NEARLY MAXIMAL REPRESENTATIONS FOR THE SPECIAL LINEAR GROUP

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1. INTRODUCTION

Since early in this century there has been a continuing interest in the following problem: For a given finite group, G, what are the maximal subgroups of G? This problem is of course most interesting when a family of groups is considered, and examples of such work are the results of Mitchell on $PSL_3(q)$, $PSU_3(q)$ and $PSp_4(q)$, q odd (see [7] and [8] resp.) and those of Hartley for $PSL_3(q)$, q even (see [3]). More recently there is the work of Mwene (see [9]). The problem of finding all the maximal subgroups of $PSL_n(q)$, or of any of the classical groups, is in general not a realistic one, since this amounts to essentially finding all irreducible subgroups of these groups (on their standard modules). A variation on this theme is the following: suppose G is a group, and H is embedded in some known way in G, what are the subgroups of G which contain H? In particular, is H maximal? Burgoyne, Greiss and Lyons [1] considered this problem for G a group of Lie type and H the fixed points of certain automorphisms of G of prime order. In [2], E. Halberstadt considers $\Sigma(X)$, the symmetric group on a finite set X, and its action on $X^{(k)}$, the k-element subsets of X, and shows that the embedding in $\Sigma(X^{(k)})$ or $A(X^{(k)})$ [alternating group] is almost always maximal and determines the exceptions. The analogue of this for linear groups is: Show $SL(V)$ is "nearly" maximal in $A(L_k(V))$ where $L_k(V)$ is the collection of k-subspaces of V. In [4], Kantor and McDonough do this problem for $k = 1$. In this paper we treat a problem similar to these. Before we get to our results we first introduce some notation.

Suppose ϕ is a homomorphism from a group G to a group X, we will say that ϕ is *maximal* if $\phi(G)$ is a maximal subgroup of X. ϕ is said to be *nearly maximal* if whenever H is a proper subgroup of X and H contains $\phi(G)$, then H normalizes $\phi(G)$. Finally, for a prime p, we say ϕ is *p-maximal,* if for any proper subgroup H of X which contains $\phi(G)$, then a p-Sylow of $\phi(G)$ is a p-Sylow of H.

Now let V be a vector space of dimension $n \geq 3$ over a field $F = \mathsf{F}_{p^e}$ and for $k \leq n-1$, let $V_K = \Lambda^k(V)$. Set $G = SL(V)$, $G_k = SL(V_k)$ and define ϕ_k, a homomorphism from G into G_k by

$$(\phi_k g)(v_1 \wedge \ldots \wedge v_k) = (gv_1) \wedge (gv_2) \wedge \ldots \wedge (gv_k).$$

Received September 14, 1977. Revision received March 14, 1979.
The author was supported in part by NSF MCS 76-07035.

Michigan Math. J. 27 (1980).

The following seems to be reasonable:

Conjecture. If $(k,p) \neq \left(\frac{n}{2}, 2\right)$, then ϕ_k is p-maximal. If $k \neq (n/2)$ or $(k,n) = (2,4)$ and $p \neq 2$, then ϕ_k is nearly maximal.

Remarks. (1) The conjecture is equivalent to the following: If $\phi_k(G) \leq H < GL(V_k)$ and $\phi_k(G) \ntrianglelefteq H$, then $H \geq SL(V_k)$.

(2) The restrictions in the conjecture are necessary: $k = (n/2)$, $p = 2$, then ϕ_k is not 2-maximal since the graph automorphism acts on V_k. Moreover, when $k = (n/2)$, V_k is self-dual as a module for $\phi(G)$ and $\phi(G) \leq \Omega^+(V_k)$ or $\phi(G) \leq Sp(V_k)$ as k is even or odd.

(3) For $p > 3$ a possible method of attack is the use of Thompson's classification of quadratic pairs [11]: if $\tau \in G$ is a transvection on V, then $\phi_k(\tau)$ acts quadratically on V_k for each k. At this time the author is uncertain whether use of this result suffices to prove the conjecture. We have chosen our method for its simplicity and geometric character.

The purpose of this paper is to prove some cases of this conjecture. We prove

THEOREM 1.1. ϕ_2, ϕ_{n-2} are *p-maximal unless* $n = 4$, $p = 2$.

THEOREM 1.2. *If* $n = 4$, $p = 2$ and $\phi_2(G) \leq H \lessdot G_2$, $\phi(G) \ntrianglelefteq H$, then $O^{2'}(H) \cong \mathrm{Sp}(V_2)$.

THEOREM 1.3. *If* $n \geq 5$, then ϕ_2, ϕ_{n-2} *are nearly maximal.*

In section two we consider the module $V \otimes V_k$ for $SL(V)$ and get an FG-decomposition. In section three we determine the dimensions of p-elements of $SL(V)$ in V_2. In section four the cases $n \leq 4$ in (1.1)–(1.3) are handled and in section five the general cases are proved. In section six some concluding remarks on what is necessary to extend our proof for the treatment of the conjecture in general.

2. THE MODULES $V \otimes V_k$ FOR $SL(V)$

In this section we get an FG decomposition for the module

$$M = V \otimes V_k \quad (1 \leq k \leq n-1),$$

where $G = SL(V)$. We only need the result for $k = n-2$; however, our proof is by induction. It is possible that the result is known, but the author has not found it accessible in the literature and so for completeness is included here.

Let $\Omega = \{1, 2, \ldots, n\}$. For $1 \leq j \leq n$, set $\Omega_j = \{\alpha \subseteq \Omega \mid |\alpha| = j\}$. When $\alpha \in \Omega_j$ we will identify α with the unique increasing map from $\{1, 2, \ldots, j\}$ to Ω with range α. Let $V = \langle v_i : i \in \Omega \rangle$ be a vector space of dimension n over an arbitrary field F. For $\alpha \in \Omega_j$, set

$$w_\alpha = v_{\alpha(1)} v_{\alpha(2)} \wedge \ldots \wedge v_{\alpha(j)} \in \Lambda^j(V) = V_j.$$

For $\alpha \in \Omega_j$, $\beta \in \Omega_{j'}$ with $\alpha \cap \beta = \emptyset$, let $\varepsilon(\alpha, \beta) \in \{1, -1\} \subseteq F^*$ be defined so

$$\varepsilon(\alpha,\beta)\, w_\alpha \wedge w_\beta = w_{\alpha\cup\beta}.$$

When $j = 1$ and $\alpha = \{i\}$, $\beta \in \Omega_{j'}$, $i \notin \beta$ we write $\varepsilon(i,\beta)$ for $\varepsilon(\{i\},\beta)$. For $i \in \Omega$, $\alpha \in \Omega_k$, let

$$x(i,\alpha) = v_i \otimes w_\alpha.$$

Now let $\gamma \in \Omega_{k+1}$ and set

$$u_\gamma = \sum_{i\in\gamma} \varepsilon(i,\gamma - \{i\})\, x(i,\gamma - \{i\}), \quad U = \langle u_\gamma : \gamma \in \Omega_{k+1} \rangle$$

The following is straightforward to prove.

LEMMA 2.1. *U is a submodule of M, and as a module for G is isomorphic to V_{k+1}.*

Next consider the map $\theta: V \otimes V_k \to V_{k+1}$ by $\theta(v \otimes w) = v \wedge w$ for $v \in V$, $w \in V_k$. This is an FG-surjection of modules, and hence M has a quotient isomorphic to V_{k+1}. Set $K = \ker\theta$. Notice $\theta(u_\gamma) = (k+1)u_\gamma$, and so $U \leq K$ if and only if char $F | k+1$. The main results of this section are as follows.

