Main Theorem will be completed provided we show that the map A is homotopic to an induced map $\hat{f}$ for $\hat{T}:\hat{P} \to J$ and that $\hat{f}$ is homotopic to an induced map f for $T: P \to J$. Since A and $\hat{f}$ are J-close, they are homotopic by (**). It is convenient to assume that the same triangulation R of S^3 was used to specify the induced maps $\hat{f}$ and f. Since the collections $\hat{P}$ and P differ only on a finite union of 3-cells (these being tubular neighborhoods of cones on d points), $\hat{f}$ and f are equal off of a regular neighborhood of this union. If Q denotes one of these 3-cells and $Q^+ = st(Q,\beta R)$, then $\hat{f} = f$ on $Fr\, Q^+$. We are assured a homotopy of $\hat{f}\,|_{Q^+}$ to $f\,|_{Q^+}$ relative to $Fr\, Q^+$ provided $\hat{f}(Q^+) \cup f(Q^+)$ misses a point of S^3; but the only suspension point in $\hat{f}(Q^+) \cup f(Q^+)$ is the one contained in the element of P which contains Q. These homotopies piece together to produce a homotopy of $\hat{f}$ and f.

7. A FINAL REFINEMENT

Each of the monotone maps $g: S^3 \to S^3$ produced by the Main Theorem has the property that there is a dense subset $D \subset S^3$ such that the restriction of g is a homeomorphism from $g^{-1}(D)$ onto D; the following adjustments are sufficient in order to insure that $g^{-1}(D)$ is also a dense subset. First, the main result in [7] yields a monotone open map h approximating g with $h^{-1}(y)$ a cellular subset for each $y \in D$. Second, incorporate the modifications described next into the process used in the proof of Proposition 4.1 to "shrink" the sets $h^{-1}(y_i)$.

(1) Choose θ_k so that $\theta_k(y_i) = \theta_{k-1}(y_i)$ for $1 \leq i \leq k-1$ (this amounts to choosing a homeomorphism α which agrees with π on the complement of a neighborhood of $\theta_{k-1}^{-1}(y_k)$).

(2) Choose a countable basis $U_1, U_2, \ldots$ for M^q and (reordering the y_i's as necessary) choose y_k so that $\theta_{k-1}^{-1}(y_k) \cap U_k \neq \emptyset$ (since h is open, each of the θ_i's is open so that such a choice is possible).

(3) If $\theta_{k-1}^{-1}(y_k)$ is a single point, then let $\theta_k = \theta_{k-1}$ and notice that $\theta_k^{-1}(y_k) \in U_k$. Otherwise, choose the homeomorphism α so that $\theta_k^{-1}(y_k) \in U_k$. The above process will produce a map g which is one-to-one over an infinite subset of the original y_i's. Let $y_1, y_2, \ldots$ denote this subset. Since $g^{-1}(y_i) \in U_i$ for each i, the collection $\{g^{-1}(y_i) : i = 1,2,\ldots\}$ forms a dense subset of M^q and, therefore, $\{y_1, y_2, \ldots\}$ also forms a dense subset of Y.

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