HEREDITARILY STRONGLY INFINITE DIMENSIONAL SPACES

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1. INTRODUCTION

Recently in [8] John J. Walsh constructed an infinite dimensional compactum Z lying in the Hilbert cube Q which contains no finite dimensional subspaces except those of dimension 0. By minor modifications, he can also eliminate all countable dimensional subsets from Z.

Prior to Walsh's result, we had the following situation. In 1932, Hurewicz [4], assuming the continuum hypothesis, gave an example of an infinite dimensional subset of the Hilbert cube such that every finite dimensional subset is countable, hence, 0-dimensional. In 1965, Henderson [3] and then Bing [1] in 1966 constructed infinite dimensional compacta all of whose closed finite dimensional subsets are of dimension 0. In 1974, Zarelua [10] constructed strongly infinite dimensional compacta all of whose closed subsets are strongly infinite dimensional, using different techniques. Following that, in 1977, Rubin, Schori and Walsh [7] developed a simpler, axiomatic approach to these constructions; we shall use their results in the sequel.

The construction given herein is a variation of Walsh's. It gives us an infinite dimensional compactum X having the property that if $Y \subset X$ and $\dim Y > 0$, then Y is strongly infinite dimensional. In a forthcoming paper, we will show that every strongly infinite dimensional space contains such a "hereditarily" strongly infinite dimensional closed subspace.

This paper is a revision of an earlier version that was incorrect. We gratefully thank John Walsh for finding the error there. We also thank Wesley Terry for many stimulating discussions that helped generate ideas for the techniques used.

2. DEFINITIONS

The spaces dealt with herein are all separable and metrizable. Let Γ denote a countable or finite indexing set.

Definition 2.1. Let X be a space and A,B be closed subsets of X. A closed set S of X *separates* A and B if $X - S$ can be written as the union to two separated sets, one containing A, the other containing B. We say S *continuum-wise separates* A and B if each continuum in X that meets A and B also meets S.

Definition 2.2 *Let* X be a space and $\{(A_k, B_k) : k \in \Gamma\}$ be a collection of disjoint pairs of closed, nonempty subsets of X. The family is called *essential* provided

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that if S_k separates A_k and B_k for each k, then $\cap \{S_k : k \in \Gamma\} \neq \emptyset$. If Γ is infinite, then X is called *strongly infinite dimensional* if it has such an essential family.

LEMMA 2.3 *Every strongly infinite dimensional space is infinite dimensional but is not countably infinite dimensional.*

Proof. Let X be such a space and suppose X can be written as the countable union $\cup \{X_i : i \in \Sigma\}$ where each X_i is 0-dimensional. For $k \in \Sigma$, choose a separator S_k of A_k and B_k such that $S_k \cap X_k = \emptyset$. It is routine to check that $\cap \{S_k : k \in \Sigma\} = \emptyset$, which is a contradiction to the definition of essential family.

For use in the sequel we state 5.5 of [7].

PROPOSITION 2.4 *Let $\{(A_j, B_j) : j \in \Gamma\}$ be an essential family in a compactum X. Let $J \subset \Gamma$ and suppose for each $j \in J$, X_j is a closed subset of X which continuum-wise separates A_j and B_j. Letting $Y = \cap \{X_j : j \in J\}$, $\{(A_k \cap Y, B_k \cap Y) : k \in \Gamma - J\}$ is an essential family in Y.*

Now suppose X is a locally connected compactum, A,B are disjoint closed subsets of X, and S is a continuum-wise separator of A and B in X. Let $F(S)$ denote the union of S and all complementary domains of S that do not intersect $A \cup B$. If Σ is a collection of continuum-wise separators of A and B, then $F(\Sigma)$ designates $\{F(S) : S \in \Sigma\}$. The following lemma can easily be proved.

