EXTENSION OF A THEOREM OF SZEGÖ

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We extend to more general measures a theorem due to Szegö [7] which states that the boundary values of a (nonidentically zero-valued) function in classical Hardy space [4] are log integrable with respect to normalized Lebesgue measure on the unit circle. Our interest in this area arose from a desire to understand the interplay between the existence of bounded point evaluations for a measure and the parts of the measure carried by the open unit disc and the boundary of the unit disc.

Let D denote the open unit disc of the complex plane, $\mathbf{C}$. In this paper all measures considered will be finite positive compactly supported Borel measures carried by $\bar{D}$. For a measure μ let $H^2(\mu)$ denote the closure in $L^2(\mu)$ of the set of polynomials in z. If for some $\lambda \in \mathbf{C}$ the point evaluation functional $p \to p(\lambda)$ defined on polynomials p is bounded, i.e.,

$$\sup\{|p(\lambda)|/\|p\|_\mu : p \text{ is a nonzero polynomial}\} < \infty,$$

then we say that μ has a *bounded point evaluation* at λ or a *b.p.e.* at λ for short.

Let K be a compact set. Then K contains *an exposed arc* J if there exists a simply connected open set E such that $E \cap K = J$ and J is the range of a smooth Jordan curve. A bounded component of K is called a *hole* of K.

For a measure μ carried by $\bar{D}$ for which

(*) there is a hole H of the support of μ so that $\bar{H}$ has an exposed arc Γ with $\Gamma \subset \partial D$

we say that μ satisfies (*) (with respect to H and Γ). Denote the open subarc obtained from Γ by the removal of the endpoints by Γ^0 (if $\Gamma = \partial D$ then let $\Gamma^0 = \partial D$). For example, $d\sigma$, normalized Lebesgue measure on ∂D, satisfies (*) (with respect to D and ∂D).

The *filling in holes* theorem due to Bram [2] interpreted in our context states that for a measure μ satisfying (*) either

(1) μ has a *b.p.e.* at every $\lambda \in H$

or else

(2) μ has no *b.p.e.*'s in H.

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Denote the absolutely continuous and singular parts of $d\mu|_{\partial D}$ with respect to σ by $d\mu_a$ and $d\mu_s$, respectively. By a result of Clary [3],

$$H^2(\mu) \approx H^2(\mu - \mu_s) \oplus L^2(\mu_s)$$

and for $\lambda \in D$, μ has a *b.p.e.* at λ if and only if $\mu - \mu_s$ has a *b.p.e.* at λ. Thus if (1) holds we must have $\mu_a(\Gamma^0) > 0$.

Whenever (1) occurs we consider the functions of $H^2(\mu)$ to have an analytic extension in H [1] and would expect the restrictions of such functions to Γ to reflect this fact. The following theorem shows that this is indeed the case.

THEOREM 1. *Let μ satisfy (*) and assume that $\mu_a(\Gamma^0) > 0$. Then the following are equivalent:*

(A) *μ has bounded point evaluations at every $\lambda \in H$.*

(B) *$L^2(\mu|_{\Gamma^0})$ is not contained in $H^2(\mu)$.*

(C) *For every $f \in H^2(\mu)$ which does not vanish μ_a a.e. on Γ^0, we have*

$$\log|f| \in L^1(d\sigma|_{\Gamma_1})$$

where Γ_1 is any closed subarc of Γ^0.

(*Note. By $L^2(\mu|_{\Gamma^0})$ in* (B) *we mean $\{f \in L^2(\mu): f \equiv 0$ off $\Gamma^0\}$. Also if $\mu_a(\Gamma^0) = 0$ then Clary's result says that* (A) *and* (B) *are equivalent.*)

We remark that our theorem implies Szegö's theorem by considering $H^2(d\sigma)$ and using the fact that $|p(0)|^2 \le \int_{\partial D} |p|^2 d\sigma$ for all polynomials p.

Before proving the theorem we need two lemmas. For the remainder of this paper μ satisfies (*).