PROPOSITION 2.2. If *W is a submodule of M, $W \not\leq U$, then $W \geq K$.*

COROLLARY 2.3. *If* char F *does not divide* $k+1$, *then K is irreducible, and $M = U \oplus K$ as an FG-module.*

COROLLARY 2.4. *If* char $F | k+1$, *then M has a unique composition series as FG-module*: $0 \subset U \subset K \subset M$.

Proof. Clearly (2.3) and (2.4) follow immediately from (2.2). (2.2) is proved by induction on n. When $k = 1$, $M = V \otimes V$ and then the result is very well known so we omit its proof. Thus we may assume $k \geq 2$.

Let $\Delta = \{1, 2, \ldots, n-1\}$. As for Ω, if $1 \leq j \leq n-1$, then Δ_j is the collection of all subsets of Δ of cardinality j. Set

$$Y = \langle v_i : i \in \Delta\rangle, \qquad Q = C_G(Y),$$
$$L = N_G(Y) \cap N_G(\langle v_n\rangle), \qquad L_1 = O^\infty(L) \cong SL(Y).$$

For $i \neq j \in \Omega$, σ_{ij} will be the transvection centralizing $\langle V_t : t \neq j\rangle$ such that $\sigma_{ij}(v_j) = v_i + v_j$. For any $i,j \in \Omega$, $E_{ij}: V \to V$ is the linear transformation such that

$$E_{ij}(v_k) = \delta_{jk} v_i.$$

Finally, if $\alpha \in \Omega_k$, $i \in \alpha$, $j \notin \alpha$, then $\alpha_i^j = \alpha - \{i\} \cup \{j\}$.

Suppose first that $k = n-1$. Then $V_k \cong V^* = \mathrm{Hom}_F(V,F)$ as FG-module and $V \otimes V^* \cong \mathrm{Hom}_F(V,V)$. Therefore we can identify G with $SL_n(F)$, M with $M_n(F)$ and let $g \in G$ act on $T \in M$ by

$$g \circ T = gTg^{-1}$$

Let

$$M_1 = \{T \in M : \operatorname{Im} T \subseteq Y,\ Y \subseteq \ker T\}$$
$$M_2 = \{T \in M : \operatorname{Im} T \subseteq Y\}$$

Then

$$[M,Q] = \langle gTg^{-1} - T : g \in Q, T \in M \rangle = (M_2 \oplus U) \cap K$$

and $U \leq [M,Q]$ if and only if char $F \mid n$. Note that $M_1 \cong Y$ as FL and FL_1-modules, and M_2/M_1 is isomorphic to $\operatorname{Hom}_F(Y,Y)$ as a module for L and L_1. $C_M(Q) = M_1 \oplus U$. Assume now that W is a submodule of M, $W \nleq U$. Suppose $C_W(Q) \neq U$, then for some $\lambda \in F$, $T \in M_1^{\#}$, $\lambda I_V + T \in C_W(Q)$. Choose $g \in L_1$, so $g \circ T \neq T$. Then $g \circ (\lambda I_V + T) = \lambda I_v + g \circ T \in C_W(Q)$, and then so is $gT - T \in M_1^{\#}$. It then follows that $M_1 \leq C_W(Q)$. In particular $E_{1n} \in C_W(Q)$. Since G is 2-transitive on the 1-spaces of V, $E_{ij} \in W$ for $i \neq j$. So $E_{21} \in W$. But then

$$\sigma_{12} \circ E_{21} = (E_{21} - E_{12}) + (E_{11} - E_{22}) \in W,$$

and hence $E_{11} - E_{22} \in W$. Again, since G is 2-transitive on 1-spaces of V, $E_{ii} - E_{jj} \in W$, W contains a base for K; and $W \geq K$. Therefore we may assume $C_W(Q) = U$. Next note that $[M,Q,Q] = M_1$, and since $M_1 \subseteq C_M(Q)$, $C_W(Q) = U$, $[W,Q,Q] = 0$. Since $W \nleq U$, we must have $[W,Q] = U$. Therefore, there is $T \in W$ and $g \in Q$, so $g \circ T - T = \lambda I_V$, some $\lambda \in F^*$. Suppose

$$T = \left(\begin{array}{c|c} T_1 & \alpha \\ \hline \gamma & a \end{array}\right) \quad \text{and} \quad g = \left(\begin{array}{ccccc|c} 1 & & & & 0 & \\ & \cdot & & & & \\ & & \cdot & & & \beta \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & 1 & \\ \hline & & 0 & & & 1 \end{array}\right)$$

Then

$$g \circ T - T = \left(\begin{array}{c|c} \beta\gamma & \begin{array}{c} a\beta - \\ T_1\beta - \\ \beta\gamma\beta \end{array} \\ \hline 0 & -\gamma\beta \end{array}\right).$$

However, if $g \circ T - T = \lambda I_V$, then

$$\beta\gamma = \begin{pmatrix} \lambda & & & & & \\ & \cdot & & & 0 & \\ & & \lambda & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & 0 & & & & \cdot \\ & & & & & & \lambda \end{pmatrix} \in M_{n-1}(F).$$

But, the row rank of $\beta\gamma \leq 1$, and we have a contradiction since $2 \leq k = n - 1$. Thus the theorem is proved when $k = n - 1$.

Now we may assume $2 \leq k \leq n - 2$. Let Δ, Y, Q, L, L_1 be as above. It is easily checked that $C_M(Q) = \langle x(i,\alpha): i \in \Delta, \alpha \subseteq \Delta_k \rangle \cong Y \otimes Y_k$ as a module for FL_1 and FL. Assume W is a submodule of M, $W \not\leq U$. If $C_W(Q) \not\leq C_U(Q)$, then by induction, $C_W(Q) \geq K \cap C_M(Q)$, and so $x(i,\alpha) \in W$ for $i \in \alpha \in \Delta_k$. However, it then follows that $x(i,\alpha) \in W$ for $i \in \alpha \in \Omega_k$ since the Weyl group of G is transitive on $\{\langle x(i,\alpha)\rangle : i \in \alpha \in \Omega_k\}$. Suppose $\gamma \in \Delta_{k+1}$, $i \neq j \in \gamma$. Then

$$\varepsilon(i,\gamma - \{i\}\, x(i,\gamma - \{i\}) - \varepsilon(j,\gamma - \{j\})\, x(j,\gamma - \{j\}) \in W.$$

But then it follows for any $\gamma \in \Omega_{k+1}$, $i \neq j \in \gamma$

$$\varepsilon(i,\gamma - \{i\})\, x(i,\gamma - \{i\}) - \varepsilon(j,\gamma - \{j\})\, x(j,\gamma - \{j\}) \in W.$$

