THREE-HOLED SPHERES AND RIEMANN SURFACES

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INTRODUCTION

Let R be a compact Riemann surface of genus g bigger than one. A Jenkins-Strebel differential φ on R is a holomorphic, quadratic differential on R all of whose noncritical, horizontal trajectories are closed. Such a differential gives a natural way of decomposing R into annuli whose boundaries consist of the critical, horizontal trajectories of φ.

In this article two procedures are given for constructing analogous holomorphic, quadratic differentials on R which are associated with a decomposition of the surface into three-holed spheres. In one case, it turns out that one again obtains Jenkins-Strebel differentials. In a second case, the form of the differentials so constructed is not known.

The first section summarizes certain facts about uniformization of three-holed spheres. Such domains can be uniformized by deleting three intervals from the real axis and there are simple inequalities for expressing compactness in terms of the endpoints of these intervals. The compactness condition is needed for a normal families argument used in section 3. That the uniformization can be achieved by removing three intervals was already observed by Jenkins in [8].

In the second section, variational formulas for certain natural functions on $T^{\#}(R)$, the reduced Teichmüller space of a Riemann surface R, are derived.

In the third section, these variational formulas are used to construct global quadratic differentials on a surface of genus g naturally associated with a partition of that surface into $2g-2$ three-holed spheres.

1. UNIFORMIZATION OF THREE-HOLED SPHERES

Let R be a Riemann surface of finite type. This means it can be obtained from a compact surface by deleting a finite number of continua. The reduced Teichmüller space of R, $T^{\#}(R)$, is defined in [6] and so is the space $Q(R)$ of holomorphic, quadratic differentials on R which are real with respect to boundary uniformizers along the boundary of R. Let g be the genus of R, m be the number of deleted continua each of which contains more than one point, and n be the number of deleted continua each of which consists of exactly one point. (m is the number of "holes" and n is the number of "punctures.") Let ρ be the real

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dimension of the continuous group of conformal self-mappings of R. In most cases $\rho = 0$. It turns out that $T^{\#}(R)$ is homeomorphic to a real Euclidean space of dimension $6g - 6 + 3m + 2n + \rho$. Our objective in this section is to describe $T^{\#}(R)$ and $Q(R)$ in the case where $g = 0$, $m = 3$, and $n = 0$. In this situation ρ is necessarily equal to zero.

It is well-known that any two three-holed spheres (Riemann surfaces for which $g = 0, m = 3$, and $n = 0$) are quasiconformally equivalent. To describe the Teichmüller space for such a surface one needs to pick a base Riemann surface. Our choice is

$$R = R(a_0, b_0, c_0) = \mathbf{C} - \{(-\infty, 0] \cup [1, a_0] \cup [b_0, c_0]\}$$

and

$$1 < a_0 < b_0 < c_0.$$

This R can be viewed as one sheet of the compact surface of genus two associated with the equation $w^2 = z(z-1)(z-a_0)(z-b_0)(z-c_0)$. In this case it is obvious that a basis for the real vector space $Q(R)$ is $\{dz^2/w^2, zdz^2/w^2, z^2dz^2/w^2\}$.

Pick any other triply connected Riemann surface R^* and a quasiconformal orientation preserving homeomorphism f from R onto R^*. Let h be a local parameter on R^*. Then the complex dilatation $\mu = (h \circ f)_{\bar z}/(h \circ f)_z$ will be a measurable function, defined independently of h, satisfying $\|\mu\|_\infty < 1$. Conversely, given any complex measurable function μ on R satisfying $\|\mu\|_\infty < 1$, there will exist a homeomorphic solution w to

$$w_{\bar z} = \mu w_z. \tag{1}$$

This solution is unique if one requires it to fix 0, 1 and ∞, [3]. From this existence theorem one realizes the abstract Riemann surface R^* as $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ with the three quasiconformal segments $w([-\infty, 0])$, $w([1, a_0])$, and $w([b_0, c_0])$ removed.

Let $M(R)$ be the set of all measurable functions μ on R satisfying $\|\mu\|_\infty < 1$, where, as usual, we think of two functions being the same function if they are equal except on a set of measure zero. $T^{\#}(R)$ is $M(R)$ factored by an equivalence relation, [6]. Let μ_1 and μ_2 be elements of $M(R)$ and w_1 and w_2 be the corresponding normalized solutions of (1). One says that μ_1 is equivalent to μ_2 if there is a quasiconformal homeomorphism h from R to R homotopic to the identity on R and a conformal mapping f from $w_1(R)$ to $w_2(R)$ such that $f \circ w_1(z) = w_2 \circ h(z)$ for all z in R.