LEMMA 1. *Let $\lambda \in H$ (H is a hole of the support of μ as in (*)). μ has a b.p.e. at λ if and only if $1/(z-\lambda) \notin H^2(\mu)$.*

Proof: This is a routine calculation and is left to the reader.

For Γ and Γ^0 as in (*) let Γ_1 be any closed subarc of Γ^0.

LEMMA 2. *Suppose that μ has bounded point evaluations in the hole H, then*

$$\int_{\Gamma_1} \log \frac{d\mu}{d\sigma}\, d\sigma > -\infty.$$

Proof: Let w be any representative of $\dfrac{d\mu}{d\sigma}$ on ∂D. By Clary's result, we may assume that $\mu_s \equiv 0$.

In our original argument we showed that if $\int_{\Gamma_1} \log w\, d\sigma = -\infty$ and if $\lambda \in H$ then a sequence of outer functions f_n can be constructed so that

$$\lim_{n\to\infty} \frac{\|f_n\|_\mu^2}{|f_n(\lambda)|^2} = 0$$

and thus μ can have no *b.p.e.* at λ. A more elegant argument has been communicated to us by J. Brennan and we shall present it instead.

For a measure μ on $\bar{D}$ let

$$\hat{\mu}(z) = \int_{\bar{D}} \frac{1}{w - z}\, d\mu(w)$$

denote the Cauchy transform of μ. Suppose $\int_{\Gamma_1} \log w d\sigma = -\infty$. Let $g \in H^2(\mu)^\perp$ and define $d\nu = g d\mu$, $d\nu_b = g w d\sigma$, and $\nu_a = \nu - \nu_b$. By Lemma 1 it suffices to show that $\hat{\nu} \equiv 0$ in H. By the choice of g, $\hat{\nu} \equiv 0$ for $|z| > 1$ so

$$-\hat{\nu}_b = \hat{\nu}_a \text{ for } |z| > 1. \tag{3}$$

Since $\hat{\nu}_a$ is analytic off the support of ν_a, it is analytic across Γ_1 so by (3) $\hat{\nu}_b$ extends continuously to Γ_1 from $|z| > 1$. For $|z| < 1$, $\hat{\nu}_b \in H^\alpha(d\sigma), \alpha < 1$ since it is the Cauchy integral of an L^1 function [4, p. 39].

Let Ω be a smoothly bounded region with $\Omega \subset H$ and

$$\partial\Omega \cap \text{support of } \mu = \partial\Omega \cap \partial D = \Gamma_1.$$

$\hat{\nu}_a$ is analytic on $\bar{\Omega}$ so $\hat{\nu} = \hat{\nu}_b + \hat{\nu}_a \in H^\alpha(\Omega)$ for $\alpha < 1$. Note that if $|dz|$ is arc length measure on $\partial\Omega$ then $H^\alpha(\Omega) = H^\alpha(|dz|)$ since $\partial\Omega$ is smooth [4, p. 173].

$$\begin{aligned}\lim_{\substack{z\to z_0\in\Gamma_1\\ z\in\Omega}} [\hat{\nu}_b(z) + \hat{\nu}_a(z)] &= \lim_{\substack{z\to z_0\in\Gamma_1\\ z\in\Omega}} [\hat{\nu}_b(z) - \hat{\nu}_b(z^*)] \\ &= g(z_0)\, w(z_0)\end{aligned}$$

for σ-*a.e.* $z_0 \in \Gamma_1$. Here z^* is the reflection of z in ∂D. The first inequality follows from (3) and the continuity of $\hat{\nu}_b$ on Γ_1. The second follows from an argument similar to that of [4, p. 39] relating Cauchy integrals to Poisson integrals and then applying Fatou's theorem. Thus

$$\begin{aligned}\int_{\Gamma_1} \log|\hat{\nu}|\, d\sigma &= \int_{\Gamma_1} \log|gw|\, d\sigma \\ &= \frac{1}{2}\int_{\Gamma_1} \log(|g|^2 w)\, d\sigma + \frac{1}{2}\int_{\Gamma_1} \log w\, d\sigma \\ &\le \frac{1}{2}\int_{\Gamma_1} |g|^2 w d\sigma + \frac{1}{2}\int_{\Gamma_1} \log w\, d\sigma \\ &= -\infty.\end{aligned}$$

Since $\partial\Omega$ is smooth, a conformal mapping argument applied to the classical Szegö's theorem shows that if $f \in H^2(|dz|), f \not\equiv 0$ then $\log|f| \in L^1(|dz|)$. Hence $\hat{\nu} \equiv 0$ in Ω and thus in H. This completes the proof.