LEMMA 2.5. *Let X be a locally connected compactum and suppose A,B are disjoint closed subsets of X. If S is a continuum-wise separator of A and B in X, then so is $F(S)$. If S is a separator of A and B in X, then so is $F(S)$.*

3. CONSTRUCTION OF CONTINUUM-WISE SEPARATORS

In this section we develop a special way to produce continuum-wise separators.

Let X be a Peano continuum, Y be a metric space and Φ: $Y \to 2^X$ (supplied with the Hausdorff metric [6]) be a function. Then Φ is *upper semi-continuous* (usc) provided: if $y_i \to y$ and U is a neighborhood in X of $\Phi(y)$, then there is an integer N such that $\Phi(y_i) \subset U$ for all $i \geq N$.

Now assume A,B are disjoint closed subsets of X and let $\mathscr{S}$ denote the collection of all continuum-wise separators of A and B in X. It is easy to show that $\mathscr{S}$ is a compact subset of 2^X. Let $\Phi : I = [-1,1] \to \mathscr{S}$ be usc and let F be as in 2.5.

LEMMA 3.1. *The composition $F \circ \Phi : I \to \mathscr{S}$ is* usc.

Proof. Let $G = F(\Phi(t))$ and let U be an open neighborhood of G. Using local connectedness, cover $X - U$ with a finite collection of closed, connected sets K_i each of which does not intersect G but does intersect $A \cup B$. Let V be the complement of $\cup \{K_i\}$. Then if $\Phi(s) \subset V$, $F(\Phi(s)) \subset V$ also.

Now let $\pi : X \to I$ be continuous and define Z to be $\cup \{\pi^{-1}(t) \cap F \circ \Phi(t) : t \in I\}$.

PROPOSITION 3.2. *Z is closed in X and $Z \in \mathscr{S}$.*

Proof. Employing the fact that $F \circ \Phi$ is usc, it is easy to show Z is closed.

For $C \in \{A,B\}$, let $U_C = \{x \in X - Z$: the component of $X - F(\Phi(\pi(x)))$ containing x meets $C\}$. Since $F \circ \Phi \circ \pi$ is usc and X is locally connected, each of U_A and U_B is open; clearly $U_A \cap U_B = \emptyset$ and $(A - Z) \subset U_A$, $(B - Z) \subset U_B$. Finally, $X - Z = U_A \cup U_B$.

This proposition and its proof were supplied to me by John Walsh. The proof greatly simplifies my original one.

4. INTERSECTION OF SEPARATORS

Let Q denote the Hilbert cube and refer to the notation at the first of Section 5. Given a sequence $\{S_i : i \geq 1\}$ of separators of opposite pairs of faces of Q, we may find it necessary to select another sequence of separators $\{S_i'' : i \geq 1\}$ so that $\cap \{F(S_i'') : i \geq 1\} \subset \cap \{S_i : i \geq 1\}$. This section is designed to show how to accomplish this.

For the next few lemmas, assume S is a separator of $A = A_m$ and $B = B_m$ in Q. In general, let $A(S), B(S)$ denote the respective complementary domains of S (i.e., components of $Q - S$) containing A and B. Let Fr U denote the topological boundary of U.

LEMMA 4.1. *The separator S contains a closed subset S' which is minimal with respect to being a separator of A and B.*

Proof. Let S' be the intersection of a maximal nest of separators of A and B that are contained in S.

LEMMA 4.2. *If U is an open neighborhood of a point $p \in S$ such that $U \cap B(S) = \emptyset$, then $S_0 = S - U$ is also a separator of A and B in Q.*

Proof. Suppose there is an arc $[a,b]$ in $Q - S_0$ from A to B. Let q be the last point of $[a,b] \cap S$. Then q lies in U. It is clear that the half open interval $q < t \leq b$ is contained in $B(S)$.

LEMMA 4.3. *Let U be a closed neighborhood in* Cl $B(S)$ *of a point $p \in S$, $U \cap B = \emptyset$. Then $S_0 = (S \cup U) -$* Int *U is a separator of A and B in Q* (Int *denotes the interior in* Cl $B(S)$).

