HARMONIC FORMS ON NONCOMPACT RIEMANNIAN AND KÄHLER MANIFOLDS

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More than thirty years ago, S. Bochner [4] proved that if a compact Riemannian manifold has nonnegative Ricci curvature (i.e., the Ricci tensor considered as a quadratic form is nonnegative) then any harmonic 1-form is parallel and if the manifold has positive Ricci curvature (i.e., the Ricci tensor is positive definite) then the only harmonic 1-form is the 0 form. The proof technique was first to calculate the Laplacian $\Delta(\langle\alpha,\alpha\rangle)$ of the inner product $\langle\alpha,\alpha\rangle$ of a harmonic 1-form α with itself. The calculation showed that when the Ricci tensor is nonnegative $\langle\alpha,\alpha\rangle$ is subharmonic (i.e., $\Delta\langle\alpha,\alpha\rangle$ is nonnegative) so that by the maximum principle $\langle\alpha,\alpha\rangle$ is constant and $\Delta(\langle\alpha,\alpha\rangle)=0$. The computation also showed that, when the Ricci tensor is nonnegative, $\Delta(\langle\alpha,\alpha\rangle)$ is positive at a point of the manifold if either the covariant differential $D\alpha$ is nonzero at that point or α is not zero at that point and the Ricci curvature is positive there, thus that $\Delta(\langle\alpha,\alpha\rangle)\equiv 0$ implies that $D\alpha\equiv 0$ and, if the Ricci curvature is positive at one point, that $\alpha=0$ at that point and hence $\alpha\equiv 0$.

In the intervening years, this technique has been applied in many other situations and has been particularly useful in the study of forms on complex manifolds (cf. [9] and [20] for extensive bibliography). During this period, the technique as described has frequently been used in a modified form in which the maximum principle is dispensed with by consideration of $\int_M \langle\Delta\alpha,\alpha\rangle$, which of course vanishes if α is harmonic (cf., e.g. [9; p. 85ff]). Essentially equivalent calculations are used in this approach, and for the purpose of this paper the direct information that $\langle\alpha,\alpha\rangle$ is subharmonic is needed.

In view of the importance of the technique for compact manifolds, it is natural to seek extensions of the technique to noncompact manifolds. For such extensions, restrictions on the forms to be considered must be imposed since, for example, any noncompact Riemannian manifold admits many nonzero harmonic 1-forms (an easy way to see this is to recall that on any open Riemannian manifold there are many nonconstant harmonic functions [15] and that the differential of a harmonic function is a harmonic 1-form since Δ and d commute). A natural type of restriction is to require the forms of be L^2 (or more generally L^p, $1\le p<\infty$), i.e. to require $\int_M \langle\alpha,\alpha\rangle$ (respectively, $\int_M \langle\alpha,\alpha\rangle^{p/2}$) to be finite. One might then hope to establish the nonexistence of L^2 or L^p harmonic forms on certain noncompact

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manifolds. This hope can be realized provided one has in hand two types of results: first, that $\langle\alpha,\alpha\rangle$ (respectively, $\langle\alpha,\alpha\rangle^{p/2}$) is subharmonic for the form being considered; and, second, that the integral over the manifold of a nonnegative subharmonic function is infinite unless the function vanishes identically.

Since the subharmonicity of $\langle\alpha,\alpha\rangle$ is a local question, the calculations for the compact case apply to yield the first type of result needed. Results of the second type have recently been obtained by the present authors [14] and by S.-T. Yau [23]:

(*) If M is a complete noncompact Riemannian manifold with Riemannian sectional curvature nonnegative outside some compact set and if f is a nonnegative subharmonic function on M, then $\int_M f^p = +\infty$ if $1 \leq p < +\infty$ unless $f \equiv 0$ [14].

(**) If M is a complete noncompact Riemannian manifold and if f is a nonnegative nonconstant subharmonic function on M then $\int_M f^p = +\infty$ if $1 < p < +\infty$ [23].

(***) If M is a complete noncompact Riemannian manifold and if u is a nonnegative C^∞ function on M with $\Delta(\log u)$ bounded below on the set $M_+ = \{x \in M: u(x) > 0\}$ and with $0 < \int_{M_+} \log u \leq \infty$, then $\int_M u^p = \infty$ for all $p > 0$ unless u is identically zero [23].

The purpose of this paper is to exhibit some theorems on the nonexistence of L^2 (or L^p) harmonic and holomorphic forms which are generalizations of the most important theorems of this type for compact manifolds; these theorems are, as indicated, to be obtained by combining local computations of subharmonicity with theorems on the infinity of integrals such as (*) or (**). The list of theorems of this type given here is by no means exhaustive: for instance, any of the numerous subharmonicity calculations made for the compact case yields a result for the noncompact case by combination with (*) or (**). The results on Riemannian manifolds are given in Section 1 and those on Kähler manifolds in Section 2. In Section 3, there is a discussion of a cohomology vanishing theorem related to the results in Section 2 (cf. the authors' announcements [11]; cf. also [21]).

Some results closely related to the present paper (in the special case of L^2) were obtained by Dodziuk [7] by method different from that used here; this method seems, however, to be usable only in the L^2 case. There are also results related to the present ones in [1], [2], and [8].

1. RIEMANNIAN MANIFOLDS

In accordance with the program already indicated, one wishes to determine when $\langle\alpha,\alpha\rangle^{1/2}$ is subharmonic, where α is a harmonic form. Since $\langle\alpha,\alpha\rangle^{1/2}$ may not be smooth (at zeroes of α), one computes instead $\Delta((\langle\alpha,\alpha\rangle + \varepsilon)^{1/2}$; if this quantity is nonnegative for all (small) $\epsilon > 0$ then $\langle\alpha,\alpha\rangle^{1/2}$ is subharmonic since the limit uniformly on compact sets of subharmonic functions is subharmonic. Let

$X_1, \ldots, X_n$ be a local orthonormal frame field, n = dimension M, in a neighborhood of a point P in M with $DX_i = 0$ at P, $i = 1, \ldots, n$. By a straightforward computation (cf. [14]),

$$\triangle(\langle\alpha,\alpha\rangle+\varepsilon)^{1/2} = (\langle\alpha,\alpha\rangle+\varepsilon)^{-3/2}\cdot\sum_i [\langle D_{X_i}D_{X_i}\alpha,\alpha\rangle(\langle\alpha,\alpha\rangle+\varepsilon) + \varepsilon\langle D_{X_i}\alpha, D_{X_i}\alpha\rangle + \langle\alpha,\alpha\rangle\langle D_{X_i}\alpha, D_{X_i}\alpha\rangle - \langle D_{X_i}\alpha,\alpha\rangle^2].$$

Since $\langle\alpha,\alpha\rangle\langle D_{X_i}\alpha, D_{X_i}\alpha\rangle - \langle D_{X_i}\alpha,\alpha\rangle^2 \geq 0$, $\triangle(\langle\alpha,\alpha\rangle+\varepsilon)^{1/2} \geq 0$ provided that $\sum_i \langle D_{X_i}D_{X_i}\alpha,\alpha\rangle \geq 0$. Now again computing relative to the frame $\{X_i\}$ as in [14], one obtains

$$\triangle\alpha|_P = \sum_i D_{X_i}D_{X_i}\alpha|_P - \sum_{i,j=1}^{n} \omega^i \wedge i(X_j)R_{X_iX_j}\alpha|_P$$

where $\triangle = -(d\delta + \delta d)$, $\{\omega^i\}$ is the dual basis of $\{X_i\}$, $i(X_j)$ denotes interior product with X_j and the curvature operator R (equal to the commutator of covariant differentiation) is extended to act on forms in the usual fashion. Since $\triangle\alpha = 0$ by hypothesis, $\left\langle\sum_i D_{X_i}D_{X_i}\alpha,\alpha\right\rangle\Big|_P = \left\langle\sum_{i,j}\omega^i\wedge i(X_j)R_{X_iX_j}\alpha,\alpha\right\rangle\Big|_P$. The quadratic form on the right hand side is denoted in previous literature by $F(\alpha)$; passage to component notation verifies that the expression here coincides with the $F(\ )$ notation of [9] and [22].