Now let $M_s = \{\mu \text{ in } M(R); \mu(\bar z) = \overline{\mu(z)}\}$. M_s consists of symmetric elements of $M(R)$. Consider the natural mapping $\Phi: M_s \to T^{\#}(R)$ which sends μ to its equivalence class. It is clear that every surface in the image of Φ can be realized as $\mathbf{C}$ with three segments on the real axis, $(-\infty, 0]$, $[1, a^*]$, and $[b^*, c^*]$ removed, where $1 < a^* < b^* < c^*$.

THEOREM 1. *Φ is a surjective, real-analytic mapping and $T^{\#}(R)$ is bi-analytically equivalent to $V = \{(a, b, c) \in \mathbf{R}^3; 1 < a < b < c\}$.*

Proof. The surjectivity of Φ is equivalent to saying that every equivalence class in $T^{\#}(R)$ is realizable by a symmetric Beltrami differential. This is a consequence of Jenkins' result in [8]. It was pointed out to me by the referee that it is also a consequence of Teichmüller's theorem and the observation that the Teichmüller-Beltrami differential $k|\varphi|/\varphi$ will be in M_s for any φ in $Q(R)$ because elements of $Q(R)$ are symmetric.

Let w_μ be the unique normalized solution of (1). The continuous and analytic dependence of $w_\mu(a_0)$, $w_\mu(b_0)$, and $w_\mu(c_0)$ on μ is shown in [3]. Obviously, for μ in M_s these are real numbers.

For μ in M_s let $[\mu]$ be its equivalence class in $T^{\#}(R)$. Let $\Psi: T^{\#}(R) \to V$ be defined by $\Psi([\mu]) = (w_\mu(a_0), w_\mu(b_0), w_\mu(c_0))$. So we have the diagram

$$\mathrm{M}_s \xrightarrow{\Phi} T^{\#}(\mathrm{R}) \xrightarrow{\Psi} \mathrm{V}. \tag{2}$$

To complete the proof of the theorem we show that Ψ is a well-defined, bi-analytic homeomorphism.

i) *Proof that Ψ is well-defined.* Suppose μ_1 and μ_2 are in M_s and w_1 and w_2 are the corresponding symmetric solutions of (1). If μ_1 and μ_2 are equivalent, then $f = w_2 \circ h \circ w_1^{-1}$ is a conformal mapping from $R(a_1, b_1, c_1)$ onto $R(a_2, b_2, c_2)$ which takes the homotopy class of α_1 into the class of α_2, the class of β_1 into β_2, and the class of γ_1 into γ_2, where α_i, β_i and γ_i are the simple loops indicated by figure 1.

Let A_i be the extremal length in $R_i = R(a_i, b_i, c_i)$ of the family of smooth curves freely homotopic to α_i. Let B_i and C_i be the extremal lengths of the corresponding families associated to β_i and γ_i, respectively. The homotopy condition on f implies that it preserves these families of curves and so $A_1 = A_2$, $B_1 = B_2$, and $C_1 = C_2$. Since there are explicit formulas for the quantities $(a-1)b/(b-a)$, $(c-b)/(b-a)$, and c in terms of the quantities A, B, and C, respectively, (this is formula (8) in section 2), one finds that

$$(a_1 - 1)b_1/(b_1 - a_1) = (a_2 - 1)b_2/(b_2 - a_2).$$

$$(c_1 - b_1)/(b_1 - a_1) = (c_2 - b_2)/(b_2 - a_2)$$

and

$$c_1 = c_2.$$

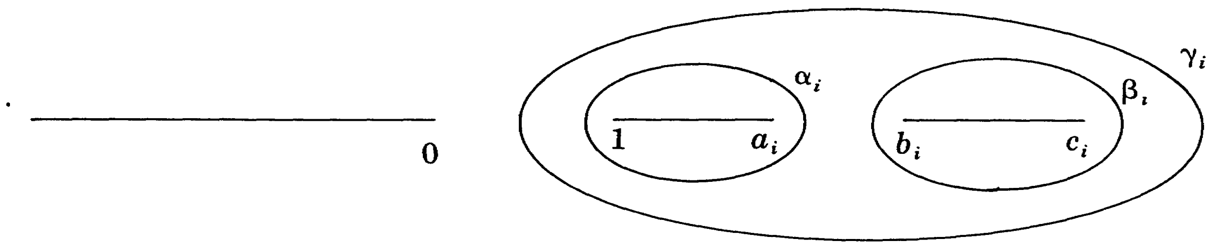

Figure 1.