Proof. Suppose $[a,b]$ is an arc in $Q - S_0$ from A to B. Let q be the last point of $[a,b] \cap S$. Then q lies in Int U, and so the arc $q \leq t \leq b$ must intersect Fr U. But Fr $U \subset S_0$, so this is a contradiction, and the lemma is proved.

LEMMA 4.4. *Suppose K is a compact subset of $B(S)$. Then there exists a continuum in $B(S)$ containing $K \cup B$ in its interior.*

LEMMA 4.5. *Assume S is minimal and let P be a compact subset of S. Let U be a closed neighborhood in* Cl $B(S)$ *of P, $U \cap B = \emptyset$ and suppose D is a complementary domain of S that does not intersect $A \cup B$ and that there exists a $p \in$* Fr *$D \cap P$. Then $S_0' = (S \cup U) -$* Int *U* (interior taken in Cl $B(S)$) *is a separator of A and B in Q, $D \subset A(S_0')$, $A(S) \subset A(S_0')$. Choosing $S_0 \subset S_0'$ to be* *minimal with respect to separating A and B, then it is also true that $D \subset A(S_0)$ and $A(S) \subset A(S_0)$.*

Proof. By 4.3, S_0' is a separator of A and B in Q. Since $p \notin S_0'$, there is a closed ball neighborhood E of p such that $E \cap S_0' = \emptyset$. Clearly $D \cap S_0' = \emptyset$, so for any $q \in D$, there is an arc $L \subset D$ from q to some point in E. On the other hand, since S is minimal, 4.2 implies E contains a point q' of $A(S)$. Since $A(S) \cap S_0' = \emptyset$, there is an arc L' in $A(S)$ from A to q' missing S_0'. Then $L \cup E \cup L'$ is a continuum meeting q and A and not intersecting S_0'. Therefore, any $q \in D$ lies in $A(S_0')$.

Example 4.6. One might expect that a minimal separator of A and B in Q would have the property that each of its complementary domains intersects A or B. That this is not necessarily the case is demonstrated as follows.

Let Y be a triod as in Figure 1.

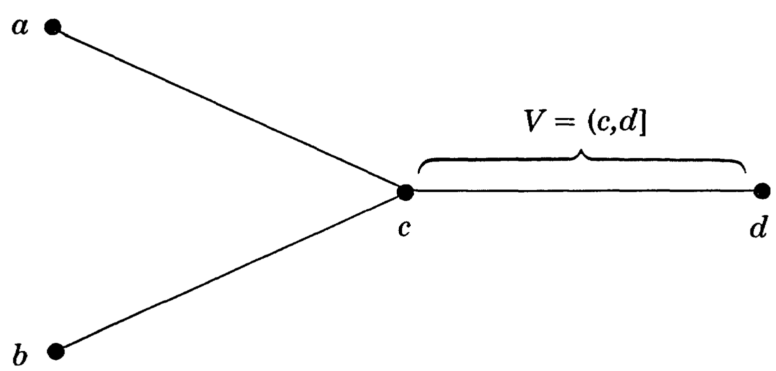

Figure 1

By [9], $Y \times Q \cong Q$. On the other hand, $A = \{a\} \times Q$ and $B = \{b\} \times Q$ are Z-sets [2] in Q, and by applying Theorem 11.1 of [2], we may assume $A = A_1$, $B = B_1$ are opposite faces of Q. Then clearly $S = \{c\} \times Q$ is a minimal separator of A_1 and B_1, while $V \times Q$ is a complementary domain of S that does not intersect A_1 or B_1. On the other hand, it is clear that S is not a Z-set in Q by the next lemma.

LEMMA 4.7. *If $K \subset Q$ is a Z-set, then $Q - K$ is connected.*

Proof. Applying Theorem 11.1 of [2], there is no loss of generality in assuming $K \subset A_1$.