Therefore W contains a base for K, so $K \leq W$ as claimed. Hence we may assume $C_W(Q) \leq C_U(Q)$. Since $C_W(Q) \neq 0$, $W \cap U \neq 0$, and $U \leq W$ by (2.1). Note that this implies char F divides $k + 1$, for otherwise $W = U \oplus (W \cap K)$, and

$$C_W(Q) \geq C_{W \cap K}(Q) \not\leq C_U(Q).$$

Next note that

$$\begin{aligned}[M,Q] &= C_M(Q) + \langle x(i,\alpha): i, n \in \alpha \in \Omega_k \rangle \\ &\quad + \langle \varepsilon(n,\alpha)\, x(n,\alpha) - \varepsilon(j,\alpha_j^n)\, x(j,\alpha_j^n): \alpha \in \Delta_k, j \in \alpha \rangle .\end{aligned}$$

Also, $[M,Q]/C_M(Q) \cong Y \otimes Y_{k\mp 1}$ as a module for FL and FL_1, and $UC_M(Q)/C_M(Q)$ plays the same role in $[M,Q]/C_M(Q)$ as U does in M. Suppose $W \cap [M,Q] \neq U$. Then by induction, for some $y \in C_M(Q)$, $\alpha \in \Omega_k$ with $j, n \in \alpha$.

$$x = x(i,\alpha) + y \in W.$$

Choose $j \notin \alpha$ and let $\sigma = \sigma_{jn}$. Then $\langle \sigma x - x \rangle = \langle x(i,\alpha_n^j) \rangle \leq C_W(Q)$, but $x(i,\alpha_n^j) \notin U$ contradicting our assumption that $C_W(Q) = C_U(Q)$. Therefore $W \cap [M,Q] = U$. But $[W,Q] \leq W \cap [M,Q] \leq U$, so Q centralizes W/U. Since $Q \not\leq Z(G)$ we must have G centralizes W/U. However, $W/U = W/[M,Q] \cap W \cong W[M,Q]/[M,Q]$ as a module for L_1. But $M/[M,Q]$ is isomorphic to $Y_{k-1} \oplus Y_k$ as a module for

L_1 and so L_1 does not centralize anything in $M/[M,Q]$, we have a contradiction, and the proof of (2.2) is complete.

3. CENTRALIZERS OF p-ELEMENTS IN V_2

In this section $V = \langle v_i : 1 \leq i \leq n \rangle$ is an n-dimensional vector space over $F = \mathbf{F}_{p^e}$, $V_2 = \Lambda^2(V)$, $G = SL(V)$. We continue to denote the transvection with axis $\langle v_k : k \neq j \rangle$ such that $v_j \to v_i + v_j$ by σ_{ij}. Let $\sigma = \sigma_{1n}$. Then

$$C_{V_2}(\sigma) = \langle v_1 \wedge v_n, v_i \wedge v_j : 1 \leq i < j \leq n-1 \rangle,$$

and so

$$\text{(3.1)} \qquad \dim C_{V_2}(\sigma) = \binom{n-1}{2} + 1.$$

LEMMA 3.2. *If $n \geq 4$, $\tau \in G$ a p-element and $C_{V_2}(\tau) \geq C_{V_2}(\sigma)$, then τ is also a transvection with axis $\langle v_1, \ldots, v_{n-1} \rangle$ and center $\langle v_1 \rangle$.*

Proof. Since τ centralizes $v_i \wedge v_j$ for $1 \leq i < j \leq n-1$, τ normalizes $\langle v_i, v_j \rangle$. For any $1 \leq i \leq n-1$ we can find j, j' distinct from i with $j, j' \leq n-1$ since $n \geq 4$. Then τ normalizes $\langle v_i, v_j \rangle$ and $\langle v_i, v_{j'} \rangle$ and so normalizes

$$\langle v_i \rangle = \langle v_i, v_j \rangle \cap \langle v_i, v_{j'} \rangle.$$

Since τ is a p-element, τ centralizes $v_1, v_2, \ldots, v_{n-1}$. Since τ also centralizes $v_1 \wedge v_n$, τ normalizes $\langle v_1, v_n \rangle$, so $[V, \tau] = \langle v_1 \rangle$ and the result is proved.

Remark 3.3. If we drop the restriction that τ be a p-element, then τ could also be $-I_v$ (and of course the product of any transvection with axis $\langle v_1, \ldots, v_{n-1} \rangle$ and center $\langle v_1 \rangle$ with $-I_V$) when $-I_v \in G$.

LEMMA 3.4. *Let $n_1, n_2, \ldots, n_r$ be positive integers with*

$$n_1 \leq n_2 \leq \ldots \leq n_r, \; n_1 + n_2 + \ldots + n_r = m.$$

Then $rn_1 + (r-1)n_2 + \ldots + 2n_{r-1} + n_r \leq \binom{m+1}{2}$ and we have equality if and only if $m = r$.

Proof. The proof is by induction on m. If $m = r$, then $n_1 = n_2 = \ldots = n_m = 1$, and then the sum is $m + (m-1) + \ldots + 2 + 1 = \binom{m+1}{2}$. Thus we may assume $r < m$ and show that we get strict inequality. Since $r < m$, $n_r > 1$. If $r = 1$, then $rn_1 + \ldots + 2n_{r-1} + n_r = m < \binom{m+1}{2}$. So we may assume $r > 1$. By induction

$$(r-1)n_1 + (r-2)n_2 + \ldots + n_{r-1} \leq \binom{m - n_r + 1}{2}.$$

Then

$$rn_1 + (r-1)n_2 + \dots + n_r \le \binom{m - n_r + 1}{2} + m \le \binom{m-1}{2} + m < \binom{m+1}{2}.$$

LEMMA 3.5. *Assume $n \ge 5$, $\sigma = \sigma_{1n}$, $\tau \in G$ so $|\tau| = p$.* Then

$$\dim C_{V_2}(\tau) \le \dim C_{V_2}(\sigma) = \binom{n-1}{2} + 1,$$

and we have equality if and only if $\sigma_G\tau$ that is, if and only if τ is also a transvection.

Proof. If we identify G with $SL_n(q)$, $(q = p^e)$, then we can identify V_2 with alternate matrices, $A = Alt_n(q) = \{(m_{ij}) \in M_n(q) : m_{ii} = 0,\ m_{ji} = -m_{ij}\}$, the action of $g \in SL_n(q)$ on $(m_{ij}) \in A$ given by:

$$g \circ (m_{ij}) = g(m_{ij})g^t \ [g^t \text{ is the transpose of } g].$$

For a positive integer k, let

$$J_k = \begin{pmatrix} 1 & 1 & & & & 0 \\ & 1 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ 0 & & & & 1 & 1 \end{pmatrix} \in M_k(q).$$

If the invariant factors of τ have degrees $d_1, d_2, \dots, d_s$ with $d_1 \le d_2 \le \dots \le d_s$ then $d_1 + d_2 + \dots + d_s = n$, and we can take

$$\tau = \begin{pmatrix} J_{d_1} & & & 0 \\ & \ddots & & \\ & & J_{d_{s-1}} & \\ 0 & & & J_{d_s} \end{pmatrix}$$

Assume $N = (N_{ab})_{1 \le a,b \le s} \in C_A(\tau)$ where $N_{ab} \in M_{d_a,d_b}(q)$. $\tau \circ N = (J_{d_a} N_{ab} J^t_{d_b})$. From this and some simple matrix calculations it follows that

$$\dim \mathrm{C_A}(\tau) = (s-1)d_1 + (s-2)d_2 + \dots + d_{s-1} + \left[\frac{d_1}{2}\right] + \left[\frac{d_2}{2}\right] + \dots + \left[\frac{d_s}{2}\right]$$