Since $1 < a_i < b_i < c_i$, it is obvious that these equalities imply $a_1 = a_2$ and $b_1 = b_2$. This shows that Ψ is well-defined.

ii) *Proof that Ψ is locally bianalytic.* Let $x = w_\mu(x_0)$ and

$$\dot{x}_\mu[\nu] = \lim_{t\to 0} \frac{w_{\mu+t\nu}(x_0) - w_\mu(x_0)}{t}$$

In the next section we will give formulas for $\dot{a}_\mu$, $\dot{b}_\mu$, and $\dot{c}_\mu$ and we will see that for every μ, $\{\dot{a}_\mu, \dot{b}_\mu, \dot{c}_\mu\}$ forms a basis for $Q(R_\mu)$. Here R_μ has the obvious definition; $R_\mu = \hat{\mathbf{C}} - ([-\infty, 0] \cup [1, w_\mu(a_0)] \cup [w_\mu(b_0), w_\mu(c_0)])$. But $Q(R_\mu)$ is the fiber of the cotangent space to $T^\#(R)$ at the point $[\mu]$, (see [6]).

iii) *Proof that Ψ is injective.* We must show that if μ_1 and μ_2 are in M_s and if $w_{\mu_1}(x) = w_{\mu_2}(x)$ for $x = \infty$, 0, 1, a_0, b_0, and c_0, then μ_1 is equivalent to μ_2. Form $h = w_{\mu_1} \circ w_{\mu_2}^{-1}$. Obviously, h preserves ∞, 0, 1, a_0, b_0, c_0 and h preserves the segments $(-\infty, 0]$, $[1, a_0]$, $[b_0, c_0]$. Hence h is homotopic to the identity of R.

iv) *Proof that Ψ is surjective:* One must show that $R(a_0, b_0, c_0)$ is quasiconformally homeomorphic to $R(a, b, c)$ by a symmetric mapping. There is, for example, the mapping $w = u + iv$ where $v(x + iy) = y$ and

$$u(x+iy) = \begin{cases} x & \text{for } -\infty \le x \le 1, \\[4pt] 1 + \dfrac{a-1}{a_0 - 1}(x-1) & \text{for } 1 \le x \le a_0, \\[4pt] a + \dfrac{b-a}{b_0 - a_0}(x - a_0) & \text{for } a_0 \le x \le b_0, \\[4pt] b + \dfrac{c-b}{c_0 - b_0}(x - b_0) & \text{for } b_0 \le x \le c_0 \\[4pt] c + (x - c_0) & \text{for } x \ge c_0. \end{cases}$$

Now suppose that w is a quasiconformal mapping from $R_1 = R(a_1, b_1, c_1)$ onto $R_2 = R(a_2, b_2, c_2)$, but that R_1 and R_2 are not necessarily conformal. Suppose w takes the class of α_1 to the class of α_2 and similarly for β_i and γ_i. Let $\mu = w_{\bar{z}}/w_z$ and $K = (1 + \|\mu\|_\infty)/(1 - \|\mu\|_\infty)$. The following lemma is a well-known property of K-quasiconformal mappings, [1].

LEMMA 1.1. *$K^{-1}A_1 \le A_2 \le KA_1$ and the same inequality is true with A replaced by B or C.*

At this point we will focus attention on just one three-holed sphere $R(a, b, c)$, so we can drop the subscripts in our notation. One has the following lemma.

LEMMA 1.2. I. $16 \exp(-2\pi/C) + 1 \le c \le \dfrac{1}{16} \exp(\pi C/2) + 1,$

$$\text{II.} \quad 16\exp(-2\pi/A) \le \frac{(a-1)b}{b-a} \le \frac{1}{16}\exp(\pi A/2),$$

$$\text{III.} \quad 16\exp(-2\pi/B) \le \frac{c-b}{b-a} \le \frac{1}{16}\exp(\pi B/2).$$

Proof. The idea is to use a basic inequality for the Teichmüller annulus, the complement in $\mathbf{C}$ of $[-1,0] \cup [r,\infty)$ where $r > 0$. Let $\Lambda(r)$ be the extremal distance from $[-1,0]$ to $[r,\infty]$. In [2, page 76] the following two inequalities are derived:

$$16r \le \exp(2\pi\Lambda(r)) \le 16(r+1) \tag{3}$$

and

$$16/r \le \exp(\pi/2\Lambda(r)) \le 16\left(1 + \frac{1}{r}\right). \tag{4}$$

The first inequality is a good one for large values of r and the second for small values of r.