LEMMA 4.8. *Suppose $\{n_i : i \geq 1\}$ is a nonredundant list of positive integers and for each $i \geq 1$, S_i is a separator of A_{n_i} and B_{n_i} in Q. Then $\cap \{S_i : i \geq 1\}$ is a Z-set in Q.*

Proof. For $\varepsilon > 0$, choose n_i so large that the canonical projection of Q onto A_{n_i} is an ε-map. Then note that $S_i \cap A_{n_i} = \emptyset$, since S_i separates A_{n_i} from B_{n_i}.

THEOREM 4.9. *Let $\{n_i : i \geq 1\}$ be a nonredundant sequence of positive integers, and for each i,* let *S_i be a separator of A_{n_i} and B_{n_i} in Q. There exists a sequence $\{S_i'' : i \geq 1\}$ of separators S_i'' of A_{n_i} and B_{n_i} in Q such that*

$$\cap \{F(S_i''): i \geq 1\} \subset \cap \{S_i : i \geq 1\}.$$

Proof. For each i, let $S_i' \subset S_i$ be minimal, and let $\mathrm{NF}(S_i')$ designate a small closed neighborhood of $F(S_i')$ so that $\mathrm{NF}(S_i')$ is a separator of A_{n_i} and B_{n_i} and $F(\mathrm{NF}(S_i')) = \mathrm{NF}(S_i')$. Let

$$K_0 = \cap \{S_i' : i \geq 1\} \subset K = \cap \{\mathrm{NF}(S_i') : i \geq 1\}.$$

Define H_i to be $K - (S_i' \cup A_{n_i}(S_i))$. Our object is to construct S_1'' so that $F(S_1'') \subset \mathrm{NF}(S)$ and $F(S_1'') \cap H = \emptyset$. If we can do this, we then can construct S_i'' similarly for each i. Then we would have

$$\cap \{F(S_i''): i \geq 1\} \subset \cap \{\mathrm{NF}(S_i') : i \geq 1\} = K.$$

However, for each $x \in K - K_0$, there is at least one i so that x lies in the H_i corresponding to S_i'. This would show that $x \notin F(S_i'')$. In turn, this would imply that $\cap \{F(S_i''): i \geq 1\} \subset K_0$, completing the proof of the theorem. Therefore, we only need to show how to construct S_1''.

To reduce notation, let $S = S_1'$, $A = A_{n_1}$, $B = B_{n_1}$, $H = H_1$. Then H is a locally compact space. For each $x \in H$ there is a compact neighborhood E_x of x in H and a complementary domain D_x of S such that $E_x \subset D_x$. Write $H = \cup \{H_i : i \geq 1\}$, a countable cover consisting of sets E_x.

Let M_A denote the closure of $A(\mathrm{NF}(S))$ and M_B denote the closure of $B(\mathrm{NF}(S))$. We note that $M_A \subset A(S)$, $M_B \subset B(S)$. If $H = \emptyset$, let $S_1'' = S$. Otherwise, let $H_1' = H_1 - B(S)$ (Note that either $H_1' = H_1$ or $H_1' = \emptyset$.). Let T_1 be a continuum in $B(S)$ containing $M_B \cup (H_1 \cap B(S)) = M_B \cup (H_1 - H_1')$ in its interior. If $H_1' = \emptyset$, let $S(1) = S$. If not, then there is a complementary domain D_1 of S containing $H_1' = H_1$ in its interior. Since K is a Z-set (4.8), use 4.7 to find a point $p \in \mathrm{Fr}\, D_1 - K$. Choose $\varepsilon > 0$ so small that the closed ε-neighborhood U of p is contained in the interior of $\mathrm{NF}(S)$ and does not intersect $K \cup T_1$. Apply 4.5 to replace S by $S(1)$. Then we have $M_A \subset A(S) \subset A(S(1))$, $D_1 \subset A(S(1))$, $T_1 \cap S(1) = \emptyset$. From this and the above it follows that $M_A \cup M_B \cup H_1 \subset A(S(1)) \cup \mathrm{Int}\, T_1$.