By the previous lemma

$$(s-1)d_1 + (s-2)d_2 + \dots + d_{s-1} \le \binom{n - d_s + 1}{2}.$$

Suppose $d_s \geq 3$. Then

$$\begin{aligned}\dim C_A(\tau) &\leq \binom{n-2}{2} + \left[\frac{d_1}{2}\right] + \dots + \left[\frac{d_s}{2}\right] \\ &\leq \binom{n-2}{2} + \left[\frac{n}{2}\right] \\ &\leq \binom{n-2}{2} + n - 2 \\ &= \binom{n-1}{2} \\ &< \binom{n-1}{2} + 1.\end{aligned}$$

Therefore we may assume $d_s = 2$. Suppose $d_1 = d_2 = \dots = d_s = 2$ so $n = 2s$. Then $s > 2$ and $\dim C_A(\tau) = 2\,[(s-1) + (s-2) + \dots + 2 + 1] + s = s(s-1) + s = s^2 < 2s^2 - 3s + 2 = \binom{2s-1}{2} + 1$. Therefore we may assume

$$d_1 = d_2 = \dots = d_r = 1,\ d_{r+1} = \dots = d_s = 2$$

and also that $r < s - 1$, since when $r = s - 1 = n - 2$, τ is a transvection and we get equality.

Now $\dim C_A(\tau) = \binom{s}{2} + 2s - 2r - 1$. Since $s - r \geq 2$, $s \leq n - 2$. Also $r + 2(s - r) = 2s - r = n$. So

$$\dim C_A(\tau) \leq \binom{n-2}{2} + n - 2 = \binom{n-1}{2} < \left(\frac{n-1}{2}\right) + 1.$$

4. THE CASES $n \leq 4$

In this section we prove (1.1) when $n \leq 4$ and (1.2). Suppose V is a three-dimensional vector space over $F = \mathsf{F}_pe$, $G = SL(V)$. Then $V_2 = \Lambda^2(V)$ is isomorphic to $V^* = \mathrm{Hom}_F(V,F)$ as a module for G. Then $G = G_2 = SL(V^*)$ and is clearly p-maximal and nearly maximal in G_2. Thus we may assume $n = 4$.

Note that $\Lambda^4(V)$ is a one-dimensional vector space over F and so may be identified with F. Consider the map $f\colon V_2 \times V_2 \to \Lambda^4(V) = F$ induced by

$$f(v_1 \wedge v_2, v_3 \wedge v_4) = v_1 \wedge v_2 \wedge v_3 \wedge v_4.$$

Then the following is well-known.

LEMMA 4.1. (i) *If $p \neq 2$, f is a non-degenerate symmetric bilinear pairing with maximal index and* $\varphi(G) \cong G/Z(G) \cong \Omega^+(V_2)$.

(ii) *When $p = 2$, f is a non-degenerate alternate bilinear pairing. In this case* $G \cong \varphi(G) \cong \Omega^+(V_2)$.

Remarks 4.2. (1) The decomposable vectors in V_2, that is those of the form $v \wedge w$, $v, w \in V$, are the singular vectors of V_2 while the remaining non-zero vectors are the non-singular vectors.

(2) In odd characteristic $\varphi(G)$ has one orbit on singular points with length $(q^2+1)(q^2+q+1)$, $(q = p^e)$ and two orbits on non-singular points each with length $(q^2(q^3-1))/2$. Note if U is a non-degenerate subspace with dimensional two and non-maximal index, then U contains $(q+1)/2$ points from each of the two orbits on non-singular points.

(3) In characteristic two there is a single orbit on singular points with length $(q^2+1)(q^2+q+1)$ and one orbit on non-singular points with length $q^2(q^3-1)$.

We next introduce some notation.

For W a vector space and $L \leq SL(W)$, U a subspace of W, set

4.3(i) $$\chi_L(U) = \{g \in L : [g, U] = 0,\ [W, g] \leq U\}.$$

For U, U' subspaces of W, set

4.3(ii) $$\chi_L(U, U') = \chi_L(U) \cap \chi_L(U').$$

Let P be a one-subspace of V, U a hyperplane. Then $|\chi_G(P)| = |\chi_G(U)| = q^3$ and $C_{V_2}(\chi_G(P))$, $C_{V_2}(\chi_G(U))$ are representatives of the two classes of maximal totally singular subspaces of V_2. Set $X_1 = \chi_G(P)$, $X_2 = \chi_G(U)$. Suppose W is a three-subspace of V_2, W not totally singular. Then if $x \in W^\#$, x not singular, $C_G(W) \leq C_G(x) \cong \Omega_5(q)$. $C_G(x)$ acts on $x^\perp$ in its natural way (irreducibly when $p \neq 2$, indecomposably with constituents of dimension 1 and 4 when $p = 2$). When $p \neq 2$, there is a unique class of elementary subgroups of order q^3 in $C_G(x)$. If E is a representative of this class, then $C_{x^\perp}(E)$ is a one subspace. Therefore, the subgroups in the G-classes of X_1 and X_2 are characterized as those elementary abelian subgroups of G of order q^3 which centralize a three-subspace of V_2. Thus we have

LEMMA 4.4. *Let $p \neq 2$, $S \in Syl_p(G)$, $P = C_{V_2}(S)$, $U = [V, S]$, $X_1 = \chi_G(P)$, $X_2 = \chi_G(U)$. If $g \in G_2$ and $X_i^g \leq S$, then $X_i^g = X_1$* or *X_2.*

We now consider the case where $p \neq 2$.

PROPOSITION 4.5. *If $p \neq 2$, then $\varphi(G)$ is p-maximal in G_2.*

Proof. Set $\bar{G} = \varphi(G)$ and assume on the contrary that $\bar{G}$ is not p-maximal in G. Thus let H be a proper subgroup of G_2 with $H \geq \bar{G}$, $|H|_p > |\bar{G}|_p$. Let $T \in Syl_p(H)$ so $T \geq S$ and set $T_1 = N_T(S)$. By (4.4) T_1 permutes X_1 and X_2. But orbits of T_1 on $\{X_1, X_2\}$ must have length a power of p, so T_1 normalizes X_1 and X_2. Set $X = X_1$, $N = N_{\bar{G}}(X)$, $W = C_{V_2}(X)$, $N_2 = N_{G_2}(X)$. Note that

$$|N{:}O^{p'}(N)| = (q-1)/(q-1,4), \qquad O^{p'}(N)/Z = O^{p'}(N/X) \cong SL_3(q)$$

and N/X is a subgroup of $GL_3(q)$. W we saw previously, is a three-subspace of X_2, $C_N(W) = X$ and as a module for N/X, W and X are isomorphic. Also $X = C_N(V_2/W)$ and

$$V_2/W \cong \mathrm{Hom}_F(X,F) = X^*$$

as a module for N/X. Let $L \leq N$ so $L \cap X = 1$, $N = XL$, and set $Q = \chi_{G_2}(W)$. Then $X \leq Q$. Set $Q_0 = Q \cap H$.

LEMMA 4.6. $Q_0 = X$.

Proof. As a module for L, $Q \cong X^* \otimes X^*$. Since the characteristic, p, is odd, L has two constituents on Q, one of which is X. Therefore, if $Q_0 > X$, then $Q_0 = Q$. However, for every point R of W and hyperplane Y of V_2 containing W,

$$\chi_{G_2}(R, Y) \leq Q.$$

If $Q_0 = Q$, then since H acts irreducibly on V_2, by [5], $H = G_2$ a contradiction. Thus $Q_0 = X$ as asserted.