In order to prove inequality (I), we delete the interval $[a,b]$ from the surface R and obtain the annulus $\hat{\mathbf{C}} - ([-\infty,0] \cup [1,c])$. Let C_0 be the extremal length of the family of smooth curves homotopic to γ in this annulus. From the comparison principle $C \le C_0$. On the other hand, the extremal metric ρ_0 for the extremal length problem C_0 is symmetric under conjugation. In fact $\rho_0 = \sqrt{|\varphi|}$ where $\varphi = dz^2/z(z-1)(z-c)$, the Jenkins-Strebel differential for this annulus. So, given any curve freely homotopic to γ, one can reflect part of this curve across [a,b] and obtain a homotopic curve of equal ρ_0-length lying in $R - [a,b]$, except for some (unimportant) boundary points. Hence, $C \ge C_0$.

By the transformation $z \mapsto (z-1)/(1-c)$, the annulus $R - [a,b]$ is taken to the Teichmüller annulus with $r = (1/(c-1))$ and $\Lambda(r) = (1/c)$. Applying (3) and (4) one obtains (I).

Actually, in [2, page 76] there is an explicit formula for r in terms of $\Lambda(r)$. We have already used this fact in the proof of theorem 1.

To prove (II) we follow the same line of argument as we did in proving (I) to see that the extremal length A is unchanged if $[c,\infty]$ is removed from R. The transformation $z \mapsto \frac{1}{a-1} \cdot \frac{z-a}{z}$ takes $R - [c,\infty]$ into a Teichmüller annulus with $r = \frac{1}{a-1} \cdot \frac{b-a}{b}$ and $\Lambda(r) = 1/A$. Then (3) and (4) yield (II).

The proof of (III) is analogous.

LEMMA 1.3. *Suppose* $R_n = R(a_n, b_n, c_n) = \hat{\mathbf{C}} - ([-\infty,0] \cup [1,a_n] \cup [b_n,c_n])$ *where* $1 < a_n < b_n < c_n$ *is a sequence of three-holed spheres. Suppose there is a constant* $\epsilon > 0$ *such that the extremal lengths* A_n, B_n*, and* C_n *all lie between* ϵ *and* $1/\epsilon$. *Then there is a surface* $R_0 = R(a_0, b_0, c_0)$ *with* $1 < a_0 < b_0 < c_0$ *which is an accumulation point of the surfaces* R_n.

Proof. This is a straightforward consequence of the inequalities in lemma 1.2. (I) says the numbers c_n are bounded, so we can pick a convergent subsequence of (a_n, b_n, c_n) which we still denote by (a_n, b_n, c_n) and let (a_0, b_0, c_0) be the limit. Obviously $1 \le a_0 \le b_0 \le c_0$ and we must show that these are strict inequalities. Inequality (I) shows that $1 < c_0 < \infty$. From (III), one sees that $a_0 = b_0 = 1$ is impossible so $b_0 > 1$. Then from (II), one obtains $1 < a_0 < b_0$. Finally, using (III) again, $b_0 < c_0$.

Remark. The modular group of $T^{\#}(R)$ is the permutation group on three letters. Viewed as acting on V, it is generated by τ and σ where

(5) $$\tau(a, b, c) = \left(\frac{c}{b}, \frac{c-1}{b-1}, \frac{c-a}{b-a}\right) \quad \text{and}$$

$$\sigma(a, b, c) = \left(\frac{c-a}{b-a} \cdot \frac{b-1}{c-1}, \frac{c-a}{b-a} \cdot \frac{b}{c}, \frac{c-a}{b-a}\right).$$

It is easy to check that $\tau^3 = \sigma^2 =$ the identity.

2. VARIATIONAL FORMULAS

If h is a differentiable function from $T^{\#}(R)$ to $\mathbf{R}$, there will be a differential φ in $Q(R)$ for which

(6) $$h(t\mu) = h(0) + \operatorname{Re}\left(t \iint_R \mu(z)\, \varphi(z)\, dxdy\right) + o(t).$$

As shorthand for this equation, we write $\dot{h} = \varphi(z)\, dz^2$.

