BOUNDED HOLOMORPHIC FUNCTIONS WITH ALL LEVEL SETS OF INFINITE LENGTH

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Let $H^\infty(\Delta)$ denote the usual space of functions bounded and analytic on the open unit disc. Two recent papers have treated the question of whether a function $F(z) \in H^\infty(\Delta)$ can have a level set $E_\alpha = \{z \in \Delta : |F(z)| = \alpha\}$ of infinite arclength, $\ell(E_\alpha) = \infty$. In [5] an example was constructed of an H^∞ function having $\ell(E_\alpha) = \infty$ for uncountably many values of α. In [1] a Blaschke product was constructed which had one level set of infinite length. The purpose of this paper is to construct the following two examples.

Example 1. There is a function $F(z) \in H^\infty(\Delta)$ such that $e^{-1} < |F(z)| < e$, $z \in \Delta$, and $\ell(E_\alpha) = \infty$, $e^{-1} < \alpha < e$.

Example 2. There is a function $G(z) \in H^\infty(\Delta)$ such that $|G(z)| < 1$, $z \in \Delta$, and $\ell(E_\alpha) = \infty$, $0 < \alpha < 1$.

Example 2 is of some interest because of its connection with the proof of the corona theorem [2], [3]. The "hard" part of the corona theorem is to show that when $f(z) \in H^\infty$ and $0 < \varepsilon < \|f\|_\infty$, there is $\delta = \delta(\varepsilon) > 0$ and $\psi(z)$ such that $0 \le \psi(z) \le 1$, $\psi \equiv 1$ when $|f| > \varepsilon$, $\psi \equiv 0$ when $|f| < \delta$, and such that $|\nabla \psi|\,dx\,dy$ is a Carleson measure. (See [2] or [3].) Example 2 shows that for f and ψ as above we cannot make ψ of the form $\psi(z) = \chi_{\{|f|>\delta\}}(z)$.

The difficult part of this paper is the construction of the function F(z) of example 1. Once this is done the function G(z) of example 2 can be constructed from F(z) using standard patching arguments.

To construct F(z) we first study the unimodular function

$$v(\theta) = \exp\left\{ i\left(\theta + \sum_{n=1}^{\infty} \frac{\cos \lambda_n \theta}{n}\right)\right\}$$

defined on $\mathbb{T}$, and its harmonic extension to the disc, v(z). The sequence $\{\lambda_n\}$ will be picked by induction, always maintaining the relationships $\lambda_1 \ge 100$,

(1) $$\lambda_n \text{ divides } \lambda_{n+1}, \qquad n \ge 1$$

and

(2) $$\lambda_{n+1} \ge \lambda_n^{\lambda_n}, \qquad n \ge 1.$$

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This high degree of lacunarity of the sequence $\{\lambda_n\}$ will force $v(z)$ to be a bad function, because it spins around so much. An induction argument allows us to pick the λ_n so that the partial products

$$v_n(\theta) = \exp\left\{ i\left(\theta + \sum_{j=1}^{n} \frac{\cos \lambda_j \theta}{j}\right)\right\}$$

have almost uniform distribution function in the following sense:

$$|\{\theta : v_n(\theta) \in I\}| \geq |I|/4 \qquad \text{for every arc } I \text{ of } \mathbb{T}. \tag{3}$$

Intuitively we can do this because the high degree of lacunarity of the sequence $\{\lambda_n\}$ causes the functions $\cos \lambda_n \theta$ to be "almost" independent random variables. Because our proof of (3) should be familiar to anyone who has read, for example, chapter VIII of [4], we merely sketch the argument.