Now set $N_2 = N_{G_2}(X)$ and $K = N_H(X)$. $O^{p'}(N_2) = QO^{p'}(L)$. Then

$$O^{p'}(QK/Q) \cong O^{p'}(L) \cong SL_3(q).$$

In particular, $|QK/Q| = q^3$. However, we have

$$|QK/Q| = |K{:}K \cap Q| = |K{:}Q_0| = |K{:}X|.$$

Therefore $|K|_p = q^3 \cdot |X| = q^6$. But $K \geq T_1$ and $|T_1| > |\bar{G}|_p = q^6$ and with this contradiction (4.5) is proved. We continue to assume $p \neq 2$ and consider the near-maximality of $\bar{G}$ in G_2.

PROPOSITION 4.7. *For p odd, $\bar{G}$ is nearly-maximal in G_2.*

Proof. Again our proof is by contradiction. Assume on the contrary that there is a subgroup H of G_2 so $\bar{G} \leq H < G_2$, $\bar{G} \not\leq H$.

LEMMA 4.8. *H is transitive on ponts of V_2.*

Proof. As we said in (4.2)(2), $\bar{G}$ has three orbits on points of V_2, the singular points, $\mathcal{S}$, and two orbits of equal length on non-singular points—η_1 and η_2. If H preserved $\mathcal{S}$, then $H \leq GO^+(V_2)$ and $\bar{G} \leq H$ contrary to assumption. So H fuses $\mathcal{S}$ to at least one of the orbits, say η_1. Suppose H did not fuse $\mathcal{S}$ to η_2 as well. Choose $R \in \mathcal{S}$ and $Y \in \eta_2$ with R and Y orthogonal and consider $\langle R, Y\rangle$. $\mathcal{S} \cap \langle R, Y\rangle = \{R\}$ and $|\eta_2 \cap \langle R, Y\rangle| = q$. Now choose $g \in H$ so $R_1 = R^g \in \eta_1$ and set $Y_1 = Y^g$. Now since H leaves η_2 invariant, $|\eta_2 \cap \langle R_1, Y_1\rangle| = q$ and

$$\{R_1\} = \eta_1 \cap \langle R_1, Y_1\rangle.$$

Therefore $\langle R_1, Y_1\rangle$ is a non-degenerate subspace of V_2 with dimension two and non-maximal index. But again by (4.2)(2), we must have

$$|\eta_1 \cap \langle R_1, Y_1 \rangle| = q + 1/2 > 1,$$

a contradiction and (4.8) is proved.

Let $x = v_1 \wedge v_2$, $y = v_1 \wedge v_2 + v_2 \wedge v_4$ where $\{v_1, v_2, v_3, v_4\}$ is a base for V. Set $X = \chi_G(\langle v_1, v_2 \rangle)$. X induces the group of all transvections of $V_2/\langle x \rangle$ with axis $x^\perp/\langle x \rangle$. $C_{\bar{G}}(\langle y \rangle)$ acts irreducibly on $V_2/\langle y \rangle$. Since H acts transitively on points, we must have that $N_H(\langle x \rangle)$ acts irreducibly on $v_2/\langle x \rangle$. Set $N = N_H(\langle x \rangle)$, and $C = C_N(v_2/\langle x \rangle)$. Then by [5] we must have $O^\infty(N/C) \cong SL(v_2/\langle x \rangle)$. Thus $|H|_p \geq |N|_p \geq q^{10} > q^6 = |\bar{G}|_p$, contradicting the p-maximality of $\bar{G}$. This completes the proof of (4.7).

We now turn our attention to the case $n = 4, p = 2$. Suppose $\bar{G} \leq H < G_2, \bar{G} \ntrianglelefteq H$. We must show $O^{2'}(H) \cong Sp(V_2)$. As we remarked in (4.2)(3), $\bar{G}$ has two orbits on points of V_2. If H normalized these two orbits, then $H \leq GO^+(V_2)$ and $\bar{G} \trianglelefteq H$ which is not the case. Therefore H is transitive on the points of V_2. As in the case of odd characteristic, if for any point A of V_2, $N_H(A)$ acts irreducibly on V_2/A, then $H = G_2$ which contradicts $H < G_2$. For if $N_H(A)$ acts irreducibly on V_2/A, setting $C = C_H(A)$, we have $N_H(A)/C \cong SL_5(q)$ and then H is flag-transitive, so by [10] $H = G_2$.

LEMMA 4.9. *H fixes the form f and therefore $H \leq SO(f) \cong Sp(V_2)$.*

Proof. It is enough to show for each point A of V_2 that N_H(A) normalizes $A^\perp$. Since H is transitive on points it suffices to consider $A = \langle y \rangle$ where

$$y = v_1 \wedge v_2 + v_3 \wedge v_4 \quad (\{v_1, v_2, v_3, v_4\} \text{ a base for } V).$$

For this A, $C = C_{\bar{G}}(A) \cong \mathrm{Sp}_4(q)$ acts indecomposably on V_2 and irreducibly on the hyperplane $A^\perp/A$ of V_2/A. If C did not normalize $A^\perp/A$, then it acts irreducibly on V_2/A which we already said is not the case. Hence C and $N_H(A)$ normalize $A^\perp$ and $H \leq SO(f)$ as claimed. If $O^\infty(H)$ normalized $\bar{G}$, then $O^\infty(H) = \bar{G}$ since $\mathrm{Aut}(\bar{G})/\bar{G}$ is solvable. But then $\bar{G} \trianglelefteq$ H, a contradiction. Therefore, we can assume $H = O^\infty(H) \leq K = Sp(V_2)$ and we must show $H = K$.

From McLaughlin [5, 6], if $\chi(A, A^\perp) \leq H$, then $H = K$ since H is transitive on points. Thus we need to show for some point A of V_2, $\chi(A, A^\perp) \leq H$. Since $|K|_2 = q^9$, it is enough to show $|H|_2 = q^9$. Now set $X = v_1 \wedge v_2$.

$$N = N_{\bar{G}}(\langle x \rangle) = N_{\bar{G}}(\langle v_1, v_2 \rangle) \geq \chi_{\bar{G}}(\langle v_1, v_2 \rangle) = X.$$

X induces transvections on $V_2/\langle x \rangle$ with axis $x^\perp/\langle x \rangle$. Let $D = C_H(\langle y \rangle)$ and $E = C_H(y^\perp/\langle y \rangle)$. Then $D/E \geq Sp(y^\perp/\langle y \rangle) \cong C = C_{\bar{G}}(\langle y \rangle)$. Therefore $|H|_2 \geq q^8$. As before, let $S \in Syl_2(\bar{G})$, $T \in Syl_2(H)$ with $T \geq S$. Set $V^i = \langle v_1, \ldots, v_i \rangle$ for $1 \leq i \leq 4$. We may assume S normalizes V^i. Then $\langle v_1 \wedge v_2, v_1 \wedge v_3 \rangle = Y$ is the only two-subspace of V_2 normalized by S, and hence by T. Then T normalizes

$$Y^\perp = \langle v_1 \wedge v_2, v_1 \wedge v_3, v_1 \wedge v_4, v_2 \wedge v_3 \rangle$$

and hence permutes the three spaces Z of V_2 with $Y \subset Z \subset Y^\perp$. Since there are $q + 1$ such three spaces, no orbit of T on these three spaces can have length greater than q. Let $Z = \langle v_1 \wedge v_2, v_1 \wedge v_3, v_1 \wedge v_4 \rangle$. Thus $|T{:}N_T(Z)| \leq q$ and so