Proof of Theorem 1. Recall that μ satisfies (*) and that $\mu_a(\Gamma^0) > 0$. We will show that $(C) \Rightarrow (B) \Rightarrow (A) \Rightarrow (C)$.

If (B) fails then $L^2(\mu|_{\Gamma^0}) \subset H^2(\mu)$. Choose any closed subarc Γ_1 of Γ^0 with $\mu_a(\Gamma_1) > 0$. Since $\sigma(\Gamma^0 - \Gamma_1) > 0$ we consider χ_{Γ_1} and see that (C) fails.

For $g \in C(\bar{D})$ with closed support in Γ^0, a version of Mergelyan's theorem [5, p. 51] shows that g can be uniformly approximated on the support of μ by rational functions with poles off the support of μ. If (A) fails, by Lemma 1 such rational functions are in $H^2(\mu)$. Hence by Lusin's theorem [6, p. 53], (B) fails.

Suppose that (A) holds and (C) fails. Since (C) fails there is an $f \in H^2(\mu)$ satisfying f restricted to Γ^0 is not zero σ *a.e.* on Γ^0, but $\log|f| \notin L^1(d\sigma|_{\Gamma_1})$ where Γ_1 is any closed subarc of Γ^0.

Consider the new measure $d\nu = |f|^2 d\mu$. Since $\int_{\Gamma_1} \log \frac{d\nu}{d\sigma} = -\infty$. ν has no b.p.e.'s in H by Lemma 2. Using Mergelyan's theorem and arguing as in $(B) \Rightarrow (A)$ we conclude that $\chi_{\Gamma_1} \in H^2(\nu)$ or

$$\chi_{\Gamma_1} f \in H^2(\mu). \tag{4}$$

Let $S_n = \{x \in \Gamma_1 : |f(x)| \geq 1/n\}$. We may choose m so that $\sigma(S_m) > 0$ since $f \not\equiv 0\ \sigma$ *a.e.* on Γ^0. Since Γ_1 is compact and properly contained in ∂D we may use the Hartog-Rosenthal theorem [5, p. 47] and Lusin's theorem to choose a bounded sequence of polynomials p_k so that

$$p_k \to \chi_{S_m}/f \text{ pointwise } \sigma\text{-}a.e. \text{ on } \Gamma_1. \tag{5}$$

Combining (4) and (5) we see that $p_k \chi_{\Gamma_1} f \in H^2(\mu)$ so the dominated convergence theorem gives $\chi_{S_m} \in H^2(\mu)$. For any $\lambda \in H$ a similar argument shows that

$$\chi_{S_m}/(z - \lambda) \in H^2(\mu).$$

By Lemmas 1 and 2, we may choose polynomials p_k approximating $1/(z - \lambda)$ in $H^2(d\mu|_{\bar{D}-S_m})$. Then $p_k(1 - \chi_{S_m}) + \chi_{S_m}/(z - \lambda) \in H^2(\mu)$ and we conclude that $1/(z - \lambda) \in H^2(\mu)$. By Lemma 1 this contradicts (B). This completes the proof.

Additional Remarks: Assume that μ satisfies (*), (1) holds, and $\mu_s|_\Gamma \equiv 0$. In a preliminary version of this paper a question was raised concerning whether functions in $H^2(\mu)$ have other local properties on Γ similar to those of classical Hardy spaces. (One example is part (C) of Theorem 1.) J. Thompson and R. Olin have informed the author that this is the case. For some related work confer [9]. Also it should be noted, as pointed out by J. Brennan, that the implication $(A) \Rightarrow (C)$ of Theorem 1 can be established by the methods used in Lemma 2.

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