Clearly, if $F(\alpha) \geq 0$ for all α and in all P in M, then $\langle\alpha,\alpha\rangle$ is subharmonic when α is harmonic. A particularly interesting case of this phenomenon is when α is a 1-form; in that case, $F(\alpha) \geq 0$ if the Ricci curvature of M is nonnegative (for the relevant—and straightforward—computation in the present notation, see [14]).

THEOREM 1. *If M is a complete noncompact manifold with nonnegative Ricci tensor, then no nonzero harmonic 1-form is in $L^p(M)$, $1 < p < \infty$. If, moreover, the sectional curvature of M is nonnegative outside of some compact set, then no nonzero harmonic 1-form is in $L^1(M)$.*

Remarks. A similar theorem holds for harmonic r-forms, $r > 1$: The only change required in hypothesis to obtain the same conclusions is that the hypothesis of nonnegativity of Ricci curvature is to be replaced by the hypothesis that F is a nonnegative quadratic form. The first statement of the Theorem is a slight improvement of a result in [23], where all the ingredients for its proof are in fact given. The second statement (for $1 \leq p < +\infty$) is essentially given in [14].

Proof of Theorem 1. Since $\langle\alpha,\alpha\rangle^{1/2}$ is subharmonic, the second statement follows from (*). The statement (**) implies that if $\int_M (\langle\alpha,\alpha\rangle^{1/2})^p < +\infty$ then $\langle\alpha,\alpha\rangle$ is constant. But the volume of a complete noncompact manifold of nonnegative Ricci curvature is infinite ([5], [23]) so that $\langle\alpha,\alpha\rangle$ must be zero constantly.

2. HERMITIAN AND KÄHLER MANIFOLDS

To extend the considerations of Section 1 to complex manifolds, it is useful to consider a somewhat more general situation, namely, to study forms with values in holomorphic vector bundles which are harmonic relative to the complex Laplacian defined in this situation. To fix a suitable notation for the necessary calculations, a brief summary of the definition of the complex Laplacian in this setting will now be given: Let M be a complex manifold and $E \xrightarrow{\pi} M$ be a holomorphic vector bundle with Hermitian metric $\langle\,,\rangle$. So if $s_1, \ldots, s_m$, $m =$ dimension of the fibre of E, is a local holomorphic frame (i.e. a set of local holomorphic sections which span the fibres as complex vector spaces) and the matrix (h_{ij}) is defined by $h_{ij} = \langle s_i, s_j \rangle$, $i,j = 1, \ldots, m$, then (h_{ij}) is an Hermitian positive definite matrix. There is a unique covariant derivative ∇ operating on (local) sections of E such that (a) $X(\langle s', s''\rangle) = \langle \nabla_X s', s''\langle + \langle s', \nabla_X s''\rangle$ for any real tangent vector X and local sections s', s'' *of* E and (b) $\nabla_X s = -\sqrt{-1}\nabla_{JX} s$ for any real tangent vector X and local holomorphic section s of E (here $J =$ the complex structure tensor of M). This covariant derivative ∇ defines, by complex linear extension $\nabla_Z s$ for any local section s of E complex tangent vector $Z = X + \sqrt{-1}Y$, namely, $\nabla_Z s = \nabla_X s + \sqrt{-1}\nabla_Y s$. Property (b) then becomes $\nabla_Z s = 0$ if Z is type (0,1) and s is holomorphic; property (a) becomes $Z(\langle s', s''\rangle) = \langle \nabla_Z s', s''\rangle + \langle s', \nabla_{\bar Z} s''\rangle$.

Now suppose M has an Hermitian metric g, i.e., a J-invariant Riemannian metric. Denote by g also the inner product on all real tensors at each point of M induced by g at that point and extend g to act on complex tensors by complex linearity. Then define the Hermitian inner product G on complex tensors (at a point of M) by $G(\varphi, \psi) = g(\varphi, \bar\psi)$. If the Riemannian covariant derivative $D_Z \varphi$, $Z = X + \sqrt{-1}Y$, $\varphi = \varphi_1 + \sqrt{-1}\varphi_2$, is defined to be

$$(D_X \varphi_1 - D_Y \varphi_2) + \sqrt{-1}(D_X \varphi_2 + D_Y \varphi_1)$$

then $ZG(\varphi, \psi) = G(D_Z \varphi, \psi) + G(\varphi, D_{\bar Z} \psi)$.

Since the transition functions of E are holomorphic, the action of $\bar\partial$ on (local) sections of E and also on i-forms $A^i(E)$ with values in E is well-defined. As usual, $\bar\partial : A^{p,q}(E) \to A^{p,q+1}(E)$ and $\bar\partial^2 = 0$. An Hermitian inner product on $A^{p,q}(E)$ is determined by forming the tensor product of the Hermitian inner products $\langle\,,\rangle$ and G, i.e. $\left\langle\!\left\langle \sum s_i \varphi_i, \sum s_j \psi_j \right\rangle\!\right\rangle = \sum_{i,j} \langle s_i, s_j\rangle\, G(\varphi_i, \psi_j)$. In the remainder of the paper for typographical convenience, the $\langle\langle\ \ \rangle\rangle$ inner product and the G inner product will be denoted by $\langle,\rangle$ since no confusion will result. There is a unique differential operator $\bar\partial^* : A^{p,q+1}(E) \to A^{p,q}(E)$ such that for E-valued forms φ, ψ of compact support of type (p,q) and $(p, q+1)$ respectively

$$\int_M \langle \bar\partial \varphi, \psi\rangle = \int_M \langle \varphi, \bar\partial^* \psi\rangle.$$

The complex Laplacian $\Box : A^{p,q}(E) \to A^{p,q}(E)$ is defined to be the operator $\bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ if $p + q > 0$; however, for $p = q = 0$ $\Box$ is defined to be $-(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)$. The unfortunate change in sign in the latter is necessitated by our wish to preserve the identity for functions on a Kähler manifold that $\square f = (1/2)\, \triangle f$, so that f is subharmonic if and only if $\square f \geq 0$. The operator $\square$ will now be computed explicitly for $p + q > 0$; the same computation applies to the case $p = q = 0$ by reversing all signs. First one needs an explicit formula for $\bar\partial^*$. For this purpose, extend ∇ to map $A^i(E)$, $i > 0$, to $A^{i+1}(E)$ by defining $\nabla\,(s\omega)$, $\omega \in A^i$, s a section of E, to be the element of $A^{i+1}(E)$ determined by

$$\nabla(s\omega)(X_0,\ldots,X_i) = \sum_{j=0}^{i} (-1)^j \nabla_{X_j} s \cdot \omega\,(X_0,\ldots,X_0,\ldots,\hat{X}_j,\ldots,X_i) + s\,d\omega\,(X_0,\ldots,X_i).$$

Then the operator $\nabla - \bar\partial$, to be denoted by ∇', maps $A^{p,q}(E)$ to $A^{p+1,q}(E)$. The complex linear extension of the Hodge $*$ operator determined by g maps $A^{p,q}$ to $A^{n-q,n-p}$. The fact that $\bar\partial^* = -*\nabla'*$ is verifiable by a standard Stokes' theorem procedure.