$|N_T(Z)|_2 \geq q^7$. Let $L = N_{\bar{G}}(\langle v_1 \rangle) \cap N_{\bar{G}}(\langle v_2, v_3, v_4 \rangle)$. Then $L \cap C_{\bar{G}}(Z) = 1$ and $L \cong GL_3(q)$. Consequently, $N_H(Z)/C_H(Z) \cong GL_3(q)$. Note that Z is a maximal isotropic subspace of V_2 and as $H \leq K$, $C_H(Z)$ is a two group $[C_K(Z) = \chi_K(Z)$, elementary abelian of order $q^6]$. Set $B = \chi_{\bar{G}}(\langle v_1 \rangle)$. $\chi_{G_2}(Z)$, as a module for L, is isomorphic to $B^* \otimes B^*$, and by (2.4), L acts indecomposably on $\chi_{G_2}(Z)$ with three constituents of dimension three, and so acts indecomposably on $\chi_K(Z)$ with two constituents of dimension three. Therefore, if $|C_H(Z)| > q^3$, then $|C_H(Z)| = q^6$. However, $|N_H(Z)|_2 \geq q^7$,

$$|N_H(Z)/C_H(Z)|_2 = |GL_3(q)|_2 = q^3,$$

and so, in fact, $|C_H(Z)| \geq q^4$. Hence $|N_H(Z)|_2 = q^9$, and as we indicated before, this suffices to complete the proof of (1.2).

5. THE GENERAL CASE

In this section we prove (1.1) for $n \geq 5$ and prove (1.3). Note that since V_2 and V_{n-2} are dual, it suffices to prove the result for V_2.

Let $v_1, v_2, \ldots, v_n$ be a base for V, and set $V^j = \langle v_i : i \leq j \rangle$ and $X_j = \chi_G(V^j)$. Let S be the p-Sylow of G which normalizes V^j for each j, $1 \leq j \leq n-1$. Suppose $g \in G_2$ and $X_1^g \leq S$. If $\sigma \in X_1^{\#}$, then

$$\dim C_{V_2}(\sigma) = \dim C_{V_2}(\sigma^g) = \binom{n-1}{2} + 1,$$

and so σ^g is a transvection by (3.5). Hence X_1^g consists entirely of transvections, so $X_1^g = X_1$ or $X_1^g = X_{n-1}$. However,

$$\dim C_{V_2}(X_1) = n-1, \dim C_{V_2}(X_{n-1}) = \binom{n-1}{2}$$

and since $n \geq 5$, $\binom{n-1}{2} > n-1$. Therefore $X_1^g = X_1$. Similarly if $X_{n-1}^g \leq S$ for some $g \in G_2$, then $X_{n-1}^g = X_{n-1}$. Consequently we have proven

LEMMA 5.1. *X_1, X_{n-1} are weakly closed in S with respect to G_2.*

We now consider the other X_j.

LEMMA 5.2. *X_j is weakly closed in S with respect to G_2 for $1 \leq j \leq n-1$.*

Proof. Suppose P is a point of V and $|\chi_G(P) \cup S| \geq q^{n-i}$. Then we have $P \subseteq V^i$. Similarly if U is a hyperplane of V and $|\chi_G(U) \cap S| \geq q^i$, then $U \supseteq V^i$. Now consider X_j, $2 \leq j \leq n-2$. Suppose $g \in G_2$ and $X_j^g \leq S$. Let P be a one-space in V^j and U a hyperplane containing V^j. $\chi_G(P)^g \cap S \geq [\chi_G(P) \cap S]^g$ and so

$$|\chi_G(P)^g \cap S| \geq q^{n-j}.$$

Also $\chi_G(P)^g \cap S$ consists entirely of transvections by (3.5). However,

$$\dim C_{V_2}(\chi_G(P)^g \cap S) \leq \dim C_{V_2}([\chi_G(P) \cap S]^g) = \dim C_{V_2}(\chi_G(P) \cap S) < \binom{n-1}{2}$$

since $|\chi_G(P) \cap S| \geq q^2$. Therefore $\chi_G(P)^g \cap S$ cannot centralize a hyperplane, so there is a point, which we denote by P^g, such that $\chi_G(P)^g \cap S \leq \chi_G(P^g)$. Since $|\chi_G(P)^g \cap S| \geq q^{n-j}$, $|\chi_G(P^g) \cap S| \geq q^{n-j}$, so $P^g \leq V_j$. Similarly, if U is a hyperplane of V, $U \geq V^j$, then $|\chi_G(U) \cap S| \geq q^j \geq q^2$ and $\chi_G(U)^g \cap S \geq [\chi_G(U) \cap S]^g$. It follows that there is a hyperplane U^g of V so $\chi_G(U^g) \cap S \leq \chi_G(U^g)$. Also, since $\chi_G(U)^g \cap S \geq [\chi_G(U) \cap S]^g$ has order at least q^j, $U^g \geq V_j$. Now for P a point, U a hyperplane with $P \leq V^j \leq U$ we have $\chi_G(P,U)^g \leq X_j^g \leq S$. But

$$\chi_G(P,U) = \chi_G(P) \cap \chi_G(U).$$

Then

$$\begin{aligned}\chi_G(P,U)^g &= \chi_G(P)^g \cap \chi_G(U)^g \cap S \\ &= [\chi_G(P)^g \cap S] \cap [\chi_G(U)^g \cap S] \\ &\leq \chi_G(P^g) \cap \chi_G(U^g) = \chi_G(P^g, U^g).\end{aligned}$$

But $X_j = \langle \chi_G(P,U) : P \leq V_1^j \leq U \rangle$ and the above implies $X_j^g = X_j$ as asserted.

Suppose on the contrary that (1.1) or (1.3) is false. Choose $n \geq 5$ minimal so that either (1.1) or (1.5) is false. Suppose (1.1) fails. Choose $H < G_2$, so $\bar{G} \leq H$, $[\bar{G}|_p < |H|_p$. Let V^j, S, X_j be defined as above. Let $T \in Syl_p(H)$ with $T \geq S$ and set $T_1 = N_T(S)$. Then T_1 normalizes X_j for each j by (5.2). In particular T_1 normalizes $X_1 = \chi_G(\langle v_1 \rangle)$. Let $L = N_{\bar{G}}(\langle x_1 \rangle) \cap N_{\bar{G}}(\langle v_2, \ldots, v_n \rangle)$ and $L_1 = L' \cong SL(\langle v_2, \ldots, v_n \rangle)$. Set $N = N_H(X_1)$, $W = C_{V_2}(X_1)$, $C_1 = C_N(W)$, $C_2 = C_N(V_2/W)$. Finally, let $Q = \chi_{G_2}(W)$. Note that L and L_1 act linearly on Q, and as a module for L and L', $Q \cong X_1^* \otimes \Lambda^{n-3}(X_1^*)$ where $X_1^* = \mathrm{Hom}_F(X_1, F)$. Therefore, by (2.2), if $\chi_N(W) = C_1 \cap C_2 > X_1$, then $|Q : C_1 \cap C_2| \leq q^{n-1}$.