Our next step will establish a recursion process. Let

$$H_2' = H_2 - [B(S(1)) \cup A(S(1))].$$

Let T_2 be a continuum in $B(S(1))$ containing $T_1 \cup (H_2 \cap B(S_1))$ in its interior. If $H_2' = \emptyset$, let $S(2) = S(1)$. If not, then there is a set D_2 consisting of a finite union of complementary domains of $S(1)$ such that $H_2' \subset D_2$. Let P be a finite subset of $S(1)$ that contains a boundary point of each component of D_2 and such that $P \cap K = \emptyset$. Choose $\varepsilon > 0$ so small that the closed ε-neighborhood U of P does not intersect $K \cup T_2$. Apply 4.5 to replace $S(1)$ by $S(2)$. Then we have $M_A \subset A(S)) \subset A(S(1)) \subset A(S(2))$, $D_2 \subset A(S(2))$, $T_1 \subset \mathrm{Int}\, T_2$, $T_2 \cap S(2) = \emptyset$. It follows that $M_A \cup M_B \cup H_1 \cup H_2 \subset A(S(2)) \cup \mathrm{Int}\, T_2$.

We continue this construction, producing a sequence $S(1)$, $S(2), \ldots$. Let S_1'' be the limit of some convergent subsequence. Then for each i,

$S''_1 \cap T_i = \emptyset$. Clearly $A(S) \subset A(S''_1)$. Hence, $(M_A \cup M_B) \cap F(S''_1) = \emptyset$, so that $F(S''_1) \subset \mathrm{NF}(S)$.

On the other hand, for any i, $H_i \subset A(S(i)) \cup \mathrm{Int}\, T_i \subset A(S''_1) \cup B(S''_1)$, so that $F(S''_1) \cap H_i = \emptyset$. This implies $F(S''_1) \cap H = \emptyset$, completing the proof of the theorem.

5. ROTATING CONTINUUM-WISE SEPARATORS

The constructions of Section 3 and Section 4 will be useful to us only if we can relate them to each other, as will be seen in subsequent developments. The proper relation will be obtained when we see how to construct the maps Φ of Section 3.

For each $n \geq 1$, let $I_n = [-1,1]$; then $Q = \Pi\{I_n : n \geq 1\}$ is the Hilbert cube and $\pi_n : Q \to I_n$ is the n-th coordinate projection. The sets $A_n = \pi_n^{-1}(-1)$ and $B_n = \pi_n^{-1}(1)$ are called the *opposite faces* of Q in the I_n direction. Using the proof on p. 49 of [5], one sees that $\{(A_n, B_n) : n \geq 1\}$ is an essential family in Q. Hence the Hilbert cube is strongly infinite dimensional.

For each $n \geq 1$, let $Q_n = \pi_n^{-1}(0)$; for $m \neq n$, let $A^n_m = A_m \cap Q_n$, $B^n_m = B_m \cap Q_n$. As a standard notation, whenever $\Delta \subset I_n$ is a Cantor set, we let $\tilde{\Delta} = \Delta \cup \{-1\} \cup \{1\}$. We use Σ^n_m to designate the collection of all continuum-wise separators of A^n_m and B^n_m in Q_n endowed with the Hausdorff metric. It is not difficult to see that Σ^n_m is a compact metric space.

5.1. Suppose $m \neq n$, $m > 1$, $n > 1$, and $\Delta \subset \mathrm{Int}\, I_n$ is a Cantor set. Let $\beta : \tilde{\Delta} \to F(\Sigma^1_m)$ be a function satisfying,

5.1.1 If (a,b) is a component of $I_n - \tilde{\Delta}$, then $\beta(a) = \beta(b)$,

5.1.2 β is usc,

5.1.3 $\beta(-1) = \beta(1) = Q_1$.

The idea of the next operation is to take $t \in \Delta$, rotate $\beta(t)$ counterclockwise through $\pi/2$ radians, make a linear adjustment to get back into Q, and then project into Q_n. The resulting set $\alpha(t)$ will be an element of $F(\Sigma^n_m)$.