To continue the calculation of $\square$ at a point $P \in M$, choose a local holomorphic frame field $e_1, \ldots, e_m$ on some neighborhood of P with $\nabla e_i = 0$ at P for each $i = 1, \ldots, m$. Such a choice can be made by multiplying an arbitrary local holomorphic frame field by a suitable holomorphically varying element of $GL(m, \mathbf{C})$. Define 1-forms ω^ρ_σ by $\nabla e_\sigma = \sum_{\rho=1}^{m} e_\rho \omega^\rho_\sigma$; these forms are of type (1,0) for any holomorphic frame and are 0 at P for the particular frame chosen. Choose also a real g-orthonormal frame $v_1, Jv_1, \ldots, v_n, Jv_n$ on M in a neighborhood of P, and set $V_i = (1/\sqrt{2})(v_i - \sqrt{-1}\,Jv_i)$ and $\bar V_i = (1/\sqrt{2})(v_i + \sqrt{-1}\,Jv_i)$ so that $G(V_i, \bar V_j) = G(\bar V_i, V_j) = 0$ and $G(V_i, V_j) = \delta_{ij}$ for all i,j. Let $\{\theta_1,\ldots,\theta_n,\bar\theta_1,\ldots,\bar\theta_n\}$ be the complex 1-forms dual to the basis $V_1, \ldots, V_n, \bar V_1, \ldots, \bar V_n$ (θ_i's are type (1,0), $\bar\theta_i$'s type (0,1)). A straightforward calculation shows that for any $\omega \in A^{p,q}(M)$, $*\,(\theta^i \wedge *\,\omega) = \iota(\bar V_i)\omega$ where ι denotes interior product. To initiate the $\square$ calculation proper, let $\varphi = \sum_{\sigma=1}^{m} e_\sigma \varphi^\sigma$; then

$$\begin{aligned}
\bar\partial\,\bar\partial^*\varphi &= -\bar\partial *\nabla' *\left(\sum e_\sigma \varphi^\sigma\right) = -\bar\partial *\nabla'\left(\sum e_\sigma *\varphi^\sigma\right)\\
&= -\bar\partial *\left(\sum (\nabla e_\sigma)(*\varphi^\sigma) + \sum e_\sigma\, \partial(*\varphi^\sigma)\right)\\
&= -\bar\partial *\left(\sum_{\rho,\sigma=1}^{m} e_\rho\,(\omega^\rho_\sigma \wedge *\varphi^\sigma) + \sum e_\sigma\, \partial(*\varphi^\sigma)\right)\\
&= -\sum_{\rho,\sigma} e_\rho\,[\bar\partial *(\omega^\rho_\sigma \wedge *\varphi^\sigma)] + \sum_{\rho} e_\sigma\, \bar\partial\,\bar\partial^*\varphi^\sigma
\end{aligned}$$

and

$$\bar\partial^*\bar\partial\varphi = -*\nabla' *\bar\partial\left(\sum_\sigma e_\sigma \varphi^\sigma\right) = -*\nabla'\left(\sum_\sigma e_\sigma\,(*\,\bar\partial\varphi^\sigma)\right)$$

$$= -*\left(\sum_{\rho,\sigma} e_\rho \omega^\rho_\sigma \wedge * \bar\partial\varphi^\sigma + \sum_\sigma e_\sigma (\partial * \bar\partial\varphi^\sigma)\right)$$

$$= \sum_\sigma e_\sigma \bar\partial * \bar\partial\varphi^\sigma - \sum_{\rho,\sigma} e_\rho * (\omega^\rho_\sigma \wedge \bar\partial\varphi_\rho)$$

At P, $\omega^\rho_\sigma = 0$ for all ρ,σ so that at P, $\Box\varphi = \sum_\sigma (e_\sigma \Box\varphi^\sigma) - \sum_{\sigma,\rho} e_\rho [\bar\partial * (\omega^\rho_\sigma \wedge *\varphi^\sigma)]$.

Write the forms ω^ρ_σ, which are of type (1,0), in terms of $\theta_1, \ldots, \theta_n$ say $\omega^\rho_\sigma = \sum_i \gamma^\rho_{\sigma i}\theta^i$. Then the fact that $*(\theta^i \wedge * u) = \iota(\bar V_i) u$ for any (p,q) form u yields after a routine calculation that at P, at which point $\omega^\rho_\sigma = 0$,

$$-[\bar\partial * (\omega^\rho_\sigma \wedge * \bar\partial\varphi^\sigma)] = \sum_i \iota(V_i)\Phi^\rho_\sigma \wedge \iota(\bar V_i)\varphi^\sigma$$

where, Φ^ρ_σ, the curvature form of E at P, is by definition $\bar\partial\omega^\rho_\sigma$. Thus at P,

$$\Box\varphi = \sum_\sigma e_\sigma(\Box\varphi^\sigma) - \sum_{i,\sigma} (\nabla_{V_i} e_\sigma)(D_{\bar V_i}\varphi^\sigma) + \sum_\rho e_\rho \left(\sum_{i,\sigma} i(V_i)\Phi^\rho_\sigma \wedge i(\bar V_i)\varphi^\sigma\right)$$

where $\Box$ in the $\Box\varphi^\sigma$ term is the complex Laplacian on $A^{p,q}$. The terms on the right hand side are in fact each independent of the choice of holomorphic frame (e_σ). Since the left-hand side is frame-independent, it is sufficient to check that the third term on the right is frame-independent. This independence is a consequence of a straightforward linear algebraic computation using the standard fact that if (in an obvious matrix notion) $\tilde e = Ae$ is another holomorphic frame then $\tilde\Phi = A^{-1}\Phi A$.

The computation of $\Box\alpha$, $\alpha \in A^{p,q}$, will be carried out only in the case of (M,g) being a Kähler manifold (a similar but more complicated calculation establishes the general Hermitian metric formula, which is stated but not explicitly computed later). In particular, it is now assumed that the local orthonormal frame $V_1, \ldots, V_n, \bar V_1, \ldots, \bar V_n$ satisfies $DV_i = D\bar V_i = 0$ at P and hence

$$[V_i, V_j] = [V_i, \bar V_j] = [\bar V_i, \bar V_j] = 0$$

at P. Such an orthonormal frame coincides at P with the coordinate frame $\dfrac{\partial}{\partial z_1}, \ldots, \dfrac{\partial}{\partial z_n}, \dfrac{\partial}{\partial \bar z_1}, \ldots, \dfrac{\partial}{\partial \bar z_n}$ of a suitably chosen holomorphic coordinate system $(z_1, \ldots, z_n)$ around P. This fact implies the formulae for any form α

$$\partial\alpha = \sum_i \theta^i \wedge D_{V_i}\alpha \qquad \bar\partial\alpha = \sum_i \bar\theta^i \wedge D_{\bar V_i}\alpha$$

$$\partial^*\alpha = \sum_i -\iota(V_i)D_{\bar V_i}\alpha \qquad \bar\partial^*\alpha = \sum_i -\iota(\bar V_i)D_{V_i}\alpha,$$

(∂^* = the adjoint of ∂): the right-hand sides are clearly frame-independent and the formulae are immediately verifiable at the point P in the coordinate frame. From these formulae,

$$\begin{aligned}\bar\partial^*\bar\partial\alpha &= -\sum_i \iota(\bar V_i)D_{V_i}\Big(\sum_j \bar\theta^j\wedge D_{\bar V_j}\alpha\Big)\\ &= -\sum_{i,j}\iota(\bar V_i)\{D_{V_i}\bar\theta^j\wedge D_{\bar V_j}\alpha+\bar\theta^j\wedge D_{V_i}D_{\bar V_j}\alpha\}\\ &= -\sum_{i,j}\iota(\bar V_i)\{D_{V_i}\theta^j\wedge D_{\bar V_j}\alpha\}-\sum_i D_{V_i}D_{\bar V_i}\alpha\\ &\quad+\sum_{i,j}\bar\theta^j\wedge\iota(\bar V_i)D_{V_i}D_{\bar V_j}\alpha\end{aligned}$$

and

$$\begin{aligned}\bar\partial\bar\partial^*\alpha &= -\sum_j \bar\theta^j\wedge D_{\bar V_j}\Big(\sum_i \iota(\bar V_i)D_{V_i}\alpha\Big)\\ &= -\sum_{i,j}\bar\theta^j\wedge D_{\bar V_j}\iota(\bar V_i)D_{V_i}\alpha.\end{aligned}$$