Consider the function $c(t\mu) = w_{t\mu}(c)$. An important formula from Teichmüller theory says that

(7) $$\dot{c} = -\frac{1}{\pi} \frac{c(c-1)}{z(z-1)(z-c)}\, dz^2.$$

Our objective is to find $\dot{C}$ for the function $C(t\mu)$ on $T^{\#}(R)$ where C is the extremal length described in section 1. As has already been pointed out, C is not affected if one deletes from R the interval $[a, b]$. Thus, there is an explicit functional relationship between C and c given by

(8) $$q = \exp(-2\pi/C)$$

$$\frac{1}{c-1} = \frac{1}{16q} \prod_{n=1}^{\infty} \left(\frac{1-q^{2n-1}}{1+q^{2n}}\right)^8 = F(q).$$

This formula is derived in [2, page 74–76]. Essentially the same formula appears in [5, page 429]. From (7) and (8), one finds that

$$\dot{C} = \frac{C^2 c}{2\pi^2 q F'(q)(c-1)} \cdot \frac{dz^2}{z(z-1)(z-c)}.$$

Since $F'(q) < 0$ (where $0 < q < 1$), we can write

$$\dot{C} = -p_3 \frac{dz^2}{z(z-1)(z-c)} \quad \text{where } p_3 \text{ is a positive constant.} \tag{9}$$

It is possible in a very similar way to derive formulas for the derivatives of the functions $A(t\mu)$ and $B(t\mu)$. The results are

$$\begin{aligned} \dot{A} &= +p_1 \frac{dz^2}{z(z-1)(z-a)(z-b)}, \\ \dot{B} &= -p_2 \frac{dz^2}{(z-a)(z-b)(z-c)} \end{aligned} \tag{10}$$

where p_1 and p_2 are positive constants.

Remarks. 1. Formula (7) remains valid if c is replaced everywhere by a or b. One therefore sees that $\{\dot{a}, \dot{b}, \dot{c}\}$ is a basis for $Q(R)$. A translation argument, as illustrated in [1, page 105], then shows that $\{\dot{a}_\mu, \dot{b}_\mu, \dot{c}_\mu\}$ is a basis for $Q(R_\mu)$.

2. We have found three functionals A, B, and C, defined in terms of extremal length, such that $\dot{A}$, $\dot{B}$, and $\dot{C}$ form a basis for $Q(R)$, where R is a triply connected domain. For a general Riemann surface, the analogous problem is unsolved. For example can one find $3g-3$ functionals $A_1, \ldots, A_{3g-3}$ on $T(S)$ such that $\dot{A}_1, \ldots, \dot{A}_{3g-3}$ forms a complex basis of $Q(S)$, where S is a compact Riemann surface of genus g and where the functionals are defined in terms of extremal length?

We now give variational formulas for functionals defined in terms of the Poincaré length on R. Let X be the Poincaré length of the Poincaré geodesic freely homotopic to α. Let Y and Z be the corresponding lengths for β and γ, respectively. Let G be the universal covering group for R and let G act on U, the upper half plane. Let $K(\alpha)$ be the multiplier of a primitive element in a conjugacy class corresponding to α, with $K(\alpha) > 1$. Let $a(\alpha)$ and $b(\alpha)$ be the attracting and repelling fixed points of such an element and $c(\alpha)$ the center of its isometric circle. Then

$$\begin{aligned} \dot{X}[\mu] &= (\log K(\alpha))^{\cdot}\,[\mu] = \frac{\dot{K}(\alpha)[\mu]}{K(\alpha)} \\ &= \frac{1}{\pi} \iint_{C} \frac{\mu(\zeta)\, d\xi\, d\eta}{(\zeta - a(\alpha))(\zeta - b(\alpha))(\zeta - c(\alpha))} \end{aligned} \tag{11}$$

One can use Poincaré theta series to express $\dot{X}$ as a quadratic differential on R. The formula is

$$\dot{X} = -\frac{1}{\pi} \ominus \left(\frac{1}{(z - a(\alpha))(z - b(\alpha))(z - c(\alpha))} \right) dz^2 \tag{12}$$

where $\ominus F(z) = \Sigma F(Az))A'(z)^2$ and where the sum is taken over all A in G. The derivation of (11) is a routine matter if one starts with the equation

$$\frac{A^{\mu}(z) - b}{A^{\mu}(z) - a} = K^{\mu} \frac{z - b}{z - a} \tag{13}$$

where $A^{\mu} = w^{\mu}A(w^{\mu})^{-1}$ and w^{μ} is the unique quasiconformal selfmapping of U normalized to fix a, b, and ∞ and where K^{μ} is the multiplier of A^{μ}. With this normalization, one has the basic formula

$$\dot{w}[\mu] = -\frac{(z - a)(z - b)}{\pi} \iint_{\mathbf{C}} \frac{\mu(\zeta)\, d\xi d\eta}{(\zeta - a)(\zeta - b)(\zeta - z)}. \tag{14}$$