LEMMA 5.3. $C_1 \cap C_2 = X_1$.

Proof. Assume on the contrary that $C_1 \cap C_2 > X$. Let R be a point of W. Then $|\chi_{G_2}(R) \cap M| = q^{(n-1)/2}$. As $n \geq 5$, $\binom{n-1}{2} > n + 1$. Therefore

$$|\chi_{G_2} \cap C_1 \cap C_2| \geq q^2.$$

In fact, from the decomposition of Q as a module for L', we see that for distinct hyperplanes Z, Z' containing R, $\chi_{G_2}(R,Z)$, $\chi_{G_2}(R,Z') \leq C_1 \cap C_2$. By [5, 6] we must have $H = SL(V_2) = G_2$, contradicting $H < G_2$. Consequently we must have $C_1 \cap C_2 = X_1$ as asserted.

LEMMA 5.4. $C_1 \cap C_2 > X_1$.

Proof. Let $\bar{N} = N/X_1$. Denote images in $\bar{N}$ by $-$. $\bar{C}_1, \bar{C}_2 \ntrianglelefteq \bar{N}$ and $\bar{C}_1 \cap \bar{C}_2 = 1$, so $\bar{C}_1\bar{C}_2 = \bar{C}_1 \times \bar{C}_2$. $\bar{N}/\bar{C}_1 = \overline{LC}_1/\bar{C}_1$. Also $\bar{N}/\bar{C}_2 \geq \overline{LC}_2/\bar{C}_2$. $\overline{LC}_2/\bar{C}_2$ acts irreducibly on V_2/W, and as a module for $\overline{LC}_2/\bar{C}_2$, V_2/W is isomorphic to $\Lambda^2(W)$. We know $|N|_p > |N_{\bar{G}}(X_1)|_p$ and therefore

$$|N/X_1|_p > |N_{\bar{G}}(X_1)/X_1|_p = |SL_{n-1}(q)|_p.$$

Since $\bar{N}/\bar{C}_1$ is a subgroup of $GL(W)$, we must have $|\bar{C}_1|_p \neq 1$. Suppose

$$\bar{L}_1\bar{C}_2/\bar{C}_2 \ntrianglelefteq \bar{N}/\bar{C}_2,$$

then by induction, $O^\infty(\bar{N}/\bar{C}_2) \cong SL(V_2/W)$ or $n = 5, p = 2$ and

$$O^\infty(\bar{N}/\bar{C}_2) \cong Sp(V_2/W).$$

But then $O^\infty(\bar{C}_1) \geq SL(V_2/W)$ or $Sp(V_2/W)$. This implies that every constituent of C_1Q/Q on Q has degree a multiple of $\binom{n-1}{2}$, so that C_1Q/Q cannot normalize X_1, a contradiction. Hence,

$$O^\infty(\bar{N}/\bar{C}_2) = \bar{L}_1\bar{C}_2/\bar{C}_2 \cong SL_{n-1}(q).$$

Then $\bar{C}_1' \cong SL(W)$. Now it follows that $|\bar{C}_2|_p \neq 1$, for otherwise $|\bar{N}/\bar{C}_2|_p > |L|_p$ and $\overline{L_1C_2/C} \ntrianglelefteq \bar{N}/\bar{C}_2$ by induction. $O^\infty(\bar{N}/\bar{C}_1) \cong SL_{n-1}(q)$ implies $\bar{C}_2 \cong SL_{n-1}(q)$. Now NQ/Q contains a subgroup isomorphic to $SL_{n-1}(q) \times SL_{n-1}(q)$ and this cannot normalize X_1. We must therefore have $H \cap Q = C_1 \cap C_2 > X_1$ as asserted.

Notice that (5.4) contradicts (5.3). With this contradiction we have shown the p-maximality of $\bar{G}$. Therefore it must be the case that $\bar{G}$ is not nearly maximal. Thus choose $H < G_2$ so $\bar{G} \leq H$, $\bar{G} \ntrianglelefteq H$. By what we have shown, $|\bar{G}|_p = |H|_p$. We now introduce some notation.

Call a vector x in V_2 rank k, $k \leq [n/2]$, if there are independent vectors $w_1, w_2, \ldots, w_{2k}$ in V so $x = w_1 \wedge w_2 + w_3 \wedge w_4 + \ldots + w_{2k-1} \wedge w_{2k}$. Let

$$\Omega = \{\langle x\rangle : x \in V_2,\ x \text{ is rank one}\},$$
$$\Omega_k = \{\langle x\rangle : x \in V_2,\ x \text{ is rank } k\}.$$

A subspace of V_2 is *pure* if all its non-zero vectors are rank 1. The maximal pure subspace of V_2 have dimensions 3 and $n-1$ and are conjugate in $\bar{G}$ to $\langle v_1 \wedge v_2, v_1 \wedge v_3, v_2 \wedge v_3\rangle$ and $\langle v_1 \wedge v_i : i = 2, 3, \ldots n\rangle$, respectively.

LEMMA 5.5. *H does not leave Ω invariant.*

Proof. Suppose $\Omega^H = \Omega$, that is, H does not fuse any vector of rank 1 to a vector of rank k for some k, $2 \leq k \leq [n/2]$. Define the following graph on Ω: $\langle x\rangle \neq \langle y\rangle$ in Ω are adjacent if and only if $\langle x, y\rangle$ is pure. This graph is isomorphic to the graph whose vertices are the two subspaces of V, two adjacent if they meet in a one-space. The automorphism group of this graph, when $n \geq 5$, is $P\Gamma L(V)$. It follows that $\bar{G} \not\leq H$, a contradiction.

Thus H does not leave Ω invariant. Hence H fuses $v_1 \wedge v_2$ to some vector $w = v_1 \wedge v_2 + v_3 \wedge v_4 + \dots + v_{2k-1} \wedge v_{2k}$ of rank k, where $2 \leq k \leq [n/2]$.

LEMMA 5.6. $n = 2k$.

Proof. Suppose $k < n/2$. Set $X = X_{2k} = \chi_G(V^{2k})$ and let $g \in H$ so $gw = v_1 \wedge v_2$. $X \leq C_H(w)$, so $X^g \leq C_H(v_1 \wedge v_2)$. However, $S \leq C_H(v_1 \wedge v_2)$, and since $|S| = |H|_p$, $S \in Syl_p(C_H(v_1 \wedge v_2))$. Therefore without loss of generality we can assume $X^g \leq S$. By (5.2), X is weakly closed in S with respect to G_2, so $X^g = X$. Let

$$W = \Lambda^2(V^{2k}) = \langle v_i \wedge v_j : 1 \leq i < j \leq 2k \rangle = C_{V_2}(X),$$
$$L = N_{\bar{G}}(V^{2k}) \cap C_{\bar{G}}(\langle v_{2k+1}, \dots, v_n \rangle).$$

Then $L \cong \mathrm{SL}(V^{2k})$. Since $g \in N_H(X)$, g normalizes W. Let $N = N_H(X)$, $C = C_N(W)$, $N_1 = N_{\bar{G}}(X)$. $C_1 = C_{N_1}(W) = C_{\bar{G}}(W)$. Clearly $|C|_p \geq |C_1|_p$ and $|N/C|_p \geq |N_1/C|_p$. However, as $N_1 \geq S \in Syl_p(N)$, we have equality in each case. However, $gN \in N/C$ and gN fuses W to $v_1 \wedge v_2$. It follows that gN does not normalize LC/C. By induction, $O^\infty(N/C) \geq SL(W)$ or $Sp(W)$ which contradicts $|N/C|_p = |N_1/C_1|_p = |L|_p$, and completes the lemma.