At P, $[D_{\bar V_j},\iota(\bar V_i)]=0$ because $D_{\bar V_j}\bar V_i=0$. Hence at P

$$\begin{aligned}\Box\alpha &= -\sum_i D_{V_i}D_{\bar V_i}\alpha+\sum_{i,j}\bar\theta^j\wedge\iota(\bar V_i)D_{V_i}D_{\bar V_j}\alpha\\ &\quad-\sum_{i,j}\bar\theta^j\wedge\iota(\bar V_i)D_{\bar V_j}D_{V_i}\alpha\\ &= -\sum_i D_{V_i}D_{\bar V_i}\alpha+\sum_{i,j}\bar\theta^j\wedge\iota(\bar V_i)\{D_{V_i}D_{\bar V_j}-D_{\bar V_j}D_{V_i}\}\alpha\\ &= -\sum_i D_{V_i}D_{\bar V_i}\alpha-\sum_{i,j}\bar\theta^j\wedge\iota(\bar V_i)R_{V_i\bar V_j}\alpha,\end{aligned}$$

last equality holding at P because there $R_{V_i\bar V_j}$, by definition being

$$D_{[V_i,\bar V_j]}-D_{V_i}D_{\bar V_j}+D_{\bar V_j}D_{V_i},$$

equals $D_{\bar V_j}D_{V_i}-D_{V_i}D_{\bar V_j}$.

The term $\sum_{i,j}\bar\theta^j\wedge\iota(\bar V_i)R_{V_i\bar V_j}\alpha$ is independent of the frames $\{V_i,\bar V_i\}$ having special properties at P. To put the first term in similar intrinsic form, define $D^2_{V,W}=D_VD_W-D_{D_VW}$ for any (complex) vector fields, V,W. Then at P

$$\sum_i D_{V_i} D_{\bar V_i} \alpha = \sum_i D^2_{V_i \bar V_i} \alpha,$$

and the right-hand side is intrinsically defined in the sense that its value is the same for all orthonormal frames $\{V_i, \bar V_i\}$. Combining these results yields the following expression for $\Box\varphi$, $\varphi = \sum_\sigma e_\sigma \varphi^\sigma$ which holds for any local orthonormal frame $\{V_i, \bar V_i\}$ with V_i's of type (1,0) and local holomorphic frame $\{e_\sigma\}$:

$$\begin{aligned}\Box\varphi = &-\left(\sum_{i,\sigma} e_\sigma (D^2_{V_i \bar V_i} \varphi^\sigma) + \sum_{\sigma, i} (\nabla_{V_i} e_\sigma)(D_{\bar V_i} \varphi^\sigma)\right)\\ &-\sum_{\sigma,i,j} e_\sigma (\bar\theta^j \wedge \iota(\bar V_i) R_{V_i \bar V_j} \varphi^\sigma)\\ &+\sum_{\sigma,\rho,i} e_\sigma (\iota(V_i)\Phi^\sigma_\rho \wedge \iota(\bar V_i)\varphi^\sigma)\end{aligned}$$

A similar calculation of $\Box\varphi^\sigma$ using the fact that on a Kähler manifold

$$\bar\partial\bar\partial^* + \bar\partial^*\partial = \partial\partial^* + \partial^*\partial$$

yields an alternative formula for $\Box\varphi$, $\varphi \in A^i(E)$,

$$\begin{aligned}\Box\varphi = &-\left(\sum_{i,\sigma} e_\sigma (D^2_{V_i \bar V_i} \varphi^\sigma) + \sum_{\sigma,i} (\nabla_{V_i} e_\sigma)(D_{\bar V_i} \varphi^\sigma)\right)\\ &-\sum_{\sigma,i,j} e_\sigma (R_{V_i \bar V_i} \varphi^\sigma)\\ &+\sum_{\sigma,i,j} e_\sigma (\theta^j \wedge \iota(V_i) R_{V_i \bar V_j} \varphi^\sigma)\\ &+\sum_{\sigma,\rho,i} e_\rho (\iota(V_i)\Phi^\rho_\sigma \wedge \iota(\bar V_i)\varphi^\sigma).\end{aligned}$$

Of course, the calculation of this second formula can also be viewed as a verification that $\partial\partial^* + \partial^*\partial = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ if the equality of the right-hand sides of the formulae is checked directly.

To compute $\Box\langle\varphi,\varphi\rangle$ for a harmonic form φ, it is convenient to introduce a covariant differentiation operation D^E on E-valued forms which is in effect the tensor product of D and ∇: precisely, define $D^E_X\left(\sum_\sigma e_\sigma \varphi^\sigma\right)$ to be

$$\sum_\sigma e_\sigma D_X \varphi^\sigma + \sum_\sigma (\nabla_X e_\sigma)\varphi^\sigma.$$

That this definition is independent of choice of the frame $\{e_\sigma\}$ is easily checked, as is the formula $X\langle\varphi,\psi\rangle = \langle D^E_X\varphi,\psi\rangle + \langle\varphi,D^E_{\bar X}\psi\rangle$ for any (complex) vector X and any $\varphi,\psi \in A^i(E)$. Thus

$$\Box\langle\varphi,\varphi\rangle = \sum_i V_i\bar V_i\langle\varphi,\varphi\rangle$$

(†)

$$= \sum_i(\langle D^E_{V_i}D^E_{\bar V_i}\varphi,\varphi\rangle + \langle D^E_{V_i}\varphi,D^E_{V_i}\varphi\rangle + \langle D^E_{\bar V_i}\varphi,D^E_{\bar V_i}\varphi\rangle + \langle\varphi,D^E_{\bar V_i}D^E_{V_i}\varphi\rangle).$$

Take now again $\{V_i\}$ and $\{e_\sigma\}$ to have the special properties at $P \in M$ indicated previously and write $\varphi = \sum_\sigma e_\sigma\varphi^\sigma$. Then at P

$$D^E_{V_i}D^E_{\bar V_i}\left(\sum_\sigma e_\sigma\varphi^\sigma\right) = \sum_\sigma(\nabla_{V_i}\nabla_{\bar V_i}e_\sigma)\varphi^\sigma + \sum_\sigma e_\sigma(D_{V_i}D_{\bar V_i}\varphi^\sigma)$$

$$= \sum_\sigma e_\sigma(D_{V_i}D_{\bar V_i}\varphi^\sigma)$$

since $\nabla_{\bar V_i}e_\sigma \equiv 0$.

Also at P

$$D^E_{\bar V_i}D^E_{V_i}\varphi = \sum_\sigma(\nabla_{\bar V_i}\nabla_{V_i}e_\sigma)\varphi^\sigma + \sum_\sigma e_\sigma(D_{\bar V_i}D_{V_i}\varphi^\sigma)$$

$$= \sum_\sigma(\nabla_{\bar V_i}\nabla_{V_i}e_\sigma)\varphi^\sigma + \sum_\sigma e_\sigma(D_{V_i}D_{\bar V_i}\varphi^\sigma) + \sum_\sigma e_\sigma(R_{V_i\bar V_i}\varphi^\sigma).$$