Differentiating (13) one finds that

$$\frac{\dot{K}}{K} = \frac{(b - a)\dot{A}}{(z - a)(z - b)A'}. \tag{15}$$

On the other hand, differentiating $w^{\mu}(A(z)) = A^{\mu}(w^{\mu}(z))$, one finds that

$$\frac{\dot{A}}{A'} = \frac{\dot{w}(A(z))}{A'(z)} - \dot{w}(z). \tag{16}$$

Substitution of (16) and (14) into (15) yields (11).

Formulas for $\dot{Y}$ and $\dot{Z}$ are obtained simply by taking formula (12) and replacing α by β or γ. It is known that the quantities X, Y, and Z are moduli for $T^{\#}(R)$, [10]. It is not obvious how to see by direct means that $\{\dot{X}, \dot{Y}, \dot{Z}\}$ is a basis for $Q(R)$. This will not be important in the application intended here.

3. QUADRATIC DIFFERENTIALS ON A COMPACT SURFACE ASSOCIATED WITH A PARTITION INTO TRIPLY CONNECTED DOMAINS

By a partition of a compact surface S of genus g bigger than or equal to 2, we mean a set of $3g - 3$ nonfreely homotopic, nonintersecting loops on S. It is obvious that $3g - 3$ is the maximal number of such loops and that the surface cut along these loops becomes a set of $2g - 2$ three-holed spheres,

$$\{\tilde{R}_i;\ 1 \leq i \leq 2g - 2\}.$$

Let each surface $\tilde{R}_i$ be marked with loops $\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_i$ homotopic to its boundary components. Another system of disjoint marked three-holed spheres

$$\{R_i; 1 \le i \le 2g-2\}$$

in S is called compatible if each R_i is homotopic to $\tilde{R}_i$ in such a way that α_i is homotopic in S to $\tilde{\alpha}_i$ and similarly for β_i and γ_i. As in section 1, A_i, B_i and C_i will be the extremal lengths in R_i of the families of curves whose elements are homotopic to α_i, β_i, and γ_i, respectively. Let $6g-6$ positive numbers q_i, r_i, and s_i, $1 \le i \le 2g-2$, be given and form the function

$$P(\{R_i\}) = \sum_{i=1}^{2g-2} q_i A_i + r_i B_i + s_i C_i. \tag{17}$$

THEOREM 2. *For a given surface S, a given system of marked, disjoint, three-holed spheres $\{\tilde{R}_i; 1 \le i \le 2g-2\}$ and a given system of positive constants $\{q_i, r_i, s_i; 1 \le i \le 2g-2\}$, there exists a compatible system $\{R_i\}$ for which the function $P(\{R_i\})$ in* (17) *is minimum.*

Proof. Step 1. The numbers A_i, B_i, and C_i may be assumed to be bounded above. To see this, take a system $\{\tilde{R}_i\}$ and let $\tilde{P} = q_i\tilde{A}_i + r_i\tilde{B}_i + s_i\tilde{C}_i$. Let $\{R_i\}$ be any compatible system for which $P(\{R_i\}) \le \tilde{P}$. Then obviously $A_i \le \tilde{P}/q_i$, $B_i \le \tilde{P}/r_i$, and $C_i \le \hat{P}/s_i$.

Step 2. The numbers A_i, B_i, and C_i are bounded below by a positive number ϵ depending only on the surface S. To see this let $\Lambda(\Gamma_i)$ be the extremal length of the family Γ_i of curves in S homotopic to α_i. By the comparison principle

$$A_i \ge \Lambda(\Gamma_i).$$

The extremal metric for the extremal length problem $\Lambda(\Gamma_i)$ is the square root of the absolute value of the Jenkins-Strebel differential of the associated annulus. The quantity $\Lambda(\Gamma_i)\pi$ equals the Poincaré length around this annulus in the Poincaré metric for this annulus. But the Poincaré metric for the annulus is bigger than the Poincaré metric λ_S for S. A geodesic in the family Γ_i for the metric λ_S will have length greater than or equal to $\epsilon > 0$. The constant $\epsilon > 0$ can be chosen independently of the family of curves Γ_i just so long as Γ_i is not homotopically trivial.