To begin, let $\alpha(-1) = \alpha(1) = Q_n$. For the sequel, treat Q as being embedded in the topological product of countably many copies of the real line. Let $t \in \Delta$. For simplicity we define the operation of changing from β to α in three steps, following a point $u = (u_1, u_2, \ldots) \in \beta(t)$.

First, $u \to v = (v_1, v_2, \ldots)$, where $v_1 = t - u_n$, $v_n = t + u_1$, $v_i = u_i$ otherwise. This operation corresponds to a re-embedding of $\beta(t)$ via a counterclockwise rotation of $\pi/2$ radians in the $(1,n)$-coordinates around the point $(0,t)$ in $I_1 \times I_n$. Most important is that $(u_1, u_n) = (0,t)$ if and only if $(v_1, v_n) = (0,t)$; i.e., in this case $u \to u$.

Unless $t = 0$, the first operation maps $\beta(t)$ outside Q; our second operation adjusts this, sending points back into Q (because $u \in Q_1$). Send $v = (v_1, v_2, \ldots)$ to $w = (w_1, w_2, \ldots)$ where $w_i = v_i$ if $i \geq 2$, $w_1 = v_1/(1-t)$ if $v_1 \geq 0$, $w_1 = v_1/(1+t)$ if $v_1 \leq 0$. Note $v_1 = 0$ if and only if $w_1 = 0$; i.e., in this case $v \to v$.

The third operation is simply projection into Q_n, given by

$$w = (w_1,w_2,\ldots) \to z = (z_1,z_2,\ldots)$$

where $z_i = w_i$ for $i \neq n$, $z_n = 0$.

We see that $[\alpha(t) \times I_n] \cap \pi_n^{-1}(t) \cap Q_1 = \beta(t) \cap \pi_n^{-1}(t)$. Also, all three operations described above are continuous. We therefore have the following result.

PROPOSITION 5.2. *Let* Δ,β *be as in* 5.1. *Then there exists a function* $\alpha : \tilde{\Delta} \to F(\Sigma_m^n)$ *such that* $\alpha(-1) = \alpha(1) = Q_n$ *and satisfying* 5.1.1, 5.1.2, *with* α *in place of* β. *Furthermore, for each* $t \in \Delta$,

$$[\alpha(t) \times I_n] \cap \pi_n^{-1}(t) \cap Q_1 = \beta(t) \cap \pi_n^{-1}(t).$$

5.3. Let us conclude this section with a lemma indicating how we might find functions β as in 5.1. Assume $\Delta \subset \operatorname{Int} I_n$ is a Cantor set and $\beta_1 : \tilde{\Delta} \to \Sigma_m^1$ is a continuous function with $\beta(-1) = \beta(1) = Q_1$. Let $\{(a_i,b_i) : i \geq 1\}$ be a nonredundant list of the complementary open intervals of $\tilde{\Delta}$ in I_n. Define $\beta_2 : \tilde{\Delta} \to \Sigma_m^1$ by $\beta_2(t) = \beta_1(t)$ if $t \notin \{a_i,b_i : i \geq 1\}$, $\beta_2(t) = \beta_1(a_i) \cup \beta_1(b_i)$ if $t \in \{a_i,b_i\}$.

5.3.1 β_2 *is* usc.

Proof. In fact, what occurs is that $\{\beta_2(t_i) : i \geq 1\}$ converges to $\beta_1(t)$ in Σ_m^1, and since $\beta_1(t) \subset \beta_2(t)$, the result follows from the nature of the Hausdorff metric.

Define $\beta : \tilde{\Delta} \to F(\Sigma_m^1)$ by the rule $\beta(t) = F(\beta_2(t))$.

5.3.2 β *is* usc.

Proof. See the proof of 3.1.

We summarize this information in the following lemma.