The term $\sum_\sigma(\nabla_{\bar V_i}\nabla_{V_i}e_\sigma)\varphi^\sigma$ is intrinsically defined. Comparing the resulting expression for $\Box\langle\varphi,\varphi\rangle$ with the $\Box\varphi$ formulae shows that if $\Box\varphi \equiv 0$, then $\Box\langle\varphi,\varphi\rangle$ is the sum of the nonnegative quantity $\sum_i\langle D^E_{V_i}\varphi,D^E_{V_i}\varphi\rangle + \sum_i\langle D^E_{\bar V_i}\varphi,D^E_{\bar V_i}\varphi\rangle$ and intrinsic terms which at a point depend only on the value of φ at that point and the curvatures R and Φ. Thus if suitable assumptions on R and Φ be made, it will follow that $\Box\langle\varphi,\varphi\rangle \geq 0$ if $\bigcirc\varphi \equiv 0$. A similar calculation shows that at P for any $\varepsilon > 0$

$$\Box(\langle\varphi,\varphi\rangle+\varepsilon)^{1/2} = \frac{1}{4}(\langle\varphi,\varphi\rangle+\varepsilon)^{-3/2}\sum_i [-(\langle D_{\bar V_i}\varphi,\varphi\rangle + \langle\varphi, D_{V_i}\varphi\rangle)(\langle D_{V_i}\varphi,\varphi\rangle + \langle\varphi, D_{\bar V_i}\varphi\rangle) + 2(\langle\varphi,\varphi\rangle+\varepsilon)(\langle D_{\bar V_i}\varphi, D_{\bar V_i}\varphi\rangle + \langle D_{V_i}\varphi, D_{V_i}\varphi\rangle)] + \frac{1}{2}(\langle\varphi,\varphi\rangle+\varepsilon)^{-1/2}\sum_i (\langle D_{V_i} D_{\bar V_i}\varphi,\varphi\rangle + \langle\varphi, D_{\bar V_i} D_{V_i}\varphi\rangle).$$

By the Schwarz inequality, the term in [ ] brackets is nonnegative. Thus

$$\Box(\langle\varphi,\varphi\rangle+\varepsilon)^{1/2} \geq \frac{1}{2}(\langle\varphi,\varphi\rangle+\varepsilon)^{-1/2}\sum_i (\langle D_{V_i} D_{\bar V_i}\varphi,\varphi\rangle + \langle\varphi, D_{\bar V_i} D_{V_i}\varphi\rangle).$$

Again, if, $\Box\varphi \equiv 0$, then the right hand side will be, at a particular point of M, dependent only on the values of φ, R and Φ at that point. If the sum $\sum_i$ is nonnegative so that $\Box(\langle\varphi,\varphi\rangle + \varepsilon)^{1/2}$ is nonnegative for all $\varepsilon > 0$, then the function $(\langle\varphi,\varphi\rangle)^{1/2}$, which is not necessarily smooth, will be subharmonic because it is then the limit uniformly on compact subsets of the subharmonic functions $(\langle\varphi,\varphi\rangle + \varepsilon)^{1/2}$ as $\varepsilon \to 0^+$.

Also for any positive C^∞ function h, $\triangle h^p = p(p-1)h^{p-2}\|dh\|^2 + ph^{p-1}\triangle h$ so that if $\triangle h \geq 0$ then $\triangle h^p \geq 0$ if $p \geq 1$. Thus $(\langle\varphi,\varphi\rangle + \varepsilon)^{p/2}$ is subharmonic if $p \geq 1$ and $(\langle\varphi,\varphi\rangle + \varepsilon)^{1/2}$ is subharmonic. Hence $\langle\varphi,\varphi\rangle^{p/2}$ is also subharmonic, a fact which also follows directly from subharmonicity of $\langle\varphi,\varphi\rangle^{1/2}$.

The following theorem is a concrete illustration of these general principles.

THEOREM 2. *If M is a complete Kähler manifold and if at each point the sum of any q eigenvalues of the Ricci tensor of M is nonnegative, then any harmonic form of type $(0,q)$ or $(q,0)$ that is in L^p for some p with $1 < p < +\infty$ is parallel. If in addition the volume of M is infinite or if the sums of any q. eigenvalues of the Ricci tensor are all positive at some point of M, then any such form is identically zero. If M is a complete Kähler manifold and if at each point the sum of any q eigenvalues of the Ricci tensor of M is nonnegative and if the Riemannian sectional curvature is nonnegative outside some compact subset of M, then no harmonic $(q,0)$ or $(0,q)$ form is in L^p, $1 \leq p < +\infty$ except the zero form.*

Proof. The case of ordinary, C-valued forms corresponds in the previous calculations to E being the trivial line bundle $M \times \mathbf{C}$ with the metric determined by $\langle 1,1\rangle = 1$ and hence $\Phi \equiv 0$. If φ is a (C-valued) form of type $(0,q)$ then

$$\theta^j \wedge \iota(V_i) R_{V_i \bar V_j}\varphi = 0$$

because $R_{V_i \bar V_j}$ preserves type and $\iota(V_i)\alpha = 0$ for any $(0,q)$ form α. Hence at p

$$D_{V_i} D_{\bar V_i}\varphi = -\Box\varphi - \sum_i R_{V_i \bar V_i}\varphi \quad \text{and} \quad D_{\bar V_i} D_{V_i}\varphi = -\Box\varphi.$$ If $\Box\varphi = 0$,

$$\langle D_{V_i} D_{\bar V_i}\varphi,\varphi\rangle + \langle\varphi, D_{\bar V_i} D_{V_i}\varphi\rangle = \sum_i \langle -R_{V_i \bar V_i}\varphi,\varphi\rangle = -\left\langle \sum_i R_{V_i \bar V_i}\varphi,\varphi\right\rangle.$$

The operator $\mathscr{R} = \sum_i R_{V_i \bar{V}_i}$ is Hermitian: for any forms α, β $\langle \mathscr{R}\alpha, \beta \rangle = \langle \alpha, \mathscr{R}\beta \rangle$. This fact is easily verified by expressing $\mathscr{R}$ in real coordinates. Thus there is an orthonormal basis $\bar{W}_1, \dots, \bar{W}_n$ for the vectors of type $(0,1)$ relative to which $\mathscr{R}\bar{W}_i = \lambda_i \bar{W}_i$, λ_i real. If $\theta_1, \dots, \theta_n, \bar{\theta}_1, \dots, \bar{\theta}_n$ is the basis dual to $W_1, \dots, W_n, \bar{W}_1, \dots, \bar{W}_n$, then $\mathscr{R}(\bar{\theta}_j) = -\lambda_j \bar{\theta}_j$ since $0 = \mathscr{R}(\bar{\theta}_j(\bar{W}_i)) = \bar{\theta}_j(\mathscr{R}(\bar{W}_i)) + \mathscr{R}(\bar{\theta}_j)(W_i)$. Consequently $\mathscr{R}(\bar{\theta}_{j_1} \wedge \dots \wedge \bar{\theta}_{j_q}) = -(\lambda_{j_1} + \dots + \lambda_{j_q})\, \theta_{j_1} \wedge \dots \wedge \theta_{j_q}$. So $\langle \mathscr{R}\varphi, \varphi \rangle \geq 0$ (>0 if $\varphi \neq 0$) at P if $\lambda_{j_1} + \dots + \lambda_{j_q} \geq 0$ (>0) at P for all $j_1, \dots, j_q$, $j_1 < \dots < j_q$. Thus if $\square\varphi = 0$,

$$(\dagger\dagger) \qquad \langle D_{V_i} D_{\bar{V}_i} \varphi, \varphi \rangle + \langle \varphi, D_{\bar{V}_i} D_{V_i} \varphi \rangle \geq 0$$

if $\lambda_{j_1} + \dots + \lambda_{j_q} \geq 0$ for all $j_1 < \dots < j_q$, and (at P)

$$(\dagger\dagger\dagger) \qquad \langle D_{V_i} D_{\bar{V}_i} \varphi, \varphi \rangle + \langle \varphi, D_{\bar{V}_i} D_{V_i} \varphi \rangle > 0$$

if $\lambda_{j_1} + \dots + \lambda_{j_q} > 0$ for all $j_1 < \dots < j_q$ and $\varphi \neq 0$ (at P). In particular $\square(\langle \varphi, \varphi \rangle + \varepsilon)^{1/2} \geq 0$ under the $\lambda_{j_1} + \dots + \lambda_{j_q} \geq 0$ condition and > 0 under the $\lambda_{j_1} + \dots + \lambda_{j_q} > 0$ condition unless $\varphi = 0$.