Step 3. Let $P = \inf \Sigma\, q_i A_i + r_i B_i + s_i C_i$ where the infimum is taken over all marked families $\{R_i\}$ compatible with $\{\tilde{R}_i\}$. We now show there is a marked family which achieves this minimum.

Pick a sequence of marked families $\{R_{in}\}$ for which

$$P = \lim_{n\to\infty} \sum q_i A_{in} + r_i B_{in} + s_i C_{in}.$$

Let $f_{in}(z)$ be the unique holomorphic, univalent function from the uniformizing domain $D_{in} = \hat{\mathbf{C}} - ([-\infty,0] \cup [1,a_{in}] \cup [b_{in},c_{in}])$ into R_{in} which makes the homotopy classes α, β, γ in figure 1 correspond to the homotopy classes α_{in}, β_{in}, and γ_{in} of R_{in}. By lemmas 1.1 and 1.3, one can pick a subsequence of (a_{in}, b_{in}, c_{in}) which converges for each i. We label the subsequence with the same letter n and let (a_{io}, b_{io}, c_{io}) be the limit of this sebsequence; $1 < a_{io} < b_{io} < c_{io}$. Now pick a further subsequence for which the functions $f_{in}(z)$ converge normally for each i in the domains $D_i = \hat{\mathbf{C}} - ([-\infty, 0] \cup [1, a_{i0}] \cup [b_{i0}, c_{i0}])$. It is obvious that the functions f_{in} form a normal family since the universal covering of S is the unit disc. Let f_i be the normal limit of f_{in}. By the Hurwitz theorem, f_i is either univalent or constant. If f_i were identically equal to a point p on the surface S, then for a small disc containing p one could find an integer n_0 such that $f_{in_0}(\alpha)$ would be contained in that disc. But then $f_{in_0}(\alpha)$ would be homotopic to a point and not homotopic to α_i. Hence f_i cannot be constant.

Step 4. Let $R_i = f_i(D_i)$. Clearly, by definition of R_i, the minimum P is achieved by $P(\{R_i\})$. It is also clear that the domains R_i are marked by letting $\alpha_i = f_i(\alpha)$, $\beta_i = f_i(\beta)$ and $\gamma_i = f_i(\gamma)$. The domains R_i are disjoint, since if R_1 meets R_2 for example, then one could pick a sufficiently large n for which R_{1n} meets R_{2n} and this would contradict the fact that $\{R_{in}\}$ forms a compatible system. Finally, each R_i is freely homotopic to $\tilde{R}_i$ in a manner which preserves the markings. To see this observe that $f_{in}(\alpha) \sim \tilde{\alpha}_i$ and for sufficiently large n, $f_{in}(\alpha)$ will lie in an annular neighborhood of α_i and will be freely homotopic to α_i. Therefore, $\alpha_i \sim \tilde{\alpha}_i$ and similarly, $\beta_i \sim \tilde{\beta}_i$ and $\gamma_i \sim \tilde{\gamma}_i$.

Definition. The positive cone of $Q(R_i)$ is the set of linear combinations of the form $a_1\dot{A}_i + a_2\dot{B}_i + a_3\dot{C}_i$ where a_1, a_2, and a_3 are positive constants.

Remark. It has been pointed out to me by Steve Kerckhoff that the trajectory structure of any quadratic differential $\varphi(z)dz^2$ in $Q(R)$ is quite easy to describe in terms of the roots and leading coefficient of the real quadratic polynomial $p(z)$ in the expression

$$\varphi(z) = \frac{p(z)}{z(z-1)(z-a)(z-b)(z-c)}.$$

In the case at hand, where $\varphi(z)$ is in the positive cone of $Q(R)$, $\varphi(z)$ is a Jenkins-Strebel differential with three characteristic annuli homotopic to the three boundary contours of R. The positive cone consists of the positive differentials defined by Jenkins in [8]. As a consequence, the differentials constructed in the following theorem turn out to be Jenkins-Strebel differentials on S.

THEOREM 3. *Let $\{R_i\}$ be a compatible system of disjoint, three-holed spheres which minimizes the quantity $P(\{R_i\})$ in formula* (17). *Then there exists a global quadratic differential φ in $Q(S)$ such that φ restricted to R_i is an element of the positive cone of $Q(R_i)$ for each i and $\cup \bar{R}_i = S$.*

Proof. We will use repeatedly theorem 1 from [7]. Let $\varphi_i = q_i\dot{A}_i + r_i\dot{B}_i + s_i\dot{C}_i$, so φ_i is in the positive cone of $Q(R_i)$.