Therefore we must have $n = 2k$ even, and H fuses $v_1 \wedge v_2$ and

$$w = v_1 \wedge v_n + v_2 \wedge v_{n-1} + \dots + v_k \wedge v_{k+1},$$

but does not fuse $v_1 \wedge v_2$ with any vector of rank j for $2 \leq j \leq k-1$. Let $g \in H$, so $gw = v_1 \wedge v_2$. Set

$$Q = \chi_{\bar{G}}(\langle v_1 \rangle, V^{n-1}) \leq C_{\bar{G}}(w). \qquad Q^g \leq C_H(v_1 \wedge v_2).$$

Since $S \in Syl_p(C_H(v_1 \wedge v_2))$ we can assume $Q^g \leq S$. Since $\bar{G}$ is transitive on incident point-hyperplane pairs of V we can assume $Q^g = Q$, $\langle g, w \rangle \in \Omega$. Note that $gw \in C_{V_2}(Q) = \langle v_1 \wedge v_n, v_i \wedge v_j : 1 \leq i < j \leq n-1 \rangle$. We need to again introduce some notation:

For a vector $v \in V^{n-1}$, set $H(v) = \langle v \wedge w : w \in V^{n-1} \rangle \leq C_{V_2}(Q)$. Let $N = N_H(Q)$, $W = C_{V_2}(Q)$, $R = \chi_{\bar{G}}(V^{n-1})$.

LEMMA 5.7. *$N_H(W)/C_H(W)$ does not act irreducibly on W.*

Proof. Note that R/Q induces transvection W with axis

$$W_1 = \langle v_i \wedge v_j : 1 \leq i < j \leq n-1 \rangle.$$

If $N_H(W)/C_H(W)$ acts irreducibly on W, then by [5],

$$O^\infty(N_H(w)/C_H(w)) \cong SL(W).$$

Then $|N|_p > |S|$, a contradiction.

Thus $N/C_N(W)$ is reducible on W. $N_G(W)$ acts indecomposably on W. $N_G(Q)$ normalizes W_1 and the $n-2$ space $H(v_1)$ and acts irreducibly on $H(v_1)$ and $W_1/H(v_1)$.

LEMMA 5.9. *If N normalizes W_1, then N normalizes $H(v_1)$.*

Proof. If N normalizes W_1, but not $H(v_1)$, then N acts irreducibly on W_1, and thus acts irreducibly on the group of all transvections with axis W_1. Since R/Q induces such transvections, we must have $|C_N(W_1)|_p = q^{(n-1)/2}$. Then $|N|_p > |S|$, a contradiction.

LEMMA 5.10. *If N normalizes $H(v_1)$, then N normalizes W_1.*

Proof. Suppose N normalizes $H(v_1)$, but does not normalize W_1. Then N acts irreducibly on $W/H(v_1)$. Let $R_1 = R\chi_{\bar{G}}(\langle v_1\rangle)$, $K = C_N(W)$, $\bar{N} = N/K$. Note that $Q \in Syl_p(K)$, otherwise $|N|_p > |S|$. $\bar{R}_1 \le \chi_{\bar{N}}(H(v_1))$. If $\bar{R}_1 \ne \chi_{\bar{N}}(H(v_1))$, then $|N|_p > |S|$. Thus $\bar{R}_1 = \chi_{\bar{N}}(H(v_1))$. Let M be the subgroup of $SL(W)$ which centralizes $H(v_1)$ and $W/H(v_1)$. Since $\bar{N}/C_{\bar{N}}(W/H(v_1))$ acts irreducibly on $W/H(v_1)$, every constituent of $\bar{N}$ in M has dimension a multiple of

$$\dim W/H(v_1) = \binom{n-2}{2} + 1.$$

However, $\bar{N}$ normalizes $\bar{R}_1 \le M$. If $n \ge 7$, then $\binom{n-2}{2} + 1 > 2(n-2) = \dim \bar{R}_1$. When $n = 6$, $\binom{n-2}{2} + 1 = 7, n-2 = 4, 2(n-2) = 8$, and we have a contradiction.

From (5.9) and (5.10) it follows that N normalizes $H(v_1)$ and W_1. Since W_1 does not have vectors of rank k, N permutes the vectors of rank 1 in W_1 among themselves. Therefore, if $v \in V^{n-1}$, $g \in N$, $H(v)^g$ is a pure subspace of W_1 with dimension $n-1$. Since $n \ge 6$, $n-1 \ge 5 > 3$, so $H(v)^g = H(v')$ for some $v' \in V^{n-1}$. We know there is a $g \in N$ so $gw \in W$ has rank 1. However, $N_{\bar{G}}(Q)$ is transitive on the vectors of rank 1 in $W - W_1$, so without loss of generality, $gw = v_1 \wedge v_n$. Let $h = g^{-1}$, so $h(v_1 \wedge v_n) = v_1 \wedge v_n + v_2 \wedge v_{n-1} + \dots + v_k \wedge v_{k+1} = w$. Choose $u \in V^{n-1}$ so $H(u)^h = H(v_2)$. Note $u \ne v_1$. Now $v_2 \wedge v_3 + w$ has rank k and this is the image of $v_1 \wedge v_n + g(v_2 \wedge v_3) \in H(u)$. If $\langle g(v_2 \wedge v_3)\rangle \ne \langle u \wedge v_1\rangle$, then $v_1 \wedge v_n + g(v_2 \wedge v_3)$ has rank two and H fuses Ω_2 to Ω_k contrary to assumption. Therefore $\langle g(v_2 \wedge v_3)\rangle = \langle u \wedge v_1\rangle$. Now also $\langle v_2 + v_{n-2} + w\rangle \in \Omega_k$.

$$v_2 \wedge v_{n-2} = h[g(v_2 \wedge v_{n-2}) + v_1 \wedge v_n].$$

But $\langle g(v_2 \wedge v_{n-2})\rangle \ne \langle u \wedge v\rangle$, so $\langle g(v_2 \wedge v_{n-2}) + v_1 \wedge v_n\rangle \in \Omega_2$, and H does fuse Ω_2 and Ω_k and this gives us a final contradiction and completes the proof of (1.1) and (1.3).

6. CONCLUDING REMARKS

(1) Thompson's theorem classifying the groups with a quadratic pair does not immediately yield the conjecture when $p > 3$ since only the groups occurring are known and not all the quadratic modules.

(2) Our method can be extended to prove the conjecture if the following generalization of (2.2)-(2.4) can be proved:

Let i, j be positive integers, $i+j \leq n$. Then as a module for $G = SL(V)$, $V_i \otimes C_j$ decomposes as follows: when $\operatorname{char} F \nmid \binom{i+j}{i}$, $V_i \otimes V_j$ is completely reducible with two constituents, one isomorphic to V_{i+j}. When $\operatorname{Char} F \mid \binom{i+j}{i}$, then G acts indecomposably on $V_i \otimes V_j$. There is a unique composition series $U \subset K \subset V_i \otimes V_j$ with $U \cong V_i \otimes V_j / K \cong V_{i+j}$ as G-modules.

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