First note that $\exp\{i\theta\}$ is evenly distributed. Suppose by induction that $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ have been chosen and that

$$|\{\theta : v_j(\theta) \in I\} \geq (1 + 2^{-n+1})\,|I|/4, \qquad j = 1, 2, \ldots, n-1, \tag{4}$$

whenever I is an arc of $\mathbb{T}$. Note that $f_n(\theta) = \theta + \sum_{j=1}^{n-1} \frac{\cos \lambda_j \theta}{j}$ is C^∞ and has first derivative bounded in modulus by $2\lambda_{n-1}$ when $n \geq 2$, and 1 when $n = 1$. We view $f_n(\theta)$ as a $\mathrm{mod}(2\pi)$ valued function. Chop $\mathbb{T}$ up into $10^4(\lambda_{n-1})^2$ intervals I_j of equal length, and let x_j denote the center of I_j. Since we need only prove (3) for intervals I of length less than $10^{-4}(\lambda_{n-1})^{-3}$, we restrict our attention to an interval $I_\delta = [\theta_0 - \delta, \theta_0 + \delta]$ of $\mathbb{R}\,\mathrm{mod}(2\pi)$, where θ_0 is arbitrary and

$$\delta < 10^{-4}(\lambda_{n-1})^{-3}.$$

Consider an interval I_j in the decomposition of $\mathbb{T}$ for which

$$|f_n(x_j) - \theta_0| \leq \frac{1}{n} - \frac{1}{1000\, n^5} \qquad \text{if } n \geq 2,$$

or $|f_n(x_j) - \theta_0)| \leq 1 - \frac{1}{1000}$ if $n = 1$. Then if λ_n is very large,

$$\left|\left\{\theta \in I_j : f_n(\theta) + \frac{\cos \lambda_n \theta}{n} \in I_\delta\right\}\right| \geq (1 - 10^{-2n})\,\frac{|I_j|}{2\pi}\cdot\left|\left\{\psi \in [0, 2\pi) : f_n(x_j) + \frac{\cos \lambda_n \psi}{n} \in I_\delta\right\}\right| \tag{5}$$

and the above estimate is independent of θ_0 and δ. Applying the induction hypothesis (4) and condition (5) we see rather easily that

(6) $$|\{\theta: v_n(\theta) \in I\}| \geq (1 + 2^{-n})|I|/4,$$

whenever I is an arc of $\mathbb{T}$.

Now let $u(\theta) = \text{Re}\, v(\theta)$ and let $F(\theta) = \exp\{u(\theta) + i\tilde{u}(\theta)\}$. Note that $-1 \leq u(\theta) \leq 1$ because $v(\theta)$ is unimodular. Also the level set $\{|F(z)| = e^\alpha\}$ is the same as the set $H_\alpha = \{u(z) = \alpha\}$. Therefore we need only show $\ell(H_\alpha) = \infty$, $-1 < \alpha < 1$. The idea is to show that there is $\delta_\alpha > 0$ and $\varepsilon > 0$ such that $u(\beta e^{i\theta}) = \alpha$ for more than $\varepsilon\lambda_n/n$ values of θ, whenever $\beta \in I_n = \left[1 - \frac{\delta_\alpha}{\lambda_n}, 1 - \frac{\delta_\alpha}{2\lambda_n}\right]$ and $n \geq n(\alpha)$ is large. For then $\ell(H_\alpha \cap \{|z| \in I_n\}) \geq \frac{\varepsilon\lambda_n}{n} \cdot \frac{\delta_\alpha}{2\lambda_n} = \frac{\varepsilon\delta_\alpha}{2n}$ and so

$$\ell(H_\alpha) \geq \sum_{n \geq n(\alpha)} \frac{\varepsilon\delta_\alpha}{2n} = \infty.$$

We now compute $\ell(H_\alpha)$ for $0 \leq \alpha < 1$; the case where $-1 < \alpha < 0$ is treated in exactly the same manner. Let $\delta_\alpha = \frac{(1-\alpha)^2}{100}$. We restrict our attention to the annulus $A_n = \{z : |z| \in I_n\}$. Because the λ_n satisfy condition (2), a well known theorem on lacunary series (*cf.* [6, page 230]) shows