Step 1. φ_i is an element of $Q(S)$. To show this let $L(R_i)$ be the space of all measurable, Beltrami differentials with support in R_i and bounded in the supremum norm. Let N_i be the set of all μ in $L(R_i)$ for which $\operatorname{Re} \iint_{R_i} \mu\varphi \, dxdy = 0$ for all φ in $Q(S)$. If we show $\operatorname{Re} \iint \varphi_i\mu \, dxdy = 0$ for all μ in N_i, then it follows that φ_i is in $Q(S)$.

Suppose, contrariwise, that there is a μ in $L(R_i)$ for which

i) $$\mathrm{Re} \iint \varphi_i \mu \, dxdy < 0$$

and

ii) $$\mathrm{Re} \iint \varphi \mu \, dxdy = 0 \qquad \text{for all } \varphi \text{ in } Q(S).$$

Then one could find a curve of Beltrami differentials $\nu(t)(z)$ which are identically zero for z not in R_i and for which $\nu(t) = t\mu + o(t)$ in the L_∞-norm and for which $S_{\nu(t)} = S$ as an element of $T(S)$, [7]. Along the curve $S_{\nu(t)}$, the terms

$$q_j A_j + r_j B_j + s_j C_j$$

remain the same for $j \neq i$ and the term $q_i A_i + r_i B_i + s_i C_i$ becomes smaller for small $t > 0$. But this contradicts the minimality of P.

Step 2. There exists a positive constant c_j such that $\varphi_j = c_j \varphi_1$. If not, there would exist μ_1 in $L(R_1)$ such that $\iint_{R_1} \varphi_1 \mu_1 dxdy < 0$ and $\iint_{R_1} \varphi_j \mu_1 \, dxdy > 0$. Let μ_j be an element of $L(R_j)$ for which $\iint_{R_j} \mu_j \varphi = - \iint_{R_1} \mu_1 \varphi$ for all φ in $Q(S)$ and let $\mu = \mu_j + \mu_1$.

Thus, i) μ is orthogonal to $Q(S)$,

ii) the support of μ is contained in $R_1 \cup R_j$,

iii) $$\iint_{R_1} \mu \varphi_1 dxdy < 0,$$

and

iv) $$\iint_{R_j} \mu \varphi_j dxdy = \iint_{R_j} \mu_j \varphi_j dxdy = - \iint_{R_j} \mu_1 \varphi_j dxdy < 0.$$

Now pick a curve $\nu(t) = t\mu + o(t)$ for which $S = S_{\nu(t)}$ as an element of $T(S)$ and supp $\nu(t) \subset R_1 \cup R_j$. Along this curve, for small enough values of t, the two terms $q_1 A_1 + r_1 B_1 + s_1 C_1$ and $q_j A_j + r_j B_j + s_j C_j$ are decreased and all the other terms are unchanged and so P is made smaller for a compatible system on a conformally equivalent, similarly marked surfaces. This contradicts the minimality of P.

Step 3, $\cup \bar{R}_i = S$. If not, let D be an open disc in$S - \cup \bar{R}_i$. Then pick μ_1 in

$L(R_1)$ for which Re $\iint \mu_1 \varphi_1 dxdy < 0$ and μ_2 in $L(D)$ for which

$$\iint \mu_2 \varphi \, dxdy = - \iint \mu_1 \varphi \, dxdy$$

for all φ in $Q(S)$. Then the differential $\mu = \mu_1 + \mu_2$ will be orthogonal to $Q(S)$ and a deformation in the direction of μ will decrease the term $q_1 A_1 + r_1 B_1 + s_1 C_1$ and leave all the other terms in (17) fixed. This would be a contradiction.

Remarks. 1. If the quantity $\Sigma\, q_i A_i + r_i B_i + s_i C_i$ is replaced by $\Sigma\, q_i X_i + r_i Y_i + s_i Z_i$, where X_i, Y_i and Z_i are the Poincaré lengths discussed in section 2, the same technique can be used to construct a global quadratic differential on S associated with this quantity. It is no longer clear however that the result will yield a Jenkins-Strebel differential.

2. It is obvious that each of the boundaries of each of the triply connected domains comes equipped with a pair of antipodal points. It would be natural to try to associate the $6g - 6$ real moduli of S with the moduli of the $2g - 2$ triply connected domains, the $3g - 3$ conditions on the lengths of their boundaries and the $3g - 3$ rotations of the boundaries.

Note added in proof: Recently, by considering Dehn Twists, Scott Wolpert has found the solution to the problem suggested in Remark 2 of Section 2.

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