By the remarks preceding the theorem, $\langle \varphi, \varphi \rangle^{p/2}$ is subharmonic under the general (first stated) hypothesis of the theorem. If $\int_M \langle \varphi, \varphi \rangle^{p/2}$ is finite for some p, $1 < p < \infty$, then (**) of the introduction implies that $\langle \varphi, \varphi \rangle$ is constant. Thus $\square \langle \varphi, \varphi \rangle = 0$. Then (†) and (††) imply that $\sum_i \langle D_{V_i} \varphi, D_{V_i} \varphi \rangle + \sum_i \langle D_{\bar{V}_i} \varphi, D_{\bar{V}_i} \varphi \rangle = 0$ and so φ is parallel. If the volume of M is infinite, then the constant $\langle \varphi, \varphi \rangle$ must be zero so $\varphi \equiv 0$. Also, regardless of the volume of M, the fact that $\square \langle \varphi, \varphi \rangle = 0$ and the formulae (†) and (†††) together imply that $\varphi = 0$ at any point where $\lambda_{j_1} + \dots + \lambda_{j_q} > 0$, for all $j_1 < \dots < j_q$. Thus $\langle \varphi, \varphi \rangle \equiv 0$ and $\varphi \equiv 0$ under either hypothesis of the second statement of the theorem. The last statement of the theorem follows immediately from (*) of the introduction and the subharmonicity of $\langle \varphi, \varphi \rangle^{p/2}$. The statements for $(q,0)$ forms follow from those for $(0,q)$ forms since a form is harmonic if and only if its conjugate is harmonic and conjugation preserves the metric norm.

On a compact Kähler manifold, a harmonic (C-valued) form of type $(q,0)$ is necessarily holomorphic. The same conclusion does not hold in general for $(q,0)$ forms on complete but noncompact Kähler manifolds. However, if a harmonic $(q,0)$ form is in $L^2(M)$, then the form is again necessarily holomorphic [2]. This fact yields a refined result in which an integrated curvature estimate suffices in contrast to the pointwise curvature condition which usually holds. To state the precise result, which is implicit in the results in [23] (see also [19] for the compact case), let for each $P \in M$, $m_q(P)$ = the minimum sum of q eigenvalues of the Ricci tensor at P.

THEOREM 3. *If M is a complete Kähler manifold with m_q bounded below and $0 < \int_M m_q \leq \infty$ then there are no harmonic $(q,0)$ or $(0,q)$ forms in $L^2(M)$ except the 0-form.*

Proof of Theorem 3. If φ is an L^2 harmonic $(q,0)$ form, then φ is holomorphic. In particular $D_{\bar{V}}\varphi = 0$ if V is a vector of type $(1,0)$: this fact follows easily from writing φ in a holomorphic normal coordinate system, for instance. Let $u = \langle\varphi,\varphi\rangle$. If $u \neq 0$ at a point $P \in M$ and if $\{V_i\}$ is a special frame at P as before, then

$$\Box \log u = \sum_i V_i\bar{V}_i(\log u) = \frac{1}{u^2}\sum_i (uV_i\bar{V}_iu - (V_iu)(\bar{V}_iu))$$

$$= \langle\varphi,\varphi\rangle^{-2}\sum_i [\langle\varphi,\varphi\rangle(\langle D_{V_i}D_{\bar{V}_i}\varphi,\varphi\rangle + \langle\varphi,D_{\bar{V}_i}D_{V_i}\varphi\rangle]$$

$$+ \langle\varphi,\varphi\rangle^{-1}\sum_i [\langle\varphi,\varphi\rangle(\langle D_{V_i}\varphi,D_{V_i}\varphi\rangle + (D_{\bar{V}_i}\varphi,D_{\bar{V}_i}\varphi))$$

$$- (\langle D_{V_i}\varphi,\varphi\rangle + \langle\varphi,D_{\bar{V}_i}\varphi\rangle)(\langle D_{\bar{V}_i}\varphi,\varphi\rangle + \langle\varphi,D_{V_i}\varphi\rangle]$$

$$= \langle\varphi,\varphi\rangle^{-2}\sum_i [\langle\varphi,\varphi\rangle\langle\varphi,D_{\bar{V}_i}D_{V_i}\varphi\rangle]$$

$$+ \langle\varphi,\varphi\rangle^{-2}\sum_i [\langle\varphi,\varphi\rangle\langle D_{V_i}\varphi,D_{V_i}\varphi\rangle - \langle D_{V_i}\varphi,\varphi\rangle^2]$$

$$\geq \langle\varphi,\varphi\rangle^{-2}\sum_i [\langle\varphi,\varphi\rangle\langle\varphi,D_{\bar{V}_i}D_{V_i}\varphi\rangle] = \langle\varphi,\varphi\rangle^{-1}\left\langle\varphi, \sum_i D_{\bar{V}_i}D_{V_i}\varphi\right\rangle.$$

Since φ is type $(q,0)$, $\theta^j \wedge \iota(\bar{V}_i(R_{V_i\bar{V}_i}\varphi = 0$ because $R_{V_i\bar{V}_i}$ preserves types. From the first formula for the complex Laplacian and the fact that at P,

$$D_{\bar{V}_iV_i} = D^2_{V_i\bar{V}_i} - R_{V_i\bar{V}_i},$$

it follows that $\sum_i D_{\bar{V}_i}D_{V_i}\varphi = -\sum_i R_{V_i\bar{V}_i}\varphi$. (Alternatively, one can deduce this formula directly from the fact that $D_{\bar{V}_i}\varphi = 0$ for any type $(1,0)$ V so that $D_{\bar{V}_i}\varphi \equiv 0$ and hence $\sum_i D_{V_i}D_{\bar{V}_i}\varphi = 0$.) By the reasoning employed in the proof of Theorem 2

$$-\left\langle\varphi, \sum_i R_{V_i\bar{V}_i}\varphi\right\rangle \geq m_q(P)\langle\varphi,\varphi\rangle$$

so that $\Box \log u \geq m_q(P)$. The conclusion of the theorem for $(q,0)$ forms follows from statement (***). The statement for $(0,q)$ forms is a consequence of that for $(q,0)$ forms by conjugation.

If X and Y are two (real) unit tangent vectors at a point of a Kähler manifold, then the holomorphic bisectional curvature determined by X and Y is by definition $R(X,JX,Y,JY)$. A Kähler manifold is said to have nonnegative (respectively, positive) bisectional curvature if this number is nonnegative (positive) for any two unit vectors X, Y. By the Bianchi identity and the Kähler properties of R,

$$\begin{aligned} R(X,JX,Y,JY) &= R(X,Y,JX,JY) + R(X,JY,X,JY) \\ &= R(X,Y,X,Y) + R(X,JY,X,JY); \end{aligned}$$

hence nonnegativity (or positivity) of sectional curvature implies nonnegativity (or positivity) of holomorphic bisectional curvature [10]

THEOREM 4. *If M is a complete Kähler manifold with everywhere nonnegative holomorphic bisectional curvature, then for any p, $1 < p < +\infty$, every harmonic form of type* $(1,1)$ *in $L^p(M)$ is parallel. If in addition the volume of M is infinite then any such form is identically zero. If (regardless of the volume of M) the holomorphic bisectional curvatures of M are all positive at one point of M, then every harmonic* $(1,1)$ *form that is in $L^p(M)$ for some p, $1 < p < +\infty$, is a constant multiple of the Kähler form of M. If M is a complete Kähler manifold with nonnegative holomorphic bisectional curvature and if the Riemannian sectional curvature of M is nonnegative everywhere outside some compact subset of M, then no harmonic* $(1,1)$ *form other than the zero form is in $L^p(M)$ for p with $1 \leq p < +\infty$.*

Note that this theorem is not quite a direct analogue of Theorems 1, 2, and 3: That M have positive holomorphic bisectional curvature does not imply vanishing of L^p harmonic $(1,1)$ forms but rather that such forms are constant multiples of the Kähler form. This corresponds to the fact in the compact case that positivity of holomorphic bisectional curvature implies not the vanishing of the second Betti number $b_2(M)$ but rather that $b_2(M) = 1$, the second cohomology group being generated by multiples of the Kähler form [10]. In the following proof of Theorem 4, it corresponds to the fact that positivity of holomorphic bisectional curvature (at a point) and vanishing of $F(\varphi)$ together imply that φ is a multiple at that point of the Kähler form but not that $\varphi = 0$ at that point.