(7) $$\left|\left\{\theta: \left|\sum_{j=n+1}^{\infty} \frac{\cos \lambda_j\theta}{j}\right| > \gamma\right\}\right| \leq c_1 \exp\{-c_2 n\gamma\},$$

where c_1 and c_2 are some absolute constants. Because $\sum_{j=n+1}^{\infty} \frac{\cos \lambda_j\theta}{j}$ is periodic of periodic $2\pi/\lambda_{n+1}$, (7) and (2) show

(8) $$|v(z) - v_n(z)| < \frac{1}{1000\,n}, \qquad z \in A_n, \text{ for n sufficiently large.}$$

Let $\theta_\alpha \in (0, \pi/2]$ be that value of θ for which $\text{Re}\, e^{i\theta} = \cos\theta = \alpha$, and let

$$J_n = \left\{\theta: |e^{i\theta_\alpha} - v_{n-1}(\theta)| \leq \frac{1}{10\,n}\right\}.$$

By condition (3), $|J_n| \geq \frac{1}{100\,n}$. Fix $\theta_0 \in J_n$. Then since $\frac{\cos \lambda_n\theta}{n}$ has amplitude $1/n$ and period $2\pi/\lambda_n$, and since by condition (2) $v_{n-1}(\theta)$ is very "flat" with respect to $\cos \lambda_n\theta$, then whenever $|\psi| < \frac{1}{2n}$ there is θ such that

(9) $$|\theta - \theta_0| < \pi/\lambda_n \quad \text{and} \quad v_n(\theta) = e^i(\theta_\alpha + \psi).$$

Standard estimates for the Poisson kernel now show that because $|J_n| \geq \frac{1}{100\,n}$ and conditions (2), (8), and (9) hold, then if n is large and $\beta \in I_n$,

(10) $\{\theta: \operatorname{Re} v(\beta e^{i\theta}) = \alpha\}$ contains more than $\frac{1}{200\,n\pi} \cdot \frac{\lambda_n}{10} = \frac{\lambda_n}{2000\,n\pi}$ points.

By our previous remarks, condition (10) implies $\ell(H_\alpha) = \infty$. Our construction of $F(z)$ is completed.

We now outline the construction of the function $G(z)$ for example 2. Consider the function $V(x) = \exp\left\{i\left(2\pi x + \sum_{n=1}^{\infty} \frac{\cos 2\pi\lambda_n x}{n}\right)\right\}$ defined for $x \in \mathbb{R}$, and let $V(w)$ be its harmonic extension to the upper halfplane $\mathbb{R}^2_+$. Let $U(w) = \operatorname{Re} V(w)$. The argument used in the construction of $F(z)$ shows that if α is some real number, $-1 < \alpha < 1$, and I is some interval of the real axis of length one, then

$$\ell(\{U(w) = \alpha\} \cap \{I \times (0,1]\}) = \infty.$$

Let $I_n = [10^n, 10^n + 1]$, $n = 1,2,\ldots.$ and let $W(x) = -\frac{1}{4}\sum_{n=1}^{\infty} n\chi_{I_n}(x)$. Repetition of the argument used for example 1 shows that whenever $-\infty < \alpha < 1$, there is an interval I_α of length one such that

$$\ell(\{U(w) + W(w) = \alpha\} \cap \{I_\alpha \times (0,1]\}) = \infty.$$

Let $\tau(z) = (-i)\frac{z+1}{z-1}$ be the usual conformal mapping of Δ onto $\mathbb{R}^2_+$ and set

$$G(z) = \exp\{-1 + U \circ \tau(z) + W \circ \tau(z) + i(\tilde{U} \circ \tau(z) + \tilde{W} \circ \tau(z))\}.$$

Then $G(z) \in H^\infty(\Delta)$, $\|G\|_\infty = 1$ and if $0 < \alpha < 1$, $\ell(\{|G(z)| = \alpha\}) = \infty$.

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