LEMMA 5.4. *Suppose* $\Delta \subset \operatorname{Int} I_n$ *is a Cantor set and* $\beta_1 : \tilde{\Delta} \to \Sigma_m^1$ *is a continuous function with* $\beta_1(-1) = \beta_1(1) = Q_1$. *Then there exists* $\beta : \tilde{\Delta} \to F(\Sigma_m^1)$ *such that* $\beta(t) = F(\beta_1(t))$ *except perhaps when t is an endpoint of a component of* $I_n - \tilde{\Delta}$. *Furthermore, we may choose* β *to satisfy* 5.1.1, 5.1.2, 5.1.3. *That is, if* (a,b) *is a component of* $I_n - \tilde{\Delta}$, *then* $\beta(a) = \beta(b)$, β *is* usc, $\beta(-1) = \beta(1) = Q_1$.

6. CONSTRUCTION OF X, MAIN RESULT

Let $\mathscr{W}$ be a countable base for the topology of $[-1,1]$. For each $W \in \mathscr{W}$, let Δ_W be a Cantor set in $W - \{-1,1\}$. Let $\mathscr{P} = \{(W,n) : W \in \mathscr{W}, n \geq 2\}$, and choose a function $\lambda : \{m \geq 3\} \to \mathscr{P}$ to be a surjection such that each $\lambda^{-1}(W,n)$ is countably infinite and $n \notin \lambda^{-1}(W,n)$.

We have the essential family, $\{(A_m^1, B_m^1) : m \geq 2\}$ in Q_1 as in Section 2, where Q_1 is playing the role of Q. For each $(W,n) \in \mathscr{P}$, let $\Lambda_{(W,n)}$ be the compactum, $\Pi\{\Sigma_m^1 : m \in \lambda^{-1}(W,n)\}$. Choose a continuous surjection $\beta_0 : \tilde{\Delta}_W \to \Lambda_{(W,n)}$ so that $\beta_0^{-1}(x)$ is uncountable for each x, $\beta_0(-1) = \beta_0(1) = (Q_1,Q_1,\ldots)$. For $m \in \lambda^{-1}(W,n)$, let $p_m : \Lambda_{(W,n)} \to \Sigma_m^1$ be the coordinate projection. Define β_m^* to be $p_m\beta_0$; thus

$\beta_m^* : \tilde{\Delta}_W \to \Sigma_m^1$ satisfies the requirements of β_1 of 5.4. Let $\beta_m : \tilde{\Delta}_W \to F(\Sigma_m^1)$ be as in the conclusion of 5.4; then m, n, $\Delta_W \subset \operatorname{Int} I_n$, β_m are as in 5.1.

Let $\alpha_m : \tilde{\Delta}_W \to F(\Sigma_m^n)$ be as α in 5.2. Now extend α_m to all of I_n by $\alpha_m(t) = \alpha_m(a)$ whenever t lies in a complementary interval (a,b) of $\tilde{\Delta}_W$ in I_n. Then $\alpha_m : I_n \to F(\Sigma_m^n)$ is also usc. Let $\mathscr{S}_m$ denote the space of all continuum-wise separators of A_m and B_m in Q. Define $\Phi : I_n \to F(\mathscr{S}_m)$ by $\Phi(t) = \alpha_m(t) \times I_n$; again, Φ is usc. Using 5.3.1 and the construction preceding 3.2, and using $X = Q$ and $\pi = \pi_n : Q \to I_n$, let $Z_m' \in \mathscr{S}_m$ be as Z in 3.2. Then $Z_m = Z_m' \cap Q_1$ lies in Σ_m^1. We have the important formula for t a nonendpoint of Δ_W:

$$Z_m \cap \pi_n^{-1}(t) = F(p_m \beta_0(t)) \cap \pi_n^{-1}(t).$$

This comes from the conclusion of 5.2 and the choice of β_m.

Define the space $X \subset Q_1$ by $X = \cap \{Z_m : m \geq 3\}$. Then by 2.4, $\{(A_2^1 \cap X, B_2^1 \cap X)\}$ is an essential family in X. Applying II.4.A of [5], we have the following result.