Proof of Theorem 4. If φ is an harmonic $(1,1)$ form, then the $(1,1)$ forms $\varphi + \bar{\varphi}$ and $(\sqrt{-1})^{-1}(\varphi - \bar{\varphi})$ are also harmonic and they are real forms. It follows easily that it suffices to prove the theorem for the case of real forms. In this case, the theorem is implied by statements (*) and (**) combined with the reasoning in Section 1 about the subharmonicity of $\langle\varphi,\varphi\rangle^{1/2}$, once it is verified that for a real harmonic $(1,1)$ form φ:

(a) The quantity $F(\varphi)$ defined in Section 1 is nonnegative if the holomorphic bisectional curvature of M is nonnegative

(b) $F(\varphi)$ is positive at a point of M if at that point the holomorphic bisectional curvature is all positive and φ is not a multiple of the Kähler form at that point.

These properties of $F(\varphi)$ are established by a direct calculation using the property that there is an orthonormal basis of the real tangent space at a point P of M of the form X_1, $X_2 = JX_1$, X_3, $X_4 = JX_3$ .... $X_{2n} = JX_{2n-1}$ such that $\varphi(X_i,X_j) = 0$ unless i,j are $2k - 1$, $2k$ in some order for some k (see [3] for proof of this property and further details; cf. also [10]).

The previous theorems give essentially complete information in the case manifolds of low dimension.

THEOREM 5. *If M is a complete Kähler manifold with* $\dim_{\mathbb{C}} M = 2$, *with nonnegative holomorphic bisectional curvature and with nonnegative sectional* *curvature everywhere outside some compact subset of M, then for any p, $1 \le p < +\infty$, there are no harmonic forms of any degree in $L^p(M)$ except forms which are identically zero.*

Theorem 5 is an immediate consequence of Theorem 2, 3 and 4 together with the standard result that a form α is harmonic if and only if each component of its (orthogonal) decomposition into forms of pure bidegree is harmonic. Of course, various other versions of Theorem 5 can be formulated by using the various statements of Theorem 2, 3, and 4.

The following result generalizes parts of the Kodaira vanishing theorem. (For a different generalization, which is more refined in its conclusions but requires more delicate hypotheses, see [1] and [2]). In the following theorem, K is the line bundle of $(n,0)$ forms, $n = \dim_{\mathbf{C}} M$ and K^* is the dual of K.

THEOREM 6. *If M is a complex Kähler manifold of complex dimension n and if $L \to M$ is an Hermitian holomorphic line bundle with the property that the curvature form of $L \otimes K^*$ has nonnegative trace, then for any p, $1 < p < +\infty$, every harmonic L-valued $(0,n)$ form in $L^p(M)$ is parallel. If in addition M has infinite volume or the trace of the curvature form of $L \otimes K^*$ is positive at one point, then any such form is zero. If M is a complete Kähler manifold, if $L \to M$ is a line bundle such that $L \otimes K^*$ has a metric with curvature of nonnegative trace, and if M has nonnegative sectional curvature everywhere outside some compact set, then for every p, $1 \le p < +\infty$ no harmonic L-valued $(0,n)$ form is in $L^p(M)$ except the zero form.*

Theorem 6 is proved by the same process as used to establish Theorems 2–5: The appropriate nonnegativity (or positivity) check is as follows: Write the harmonic form φ as $e\varphi^1$ where e is a local nonvanishing holomorphic section of L. Then what is required is to verify the nonnegativity (or positivity) of

$$\left\langle \sum_i D^L_{V_i} D^L_{\bar{V}_i} \varphi, \varphi \right\rangle + \left\langle \varphi, \sum_i D^L_{\bar{V}_i} D^L_{V_i} \varphi \right\rangle$$

or equivalently of

$$\langle e,e\rangle \left\langle \sum_i D_{V_i} D_{\bar{V}_i} \varphi^1, \varphi^1 \right\rangle + \left\langle e\varphi^1, \left(\sum_i \nabla_{\bar{V}} \nabla_{V_i} e \right) \varphi^1 + e \left(\sum_i D_{\bar{V}_i} D_{V_i} \varphi^1 \right) \right\rangle$$

The first term is equal (since φ is harmonic) to

$$\langle e,e\rangle \left\langle -\sum_i R_{V_i \bar{V}_i} \varphi^1 + \sum_i \iota(V_i) \Phi^1_1 \wedge \iota(\bar{V}_i) \varphi^1, \varphi^1 \right\rangle.$$

The second term is equal to

$$\sum_i - \langle e,e\rangle \langle \varphi^1, \varphi^1 \rangle \left(\sum_i \Phi^1_1 (V_i, \bar{V}_i) \right) + \sum_i \langle e,e\rangle \langle \varphi, \iota(V_i) \Phi^1_1 \wedge \iota(\bar{V}_i) \varphi^1 \rangle.$$

Now if φ^1 is a $(0,n)$ form, then

$$\sum_i \iota(V_i)\Phi_1^1 \wedge \iota(\bar{V}_i)\varphi^1 = \sum_i \Phi_1^1(V_i, \bar{V}_i)\varphi^1,$$

since φ^1 is a multiple of $\bar{\theta}^1 \wedge \dots \wedge \bar{\theta}^n$. Thus the second term is zero so that what is required is to check the nonnegativity (or positivity) of

$$\left\langle -\sum R_{V_i \bar{V}_i}\varphi^1 + \sum_i \Phi_1^1(V_i, \bar{V}_i)\varphi^1, \varphi^1 \right\rangle.$$

By the calculations for Theorem 2 this expression is greater than or equal to $\langle \varphi^1, \varphi^1 \rangle \times$ (the trace of the Ricci tensor $+\sum_i \Phi_1^1(V_i, \bar{V}_i)$), the second factor being nonnegative (or positive) precisely by the hypothesis of the theorem. The remainder of the proof of the theorem follows a by now familiar pattern and is omitted.

Although it will not be used further here, the following formula for the complex Laplacian on E-valued forms when M is an Hermitian but not necessarily Kähler manifold is useful for, among other purposes, illustrating the simplification that the Kähler condition generates. To state the formula (given here in the form calculated in [17]; see also [2]) let S and T be the operators on ordinary forms.

$$\left(\sum_i \bar{\theta}_i \wedge D_{\bar{V}_i}\right) - \bar{\partial} \quad \text{and} \quad \left(\sum_i i(\bar{V}_i) D_{V_i}\right) - \bar{\partial}^*,$$

respectively, where $\bar{\partial}^*$ is the adjoint of $\bar{\partial}$ on ordinary forms. S and T are tensors for any Hermitian manifold; they are identically zero on Kähler manifold. Then for a E-valued form $\varphi = \sum e_\sigma \varphi^\sigma$,

$$\Box\varphi = \sum_\sigma e_\sigma \left(\sum_{i,\rho} \iota(V_i)\Phi_\rho^\sigma \wedge \iota(\bar{V}_i)\varphi^\rho\right) + \sum_\sigma e_\sigma(\Box\varphi^\sigma),$$

as before. And for any ordinary form α,

$$\begin{aligned}
\Box\alpha = & -\sum_k D_{V_k} D_{\bar{V}_k}\alpha - \sum_{k,i} \theta^i \wedge \iota(V_k) R_{V_k \bar{V}_i}\alpha \\
& - \sum_{k,i} \iota(\bar{V}_k)(D_{V_k}\bar{\theta}^i \wedge D_{\bar{V}_i}\alpha) - \sum_{k,i} \bar{\theta}^k \wedge [D_{\bar{V}_k}, \iota(\bar{V}_i)]\, D_{V_i}\alpha \\
& + \sum_k \iota(\bar{V}_k)\, D_{V_k}(S\alpha) - \sum_i T((\bar{\theta}^i \wedge D_{\bar{V}_i}\alpha) + TS\alpha \\
& + \sum_k \bar{\theta}_k \wedge D_{\bar{V}_k}(T\alpha) + \sum_i S(\iota(\bar{V}_i) D_{V_i}\alpha) + ST\alpha.
\end{aligned}$$

3. COHOMOLOGY VANISHING

The results of Section 2 do not yield directly results on cohomology vanishing for noncompact manifolds because it might happen that a particular deRham or Dolbeault cohomology class contained no L^p harmonic form. There is a cohomology vanishing theorem related to Theorem 1 given in [23] (Theorem 6 there). Also, there is a cohomology vanishing theorem (a weaker form of which was announced by the authors in [11] as Theorem 1 there; cf. [21]) which is closely related to Theorem 6:

THEOREM 7. *If M is a complete Kähler manifold with everywhere nonnegative sectional curvature and if L is an Hermitian holomorphic line bundle with the property that the curvature of $L \otimes K^*$ is positive, then $H^q(M, \mathcal{O}(L)) = 0$ for all $q \geq 1$.*

Proof of Theorem 7. By the Dolbeault isomorphism theorem, it is enough to show that if φ is a $\bar\partial$-closed L-valued $(0,q)$ form then there is an L-valued $(0,q-1)$ form ψ with $\bar\partial\psi = \varphi$. To solve this $\bar\partial$ problem, define, for each point $P \in M$, the number $c(P)$ to be the minimum among the eigenvalues of the curvature of $L \otimes K^*$. Then $P \to c(P)$ is a positive continuous function on M. There is a nonnegative continuous plurisubharmonic function $\tau : M \to \mathbf{R}$ such that $e^{-\tau/2}\varphi/\sqrt{c}$ is in $L^2(M)$, i.e. $\int c^{-1}e^{-\tau}\langle\varphi,\varphi\rangle < +\infty$. The existence of such a function τ is a consequence of the existence of a convex exhaustion function on M [6] say $\tau_1 : M \to \mathbf{R}$. If $f : \mathbf{R} \to \mathbf{R}$ is a suitable (rapidly increasing) nondecreasing convex function, then $\tau = f \circ \tau_1$ will be a convex function such that $c^{-1/2}e^{-\tau/2}\varphi$ is in $L^2(M)$. This function τ is plurisubharmonic because any convex function on a Kähler manifold is plurisubharmonic [12]. Some technical difficulty arises in the following argument from the possible nondifferentiability of τ. Since the method of disposing of this difficulty is given in detail in [13], it will not be considered further here and the rest of the argument will be given as if τ were C^∞.

Define a new metric h_τ on the bundle L by multiplying the original metric h by $e^{-\tau}$. If $\langle\ \rangle_\tau$ is the metric on L-valued forms determined by this new metric h_τ (and by the metric on M) then $\int_M 1/c\langle\varphi,\varphi\rangle_\tau < +\infty$.

It is a standard result ([2] [16] and especially [13] for the present version of the L^2 theory of the $\bar\partial$-operator) that an appropriate form ψ will exist if for all $(0,q)$ L-valued forms ω of compact support

$$q\int_M c\langle\omega,\omega\rangle_\tau \leq \int_M \langle\Box\omega,\omega\rangle_\tau.$$

From a previous formula for $\Box$,

$$\langle\Box\omega,\omega\rangle_\tau = -\sum_i \langle eD^2_{V_i\bar V_i}\omega^1, e\omega^1\rangle_\tau + \left\langle e\sum_i - R_{V_i\bar V_i}\omega^1, e\omega^1\right\rangle_\tau$$
$$+ \sum_i \langle e(\iota(V_i)\Phi^1_{1,\tau} \wedge \iota(\bar V_i)\omega^1), e\omega^1\rangle_\tau$$

where $\Phi^1_{1,\tau}$ is the curvature of L with metric h_τ and the $\omega = e\omega^1$ notation of Theorem 6 is continued. Now if Φ^1_1 denotes the curvature of L with its original metric, then the form $\Phi^1_{1,\tau}$ is $\Phi^1_1 + \bar{\partial}\partial\tau$ and hence $\Phi^1_{1,\tau} \geq \Phi^1_1$. Then calculations as in Theorems 3 and 6 show that

$$\left\langle e\sum_i - R_{V_i\bar{V}_i}\omega^1, e\omega^1\right\rangle_\tau + \sum_i \langle e(\iota(V_i)\Phi^1_{1,\tau} \wedge \iota(\bar{V}_i)\omega^1, e\omega^1\rangle_\tau$$

$$\geq \left\langle e\sum_i - R_{V_i\bar{V}_i}\omega^1, e\omega^1\right\rangle_\tau$$

$$+ \sum_i \langle e(\iota(V_i)\Phi^1_1 \wedge \iota(\bar{V}_i)\omega^1), e\omega^1\rangle_\tau$$

$$\geq qc\langle e\omega^1, e\omega^1\rangle_\tau = qc\langle \omega, \omega\rangle_\tau .$$

Suppose for the moment that $\int_M -\left\langle e\sum_i D^2_{V_i\bar{V}_i}\omega^1, e\omega^1\right\rangle_\tau \geq 0$. Then

$$\int_M \langle \Box\omega, \omega\rangle \geq q\int_M c\langle\omega,\omega\rangle_\tau .$$

The theorem then follows as already indicated.

Thus is remains only to show that $\int -\left\langle e\sum_i D^2_{V_i\bar{V}_i}\omega^1, e\omega^1\right\rangle_\tau \geq 0$. This inequality holds in fact independently of the curvature assumptions. To check the inequality set $\beta = \sum_i \langle e(D_{\bar{V}_i}\omega^1), e\omega^1\rangle_\tau \bar{\theta}_i$. β is a globally defined (ordinary) 1-form because its definition is independent of choice of the local holomorphic section e and the frame $\{V_i\}$. A direct calculation of $*d*\beta$ at the point P (where e and $\{V_i\}$ satisfy their previously indicated special properties) yields the result that

$$\int * d * \beta = \int_M \left[-\left\langle e\sum_i D^2_{V_i\bar{V}_i}\omega^1, e\omega^1\right\rangle_\tau - \sum_i \langle eD_{\bar{V}_i}\omega^1, eD_{\bar{V}_i}\omega^1\rangle_\tau \right]$$

Since $\int_M * d * \beta = 0$ for any 1-form β,

$$\int_M -\left\langle e\sum_i D^2_{V_i\bar{V}_i}\omega^1, e\omega^1\right\rangle_\tau = \int_M \sum_i \langle eD_{\bar{V}_i}\omega^1, eD_{\bar{V}_i}\omega^1\rangle_\tau$$

so

$$-\int_M \sum_i \langle eD^2_{V_i\bar{V}_i}\omega^1, e\omega\rangle_\tau \geq 0,$$

as required. Incidentally, the inequality just established in the case of line bundles holds also (and by essentially the same proof) for arbitrary holomorphic vector bundles. Thus under suitable curvature assumptions there are vanishing theorems for forms with values in a bundle of higher dimension than 1, but no detailed formulation of such results will be given here (see [11], in which some further remarks on the noncompact case are given.)

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