PROPOSITION 6.1. $\dim X \neq 0$.

We now state and prove our main result.

THEOREM 6.2. *If $G \subset X$ and $G \neq 0$, then G is strongly infinite dimensional.*

Proof. Since $G \subset Q_1$ and $\dim G \neq 0$, there exists $n > 1$ such that $\pi_n(G)$ contains a $W \in \mathscr{W}$ (It is easy to show that a countable product of 0-dimensional spaces is 0-dimensional). For each $m \in \lambda^{-1}(W,n)$, let $\mathscr{A}_m = \pi_m^{-1}[-1,-1/2] \cap Q_1$, $\mathscr{B}_m = \pi_m^{-1}[1/2,1] \cap Q_1$. Let $\mathscr{F} = \{\mathscr{A}_m \cap G, \mathscr{B}_m \cap G) : m \in \lambda^{-1}(W,n)\}$. We want to show that $\mathscr{F}$ is an essential family for G. It is clear that each element of $\mathscr{F}$ is a disjoint pair of closed subsets of G. Note that $A_m^1 \cap G \subset \mathscr{A}_m$, $B_m^1 \cap G \subset \mathscr{B}_m$. Suppose for each $m \in \lambda^{-1}(W,n)$, S_m is a separator of $\mathscr{A}_m \cap G$ and $\mathscr{B}_m \cap G$ in G. Write $G - S_m = H_m \cup K_m$ where H_m, K_m are disjoint and open in G,

$$\mathscr{A}_m \cap G \subset H_m, \ \mathscr{B}_m \cap G \subset K_m.$$

Let $H_m^* = H_m \cup \operatorname{Int} \mathscr{A}_m$, $K_m^* = K_m \cup \operatorname{Int} \mathscr{B}_m$ (interior taken in Q_1). Then H_m^* and K_m^* are separated in Q_1, so there exist disjoint open sets U_m, V_m of Q_1 such that $H_m^* \subset U_m$, $K_m^* \subset V_m$. Define $\tilde{S}_m$ to be $Q_1 - (U_m \cup V_m)$. Note that $\tilde{S}_m \cap G \subset S_m$. Since $A_m^1 \subset U_m, B_m^1 \subset V_m$, $\tilde{S}_m$ separates A_m^1 and B_m^1 in Q_1. Thus $\tilde{S}_m \in \Sigma_m^1$.

For each m, let S_m' be chosen as in 4.9, with Q_1 replacing Q, and $\tilde{S}_m$ replacing S_{n_i} there. Thus $\cap \{F(S_m''): m \in \lambda^{-1}(W,n)\} \subset \cap \{\tilde{S}_m : m \in \lambda^{-1}(W,n)\}$. There is a point $r \in \Lambda_{(W,n)}$ such that $p_m(r) = F(S_m'')$ for each m. Choose a nonendpoint $t \in \Delta_W$ such that $\beta_0(t) = r$. Hence $\beta_m^*(t) = p_m \beta_0(t) = F(S_m'') = \beta_m(t)$. From a previously noted formula, we conclude that

$$Z_m \cap \pi_n^{-1}(t) = p_m \beta_0(t) \cap \pi_n^{-1}(t) = F(S_m'') \cap \pi_n^{-1}(t).$$

Therefore,

$$\emptyset \neq \pi_n^{-1}(t) \cap G \subset \cap \{F(S''_m) : m \in \lambda^{-1}(W,n)\} \cap G \subset \cap \{\tilde{S}_m : m \in \lambda^{-1}(W,n)\} \cap G$$
$$= \cap \{\tilde{S}_m \cap G : m \in \lambda^{-1}(W,n)\} \subset \cap \{S_m : m \in \lambda^{-1}(W,n)\},$$

giving us the desired result.

COROLLARY 6.3. *The only finite dimensional subsets of X are those of dimension* 0, *X contains no countable dimensional sets, and X itself is infinite dimensional.*

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